POISSON BRACKET IN DIFFEOLOGY

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In this note we show how to understand the Poisson bracket in diffeology, working directly on the group of Hamiltonian diffeomorphisms without involving tangent spaces and Hamiltonian gradients. Indeed, as it is usually defined, Poisson bracket seems to be a contravariant object and therefore not really adapted to diffeology, but it can be defined in a covariant way, which is more adequate with the diffeology framework.

Classical Poisson brackets

Poisson bracket are generally introduced in symplectic geometry as a binary operation on the space of functions. Let (M, ω) be a symplectic manifold, that is,

$$\omega \in \Omega^2(M), \quad d\omega = 0 \quad \text{and} \quad \ker(\omega) = 0.$$

Let $x \mapsto u$ be a smooth real function on M, we denote by

$$\operatorname{grad}_\omega(u) = \omega^{-1}(du)$$

is *symplectic gradient*. Here ω is regarded as a linear isomorphism from TM to T^*M, so $\omega^{-1}(du)$ is a tangent vector field. The Poisson bracket is usually defined in the following way [Sou70]:

Definition [Souriau] *Let* $x \mapsto u$ *and* $x \mapsto v$ *be two smooth real functions on* M,[1] *The* Poisson bracket *of* u *and* v *is denoted and*

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[1]They are called *dynamic variables* by Souriau.

defined by:[2]

$$\{u, v\} = \omega(\mathrm{grad}_\omega(u), \mathrm{grad}_\omega(v)).$$

The Poisson bracket as it is so defined is a bilinear map

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(M, \mathbf{R})^2 \to \mathcal{C}^\infty(M, \mathbf{R})$$

That satifies some classical relations called Jacobi identity we do not discuss here.

Next, by applying the definition above of the symplectic gradient, we have also the equivalent definition:

$$\{u, v\} = du(\mathrm{grad}_\omega(v)) \quad \text{also denoted by} \quad \frac{\partial u}{\partial x}(\mathrm{grad}_\omega(v)).$$

Poisson brackets in Diffeology

The problem with all these concept involving tangent vector fields and Lie algebras is that they have not one only interpretation in diffeology. That is why we have to bypass as much as possible the introduction of tangent spaces in their generalizations in diffeology. That works for the moment map, relatively well, as the many examples in [PIZ10] have shown. This is what we propose in the case of the Poisson bracket, to get a definition that covers the classical case and satifies the constraints of a good diffeological equivalent.

Let us now come back to the classical picture for a while. Assume that the vector field

$$x \mapsto \mathrm{grad}_\omega(u)$$

is integrable. That is, it defines a 1-parameter group of diffeomorphisms

$$t \mapsto e^{\mathrm{grad}_\omega(u)}.$$

Let us change our notation to:

$$Z_M : x \mapsto Z_M(x) = \mathrm{grad}_\omega(u) \quad \text{and} \quad Z'_M : x \mapsto Z'_M(x) = \mathrm{grad}_\omega(u').$$

We have

$$\{u, u'\} = \omega_x(Z_M(x), Z'_M(x))$$

Assume now that Z and Z' belong to the Lie algebra $\mathcal{G}$ of a Lie group G

$$Z, Z' \in \mathcal{G}.$$

[2]It is denoted by $[u, v]_P$ by Souriau.

The 1-parameter groups $\{e^{t\,\mathrm{grad}_\omega(u)}\}_{t\in\mathbf{R}}$ and $\{e^{t\,\mathrm{grad}_\omega(u')}\}_{t\in\mathbf{R}}$ are just the action of two 1-parameter group in G

$$\{e^{tZ}\}_{t\in\mathbf{R}} \quad \text{and} \quad \{e^{tZ'}\}_{t\in\mathbf{R}} \quad \text{belong to} \quad \mathrm{Hom}^\infty(\mathbf{R}, G),$$

and $Z_M(x)$ and $Z'_M(x)$ are the fundamental vector fields associated with Z and Z′ in $\mathcal{G}$. Thus,

$$\omega_x(Z_M(x), Z'_M(x)) = \hat{x}^*(\omega)(Z, Z'),$$

where

$$\hat{x} : g \mapsto g_M(x)$$

is the *orbit map* at the point x and $g_M \in \mathrm{Diff}(M)$ denotes the action of G on M. Indeed

$$\begin{aligned}\hat{x}^*(\omega)(Z, Z') &= \omega_x(D(\hat{x})_{1_G}(Z), D(\hat{x})_{1_G}(Z')) \\ &= \omega_x(Z_M(x), Z'_M(x)),\end{aligned}$$

because

$$Z_M(x) := \left.\frac{\partial e^{tZ}_M(x)}{\partial t}\right|_{t=0} = D(\hat{x})_{1_G}(Z),$$

where 1_G denotes the identity in G. Now,

Proposition. *If the group* G *preserves* ω*, that is, if*

$$\forall g \in G, \quad g^*_M(\omega) = \omega,$$

then

$$\forall g \in G, \quad L(g)^*(\hat{x}^*(\omega)) = x^*(\omega).$$

where $L(g)$ *is the left multiplication in* G*. Thus the map* $x \mapsto \hat{x}^*(\omega)$ *defined on* M *takes its values in the vector space of left invariant 2-forms on* G*.*

