# SMOOTH ORBIFOLDS

PATRICK IGLESIAS-ZEMMOUR

ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-SO.pdf

We discuss the smooth structure of orbifolds in the context of Diffeology. We illustrate the definition of Diffeological Orbifolds with two simple examples: the Cone Orbifold and the Teardrop.

The word *orbifold* has been coined by Thurston [WT78] in 1978 as a replacement for *V-manifold*, concept invented by Ishiro Satake in 1956 [IS56]. The concept was introduced to describe the smooth structure of spaces that look like manifolds, except on a few subsets, where they look like quotients of linear domains by a finite group of linear transformations.

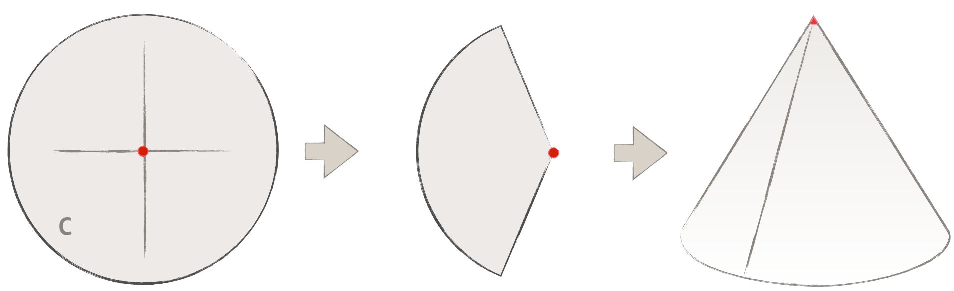


Figure 1. The Cone Orbifold viewed by a topologist

The typical example is the quotient of the field $\mathbf{C}$ by a group of roots of unity[1]. The quotient space is always drawn as a cone, to suggest the singularity of the point 0. But how do we capture the

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[1]We consider $\mathbf{C}$ for its field structure: addition and multiplication, not for its complex structure which is anecdotic here.

smooth structure around the singular point? That is the whole question.

**1. The Satake Model** — Satake defined the smooth structure of an orbifold by a family of compatible *local uniformizing systems*.

