SMOOTH EMBEDDINGS AND SMOOTHLY EMBEDDED SUBSETS

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-SEASES.pdf

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Since Henri Joris we potentially know that, paradoxically, the semi-cubic $y^2 - x^3 = 0$ is an embedded submanifold of $\mathbf{R}^2$. So, how to understand the singularity of the cusp? This is because, if the cusp is embedded, it is not smoothly embedded. And this note will detail this aspect.

Smooth Embeddings

The semi-cubic is the subset $\mathfrak{S} \subset \mathbf{R}^2$ of equation $y^2 - x^3 = 0$. It is represented in Figure 1. It is the image of the map

$$j : \mathbf{R} \to \mathbf{R}^2, \quad j : t \mapsto \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}.$$

The map j is not an immersion since $j'(0) = 0$, where j' is the derivative of j. But a longtime question was : is j an induction?

I recall that the map j is an induction [PIZ13, §1.29] if and only if: j is smooth, injective and the inverse $j^{-1} : \mathfrak{S} \to \mathbf{R}$ is smooth, where $\mathfrak{S} = \mathrm{val}(j)$ is equipped with the subset diffeology.

In that case, that is equivalent to the following:

Question. For any smooth parametrization $P : r \mapsto (x(r), y(r))$ with values in $\mathfrak{S}$, that is, $y(r)^2 - x(r)^3 = 0$ for all r, is the map

$$t(r) = \frac{y(r)}{x(r)}$$

smooth.

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The solution to this question comes from a theorem due to Henri Joris [Jor82], which was pointed out to me by Yael Karshon.[1] In a simple version:

Theorem. [H. Joris] *If a real map* f *is such that* $t \mapsto f(t)^2$ *and* $t \mapsto f(t)^3$ *are smooth, then* f *is smooth.*

Corrolary. *The map* $j : t \mapsto (t^2, t^3)$ *is an induction from* $\mathbf{R}$ *into* $\mathbf{R}^2$*. The semi-cubic* $\mathfrak{S}$ *image of* j *is an embedded submanifold of* $\mathbf{R}^2$*.*

Proof. Clearly $j : t \mapsto (x = t^2, y = t^3)$ is smooth, and injective: $t = \sqrt[3]{y}$. Let $r \mapsto P(r) = (x(r), y(r))$ be a plot in $\mathbf{R}^2$ with value in $\mathfrak{S}$. Then, $j^{-1} \circ P(r) = t(r)$ such that $r \mapsto t(r)^2$ and $r \mapsto t(r)^3$ are smooth. To apply Joris theorem we need to come back to a map from $\mathbf{R}$ to $\mathbf{R}$. We can use Boman's theorem][Bom67]

Theorem. [J. Boman] *A continuous parametrization* $P : U \to \mathbf{R}^m$*, is smooth if and only if, for any smooth path* γ *in U, the composite* $P \circ \gamma$ *is smooth.*

But, for each such smooth path, the composite $s \mapsto t(r(s))$ is smooth, thanks indeed to Joris theorem. Thus, $r \mapsto P(r)$ is smooth. To finish j is an embedding because $y \mapsto \sqrt[3]{y}$ is an homeomorphism of $\mathbf{R}$. ☐

So, we have a figure, the semi-cubic, image of $\mathbf{R}$ by an induction, which is by construction a submanifold of $\mathbf{R}^2$ because its subset diffeology is equivalent to $\mathbf{R}$. There is clearly something weird in that situation, the cusp at $(0, 0)$ is obviously a singularity. But since it is transparent to the subset diffeology, how to capture it? That is the question.

The answer lies in the relationship between the ambient space $\mathbf{R}^2$ and the subspace $\mathfrak{S}$. Consider the pseudo-group of diffeomorphisms of $\mathbf{R}^2$ that preserve $\mathfrak{S}$, that is, that fix globally $\mathfrak{S}$,

$$\mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S}) = \{\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2) \mid \phi(\mathfrak{S}) \subset \mathfrak{S}\}$$

Since ϕ is a local diffeomorphism $\phi(\mathfrak{S}) = \phi(\mathfrak{S} \cap \mathrm{dom}(\phi))$. If $\mathfrak{S} \cap \mathrm{dom}(\phi)$ is empty, there is nothing to check. Then,

Proposition. *Every local diffeomorphism* $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S})$ *fixes the point* $(0, 0)$*,*

$$\forall \phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S}),\ \phi(0, 0) = (0, 0).$$

[1] In a private discussion, but since commented in [KMW22].

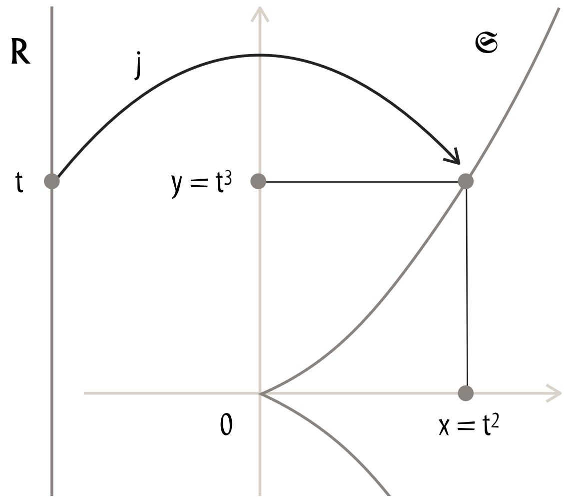

Figure 1. The semi-cubic $y^2 = x^3$.