Let us denote for now

$$\mathcal{G}^*_k = \{\epsilon \in \Omega^k(G) \mid \forall g \in G, \quad L(g)^*(\epsilon) = \epsilon\}.$$

Thus,

$$\{\cdot,\cdot\} = [x \mapsto \hat{x}^*(\omega)] \in \mathcal{C}^\infty(X, \mathcal{G}^*_2).$$

Such that

$$\{\cdot,\cdot\}_x(Z, Z') = \omega_x(Z_M(x), Z'_M(x)).$$

Now, let us come back to our symplectic gradients. The flow $\{e^{t\,\mathrm{grad}_\omega(u)}\}_{t\in\mathbf{R}}$ they define is called *Hamiltonian* because its Hamiltonian [PIZ13, §9.15] is $[x \mapsto u]$. Of course the group

$$H_\omega = \mathrm{Ham}(M, \omega)$$

is not a Lie group, but it is a diffeological group and as such obeys to all diffeological constructions, in particular the diffeological vector space of left-invariant k-forms on H_ω is well defined

$$\mathcal{H}^*_{k,\omega} = \{\epsilon \in \Omega^k(H_\omega) \mid L(g)^*(\epsilon) = \epsilon\},$$

for any diffeological space X equipped with a closed 2-form ω. The space $\mathcal{H}^*_\omega = \mathcal{H}^*_{1,\omega}$ has been already defined as the space of *momenta* of the group H_ω [PIZ13, §7.12].

Therefore, we can define now the Poisson bracket in full generality:

Definition [Poisson Bracket] *Let* X *be a diffeological space and* ω *be a closed* 2*-form on* X*. Let* $\mathrm{Ham}(X,\omega)$ *be the group of Hamiltonian diffeomorphisms* [PIZ13, §9.15]*, and* $\mathcal{H}^*_2$ *be the diffeogical vector space of left-invariant* 2*-form on* X*. We call* Poisson bracket *the map*

$$\{\cdot,\cdot\} : x \mapsto \hat{x}^*\omega, \quad \{\cdot,\cdot\} \in \mathcal{C}^\infty(X, \mathcal{H}^*_2).$$

Remark The poisson bracket is not only a map from X to left-invariant 2-forms on $\mathrm{Ham}(X,\omega)$ but to closed 2-forms. Indeed $d[\hat{x}^*\omega] = \hat{x}^*[d\omega] = 0$. Developed, that gives the Jacobi identity.

Now, let us check that this definition fits the concept of Poisson bracket when X is a symplectic manifold. For that we consider a n-plot $P : U \to H_\omega$ with $P : r \mapsto g_r$, centered at $1_G = P(0)$. Then, for all $x \in X$, for all $r \in U$ and $\delta r, \delta' r \in \mathbf{R}^n$:

$$\begin{aligned}\{\cdot,\cdot\}_x(P)_r(\delta r, \delta' r) &= \hat{x}^*\omega(P)_r(\delta r, \delta' r)\\ &= \omega(\hat{x} \circ P)_r(\delta r, \delta' r)\\ &= \omega(r \mapsto g_r(x))_r(\delta r, \delta' r).\end{aligned}$$

Since $\{\cdot,\cdot\}_x$ is left-invariant it is defined by its value at the identity [PIZ13, §6.40, 7.18]. Thus, let us compute $\{\cdot,\cdot\}_x(P)$ at $r = 0$ for two vector $v, v' \in \mathbf{R}^n$.

$$\begin{aligned}\omega(r \mapsto g_r(x))_0(v, v') &= \omega(r \mapsto g_r(x))_0(v, v')\\ &= \omega_{g_0(x)}\Big(\frac{\partial g_r(x)}{\partial r}\Big|_{r=0}(v), \frac{\partial g_r(x)}{\partial r}\Big|_{r=0}(v')\Big)\\ &= \omega_x(Z_M(x), Z'_M(x)),\end{aligned}$$

where

$$\begin{aligned}\left.\frac{\partial g_r(x)}{\partial r}\right|_{r=0}(v) &= D(s \mapsto g_{sv}(x))_{s=0}(1), \quad (s \in \mathbf{R}) \\ &= D(\hat{x})_{1_G}\Big(\left.\frac{\partial g_{sv}}{\partial s}\right|_{s=0}\Big) \\ &= D(\hat{x})_{1_G}(Z),\end{aligned}$$

with

$$Z = \left.\frac{\partial g_{sv}}{\partial s}\right|_{s=0}.$$

Indeed $s \mapsto g_{sv}$ is a centered path at the identity $\mathbf{1}_{1_G}$, then its derivative defines a tangent vector $Z \in T_{1_G}(G)$, that is an element of the Lie algebra $\mathcal{G}$. If the action of G on M is Hamiltonian then $Z = \mathrm{grad}_\omega(u)$ and $Z' = \mathrm{grad}_\omega(u')$ Therefore:

$$\omega(r \mapsto g_r(x))_0(v, v') = \omega_x(Z_M(x), Z'_M(x)) = \{u, u'\}.$$

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