EMBEDDING A DIFFEOLOGICAL SPACE INTO ITS POWERSET

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In this note we prove that the natural inclusion of a diffeological space into its powerset is an embedding. And a closed embedding if the space is Hausdorff.

The main ingredient is the *Powerset Diffeology* of a diffeological space, defined in the the exercise 62 page 60 of the textbook [DBook]. We recall, in the first article, the results of this exercise.

1. **The inclusion map.** — Let X be a diffeological space and $\mathfrak{P}(X)$ be its powerset, equipped with the powerset diffeology. The natural *inclusion map*

$$j\colon X \to \mathfrak{P}(X) \quad \text{with} \quad j\colon x \mapsto \{x\},$$

is smooth and is an induction [DBook, §1.29].

Proof. Let $P\colon U \to X$ be a plot in X, we have to check that $j \circ P\colon r \mapsto j(P(r)) = \{P(r)\}$ is a plot for the powerset diffeology.
For all $r_0 \in U$ and for all plot Q_0 with value in $\{P(r_0)\}$, which is necessarily constant, we define the following family of constant plots:

$$r \mapsto Q_r : \mathrm{dom}(Q_0) \to X, \text{ with } \; Q_r(s) = P(r).$$

Since $\mathrm{val}(Q_r) \subset j(P(r)) = \{P(r)\}$, and $(r,s) \mapsto Q_r(s) = P(r)$ is clearly smooth, the conditions to be a powerset plot are satisfied by $j \circ P$. Thus, j is smooth.
Let us check that j is now an induction. Let $P\colon U \to \mathfrak{P}(X)$ be a powerset plot with value in $j(X)$, we have to check that $j^{-1} \circ P$ is

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a plot in X. Indeed, for all $r \in U$ there exists $x_r \in X$ such that $P(r) = \{x_r\}$. For any fixed $r_0 \in U$ and for any plot Q_0 such that $\mathrm{val}(Q_0) \subset P(r_0) = \{x_{r_0}\}$, there exists an open neighborhood V of r_0 and a smooth family $r \mapsto Q_r$ of plots, defined on V, such that $\mathrm{val}(Q_r) \subset P(r) = \{x_r\}$. Thus, $(s, r) \mapsto Q_r(s) = x_r = j^{-1} \circ P$ is locally smooth, and $r \mapsto x_r$ is a plot of X. ☐

2. Embedding X in its Powerset. — The inclusion $j\colon X \to \mathfrak{P}(X)$ is not just an induction, it is an embedding [DBook, §2.13]. That is, the pullback of the D-topology of $\mathfrak{P}(X)$ coincides with the D-topology of X.

Proof. We have to prove that, for any D-open subset $\mathcal{O} \subset X$, there exists an D-open set $\mathcal{O}'$ in $\mathfrak{P}(X)$ such that $j(\mathcal{O}) = j(X) \cap \mathcal{O}'$.

Let us define

$$\mathcal{O}' = \{A \in \mathfrak{P}(X) \mid A \cap \mathcal{O} \neq \varnothing\}.$$

Clearly $j(\mathcal{O}) = j(X) \cap \mathcal{O}'$. It remains to prove that $\mathcal{O}'$ is D-open in $\mathfrak{P}(X)$. Then, for any plot $P\colon U \to \mathfrak{P}(X)$, we have to prove that

$$P^{-1}(\mathcal{O}') = \{r \in U \mid P(r) \cap \mathcal{O} \neq \varnothing\}$$

is open. Let $r_0 \in P^{-1}(\mathcal{O}')$, thus $P(r_0) \cap \mathcal{O} \neq \varnothing$. Pick $x_0 \in P(r_0) \cap \mathcal{O}$ and the constant plot $Q_0(s) = x_0$, there exists a smooth family of plots Q_r for r near r_0 such that $\mathrm{val}(Q_r) \subset P(r)$ and $(s, r) \mapsto Q_r(s)$ is smooth. Since $\mathcal{O}$ is open, by continuity of $Q_r(s)$, it exists a product of two balls $\Omega = B(s, \varepsilon) \times B(r_0, \eta)$ (s is arbitrary choosen) such that for all $(s, r) \in \Omega$, $Q_r(s) \in \mathcal{O}$, the condition $\mathrm{val}(Q_r) \subset P(r)$ implies that $P(r) \cap \mathcal{O} \neq \varnothing$. Thus, $B(r_0, \eta) \subset P^{-1}(\mathcal{O}')$, that is, $P^{-1}(\mathcal{O}')$ is open, and therefore, $\mathcal{O}'$ is D-open. ☐