Proof. Let $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S})$ such that $\phi(0,0) = (x,y)$ and $(x,y) \neq (0,0)$. The restriction $\phi \restriction \mathfrak{S}$ belongs to $\mathrm{Diff}_{\mathrm{loc}}(\mathfrak{S})$. Thus, $\varphi = j^{-1} \circ (\phi \restriction \mathfrak{S}) \circ j$ belongs to $\mathrm{Diff}_{\mathrm{loc}}(\mathbf{R})$.

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\ j\ } & \mathfrak{S} \subset \mathbf{R}^2 \\ {\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \phi \restriction \mathfrak{S}} \\ \mathbf{R} & \xrightarrow[\ j\ ]{} & \mathfrak{S} \subset \mathbf{R}^2 \end{array}$$

Let us derive $j \circ \varphi = (\phi \restriction \mathfrak{S}) \circ j$ at the point 0, and let $j(t) = (x,y) = \phi(0,0)$:

$$\begin{aligned} D(j \circ \varphi)(0) &= D((\phi \restriction \mathfrak{S}) \circ j)(0) \\ D(j)(t) \circ D(\varphi)(0) &= D(\phi \restriction \mathfrak{S})(0,0) \circ D(j)(0) \\ D(j)(t) \circ D(\varphi)(0) &= 0. \end{aligned}$$

But $t \neq 0$ implies $\mathrm{rank}(D(j)(t)) = 1$, and $D(\varphi)(0)$ is a non-zero number, thus the composite $D(j)(t) \circ D(\varphi)(0)$ cannot be zero. ☐

So, although the local diffeomorphisms are transitive on $\mathfrak{S}$, the local diffeomorphisms of $\mathbf{R}^2$ preserving $\mathfrak{S}$ are not. This is this iatus which capture the singularity at 0. This remark leads then to a refinement of the concept of embedding. Let X be a diffeological space and $\mathfrak{S} \subset X$ an embedded subset [PIZ13, §2.13].

Let us begin by defining two groupoids:

A) Let $K_{\mathfrak{S}}$ be the groupoids of germs of local diffeomorphisms of $\mathfrak{S}$.

$$\begin{cases} \mathrm{Obj}(K_{\mathfrak{S}}) = \mathfrak{S} \\ \mathrm{Mor}_{\mathfrak{S}}(x, x') = \{\mathrm{germ}(\varphi)_x \mid \varphi \in \mathrm{Diff}_{\mathrm{loc}}(\mathfrak{S}),\ \varphi(x) = x'\} \end{cases}$$

B) Let $K_{X,\mathfrak{S}}$ be the groupoids of germs of local diffeomorphisms of X preserving $\mathfrak{S}$.

$$\begin{cases} \mathrm{Obj}(K_{X,\mathfrak{S}}) = \mathfrak{S} \\ \mathrm{Mor}_{X,\mathfrak{S}}(x, x') = \{\mathrm{germ}(\phi)_x \mid \phi \in \mathrm{Diff}_{\mathrm{loc}}(X, \mathfrak{S}),\ \varphi(x) = x'\} \end{cases}$$

There is a natural morphism restriction Phi from $K_{X,\mathfrak{S}}$ to $K_{\mathfrak{S}}$:

$$\Phi_{\mathrm{Obj}} = \mathbf{1}_{\mathfrak{S}} \quad \text{and} \quad \Phi_{\mathrm{Mor}} : \mathrm{germ}(\phi)_x \mapsto \mathrm{germ}(\phi \upharpoonright \mathfrak{S})_x.$$

That leads to the following defintion:

Definition 1. *We shall say that an embedded subset* $\mathfrak{S} \subset X$ *is* smoothly embedded *if the morphism* Φ, *from* $K_{X,\mathfrak{S}}$ *to* $K_{\mathfrak{S}}$ *is surjection on the arrows. That is, if* Φ *is a full functor*

In other words, the embedded subset $\mathfrak{S}$ is smoothly embedded if it is embedded and the germ of any local diffeomorphism of $\mathfrak{S}$ can be extended to a local diffeomorphism of X; or again if the germ of any local diffeomorphism of $\mathfrak{S}$ is the imprint, or the trace, of a local diffeomorphism of X.

So, the semi-cubic is indeed and embedded submanifold, but not smoothly embedded.

This notion of embedded subset goes back to the embeddings. We get the definition of smooth embeddings

Definition 2. *Let* $j : \Sigma \to X$ *be an embedding. we shall say that* j *is a smooth embedding if* $\mathfrak{S} = j(\Sigma)$ *is smoothly embedded in* X.

So, $j : t \mapsto (t^2, t^3)$ is an embedding, but not a smooth embedding.

References

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