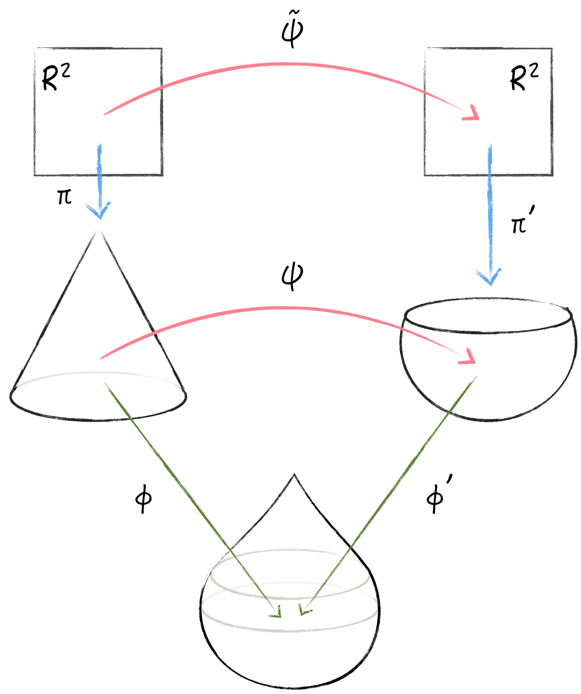


Figure 2. Satake's local uniformizing systems

Given a topological space M, a *local uniformizing system* for an open subset $\mathcal{O} \subset \mathrm{M}$ is a triple $(\mathrm{U}, \Gamma, \varphi)$, where U is a connected open subset of $\mathbf{R}^n$ for some $n$, $\Gamma$ is a finite group of diffeomorphisms[2] of U, and where $\varphi\colon \mathrm{U} \to \mathcal{O}$ is a map which induces a homeomorphism between $\mathrm{U}/\Gamma$ and $\mathcal{O}$. Then, every point of M must belong to some uniformized open subset, and the local uniformizing systems covering M must be compatible, that is, patched together by smooth *injections*, or *transition maps*. An *injection* from a local uniformizing system $(\mathrm{U}, \Gamma, \varphi)$ to another $(\mathrm{U}', \Gamma', \varphi')$ is a diffeomorphism $\lambda$ from U onto an open subset of $\mathrm{U}'$ such that $\varphi = \varphi' \circ \lambda$. Read the precise description detailed in the paper *Orbifolds as Diffeologies* [IKZ10].

[2]Without loss of generality, one can assume $\Gamma \subset \mathrm{GL}(n, \mathbf{R})$.

**The main problem** with Satake's approach is that it does not lead to a satisfactory notion of smooth maps, and therefore prevents the conception of a category of orbifolds. Indeed, in [IS57, page 469], Satake writes this footnote:

> "*The notion of* $C^\infty$*-map thus defined is inconvenient in the point that a composite of two* $C^\infty$*-maps defined in a different choice of defining families is not always a* $C^\infty$ *map.*"

And, for a mathematician, that is very annoying.

We have solved this delicate problem, ten years ago in our paper[3] [IKZ10], by embedding Satake's V-manifolds into the category of diffeological spaces.

**2. The Diffeology Approach** — The diffeological framework is well adapted to formalize the intuition of orbifolds. Indeed, what would be an orbifold from the point of view of diffeology?

**Definition** *An orbifold is a diffeological space which is localy diffeomorphic, at each point, to some quotient space* $\mathbf{R}^n/\Gamma$*, for some finite subgroup* $\Gamma$ *of the linear group* $\mathrm{GL}(n, \mathbf{R})$.

And that works. We could show [IKZ10] that, according to this definition, it was possible to associate to every defining family of local uniformizing system, a diffeology on the underlying set; and conversely, to every diffeological orbifold, a family of local uniformizing system defining a Satake's V-manifold. And we could show that these constructions are inverse from each other, modulo equivalence.

Therefore, it is reasonable and advantageous to declare the smooth structure of the orbifold to be defined by its diffeology, that is, by all its smooth parametrizations (plots).

This approach reverses the usual point of view, as it often happens in diffeology. We don't build a struture on top of a topological set by gluing together local uniformizing system. The sets comes equipped with a diffeologies and we check wether or not this diffeology is an orbifold diffeology.

**3. First example: The Cone Orbifold** — The cone orbifold is the quotient

$$\mathcal{C}_m = \mathbf{C}/\mathbf{Z}_m,$$

[3] But published later.

where $m \in \mathbf{N}$, $m \neq 0$ and

$$\mathbf{Z}_m = \{\varepsilon \in \mathbf{C} \mid \varepsilon^m = 1\}.$$

The diffeological space $\mathcal{C}_m$, equipped with the quotient diffeology, it is by definition an orbifold. The topologists are used to represent this orbifold by gluing the two sides of a fundamental domain, as it is illustrated in Figure 1. But that representation disservices the diffeological intuition. We will show now how the orbifold $\mathcal{C}_m$ can be represented as a special diffeology on the field $\mathbf{C}$ itself.

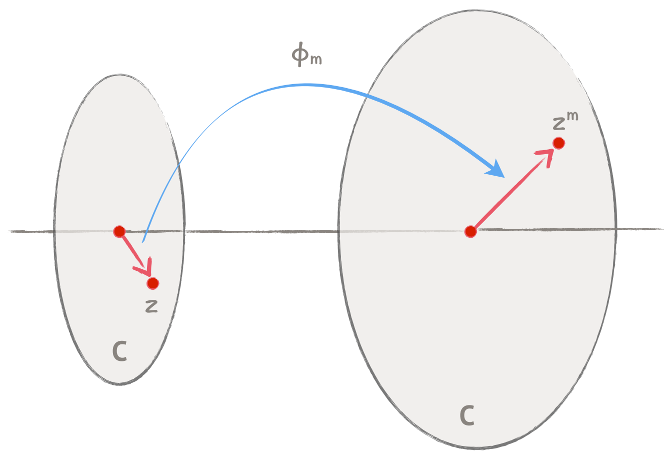


Figure 3. The Cone Orbifold viewed by a diffeologist

Considering then the map

$$\varphi_m : \mathbf{C} \to \mathbf{C} \quad \text{with} \quad \varphi_m(z) = z^m,$$