3. The case of the empty set. — Let $\mathfrak{P}(X)^\star$ be the subset of non empty sets in X,

$$\mathfrak{P}(X)^\star = \mathfrak{P}(X) - \big\{\{\varnothing\}\big\}.$$

Then, equipped with the powerset diffeology, $\mathfrak{P}(X)$ is the vague adjunction of the singleton $\{\varnothing\}$ to $\mathfrak{P}(X)^\star$, as defined in [PB17]. That implies in particular that, for the D-topology, $\{\varnothing\} \in \mathfrak{P}(X)$ is closed, and $\mathfrak{P}(X)$ is the only neighborhood of $\{\varnothing\}$.

Proof. Indeed, for a parametrization P in $\mathfrak{P}(X)$ and a point $r_0 \in \mathrm{dom}(P)$, if $P(r_0) = \{\varnothing\}$ then the condition of the Powerset Diffeology is empty. ☐

4. X in $\mathfrak{P}(X)$. — If X is Hausdorff for the D-topology, then its image $J(X)$ in $\mathfrak{P}(X)^\star$ is closed. Or, $J(X) \cup \{\varnothing\}$ is closed in $\mathfrak{P}(X)$.

Proof. Let us show that $\mathfrak{P}(X)^\star - j(X)$ is open,

$$\mathfrak{P}(X)^\star - j(X) = \left\{ A \subset X \mid \exists \{x, y\} \subset A, x \neq y \right\} .$$

Consider a plot $P : U \to \mathfrak{P}(X)^\star$, and $r_0 \in P^{-1}(\mathfrak{P}(X)^\star - j(X))$. Suppose now $\{x_0, y_0\} \subset P(r_0)$ with $x_0 \neq y_0$. Let Q_0 and Q'_0 be two constant plots such that : $\mathrm{val}(Q_0) = \{x_0\}$ and $\mathrm{val}(Q'_0) = \{y_0\}$, so there exists two smooth family $U \supset V \ni r \mapsto Q_r$ and $U \supset V' \ni r \mapsto Q'_r$ such that $\mathrm{val}(Q_r) \subset P(r)$ and $\mathrm{val}(Q'_r) \subset P(r)$. As X is supposed to be Hausdorff, we may choose $\mathcal{O}$ and $\mathcal{O}'$ two disjoint D-open neighborhoods of respectively x_0 and y_0. Given s_0 in a common domain of Q_r and Q'_r, the intersection of the preimages of $\mathcal{O}$ and $\mathcal{O}'$ by the continuous maps $(r, s) \mapsto Q_r(s) \in X$ and $(r, s) \mapsto Q'_r(s) \in X$ defines an open neigborhood $W \times S$ of (r_0, s_0), with $W \subset V \cap V'$, such that $Q_r(s) \in \mathcal{O}$ and $Q'_r(s) \in \mathcal{O}'$ for any $(r, s) \in W \times S$. As $\mathrm{val}(Q_r) \subset P(r)$ and $\mathrm{val}(Q'_r) \subset P(r)$, the condition $\mathcal{O} \cap \mathcal{O}' = \varnothing$ insures that $P(r)$ includes at least two distinct points of X for all $r \in W$. Hence $W \subset P^{-1}(\mathfrak{P}(X)^\star - j(X))$. Thus, $P^{-1}(\mathfrak{P}(X)^\star - j(X))$ is open, as a union of open subsets. Therefore, $\mathfrak{P}(X)^\star - j(X)$ is D-open and $j(X)$ is closed in $\mathfrak{P}(X)^\star$. ☐

References

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