it is clear that the preimages of the points $\zeta \in \mathbf{C}$ are exactly the orbits of the group $\mathbf{Z}_m$. Since $\varphi_m$ is surjective, $\mathbf{C}$ can be identified with the quotient set $\mathbf{C}/\mathbf{Z}_m$, with canonical projection $\varphi_m$. The last question is what diffeology on $\mathbf{C}$ represents the quotient diffeology? And naturally, it is the pushforward

$$\mathcal{C}^\infty_m = \varphi_{m*}(\mathcal{C}^\infty)$$

of the standard diffeology $\mathcal{C}^\infty$ on $\mathbf{C}$.

**Property** A parametrization $\mathrm{P} : \mathrm{U} \to \mathbf{C}$ belongs to $\mathcal{C}^\infty_m$ if an only if, for all point $r_0 \in \mathrm{U}$, there exist a small ball $\mathcal{B}$ centered at $r_0$ and a smooth parametrization $\mathrm{Q} : \mathcal{B} \to \mathbf{C}$ such that $\mathrm{P}(r) = \mathrm{Q}(r)^m$, for all $r \in \mathcal{B}$.

Thus, as we can see in this simple example, the same set $\mathbf{C}$ can be equipped with an inifinity of orbifold diffeologies, one for each integer, without altering the underlying space.

**4. Second example: The Teardrop** — The teardrop is the orbifold sketched in Figure 2. It is a sphere with a conic singular point on top, at the north pole N. We describe this orbifold as a diffeology on the underlying space $S^2$. By convenience, $S^2$ is regarded as a subset of $\mathbf{C} \times \mathbf{R}$. Now, the set of parametrizations

$$\zeta : U \to S^2 \quad \text{with} \quad \zeta(r) = \begin{pmatrix} Z(r) \\ t(r) \end{pmatrix}, \quad \text{and} \quad |Z(r)|^2 + t(r)^2 = 1.$$

that satisfy, for all $r_0 \in U$,

(1) if $\zeta(r_0) \neq N$, then there exists a small ball $\mathcal{B}$ centered at $r_0$ such that $\zeta \restriction \mathcal{B}$ is smooth.
(2) If $\zeta(r_0) = N$, then there exist a small ball $\mathcal{B}$ centered at $r_0$ and a smooth parametrization $z$ in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,

$$\zeta(r) = \frac{1}{\sqrt{1 + |z(r)|^{2m}}} \begin{pmatrix} z(r)^m \\ 1 \end{pmatrix}$$

is a diffeology of orbifold with a unique singularity, of conic type and structure group $\mathbf{Z}_m$, at the north pole. This is illustrated by the Figure 4.

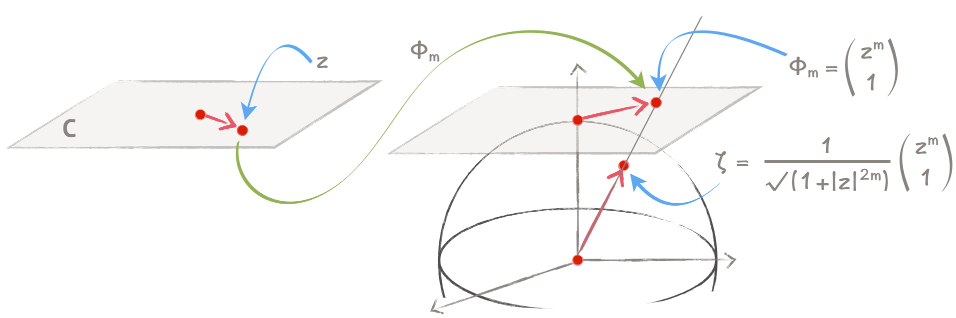


Figure 4. The diffeology of the teardrop

**5. Conclusion** — It is not hard now, to imagine many other examples based on the construction of previous two: a sphere, or a plane, with as many different singular conic (or not) points we wants. I leave that as an exercise.

A last remark, however. The fact that, contrarily to manifolds, orbifolds may have a rich set of smooth local invariants, permits to build easily more different orbifold strutures on the same underlying space. In our examples above, we just picked up a diffeology finer than the standard manifold diffeology which happens to be an orbifold diffeology.

In particular, we have seen earlier that the quotient space of a disc by $\mathbf{Z}_m$ is equivalent to the same disc but equipped with a finer diffeology. It is then easy to extract from a manifold diffeology, a finer orbifold diffeology with as many conic singularities as we want, with arbitrary structure groups. Let's build a simple example: Consider the plane $\mathbf{C}$ and let $\mathrm{L} : \mathbf{Z} + i\mathbf{Z} \to \mathbf{N}$ be the bijection described by the figure 5.

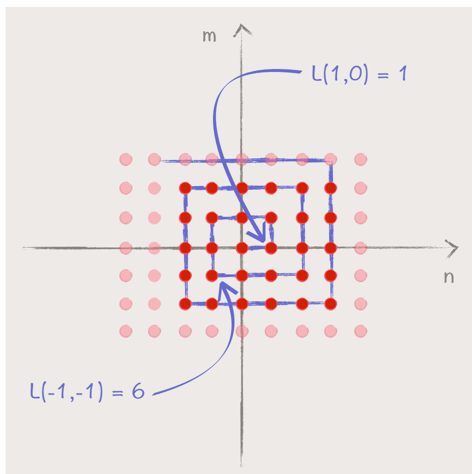


Figure 5. The function $\mathrm{L}(n, m)$.

Then, let us define the following diffeology, with parametrizations $\zeta : \mathrm{U} \to \mathbf{C}$ such that, for all $r_0 \in \mathrm{U}$:

(1) if $\zeta(r_0) \notin \mathbf{Z} + i\mathbf{Z}$, then there exists a small ball $\mathcal{B}$ centered at $r_0$ such that $\zeta \upharpoonright \mathcal{B}$ is smooth.
(2) If $\zeta(r_0) = n + im$, with $n, m \in \mathbf{Z}$, then there exist a small ball $\mathcal{B}$ centered at $r_0$ and a smooth parametrization $z$ in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,
$$\zeta(r) = n + im + z(r)^{1+\mathrm{L}(n,m)}.$$

In this example, the integer points $n + im \in \mathbf{C}$ are conic with cyclic groups all different, equal to $\mathbf{Z}_{1+\mathrm{L}(n,m)}$.

Note finally that — with all the transition functions to specify — a description of this orbifold using the original Satake's defining family would be, for the least, laborious. We can appreciate, on this example, the simplification brought by the diffeological approach.

## References


[IS56] Ishiro Satake. *On a generalization of the notion of manifold*, Proceedings of the National Academy of Sciences **42** (1956), 359–363.

[IS57] Ishiro Satake. *The Gauss-Bonnet Theorem for V-manifolds*. Journal of the Mathematical Society of Japan, Vol. 9, No. 4, (1957), 464–492.

[WT78] William Thurston. *The Geometry and Topology of Three-Manifolds* (Chapter 13). Princeton University lecture notes, 1978–1981.

[IKZ10] Patrick Iglesias, Yael Karshon and Moshe Zadka. *Orbifolds as Diffeology*. Transactions of the American Mathematical Society, Volume 362, Number 6, Pages 2811–2831. Providence R.I. June 2010.


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