DIFFEOLOGICAL SPACES ARE LOCALLY CONNECTED

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-KSODS.pdf

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Every diffeological space is locally connected for the D-Topology,

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Every diffeological space X naturally has a topology: the D-topology, defined originally in [Igl85]. This is the finest topology for which the plots are continuous. A subset $\mathcal{O} \subset X$ is open for the D-topology, or is a *D-open*, if $P^{-1}(\mathcal{O})$ is open in dom(P) for all plots P in X.

It has been shown that a the connected components for the D-topology are the path-connected components, and that the space X is the sum of its connected components [PIZ13, §5.7, 5.8]. We can add moreover:

Proposition. *Every diffeological space is locally connected. That means that every D-open subset is the sum of its connected components for the D-topology. Since connected components for the D-topology coincide with pathwise connected components, Every connected diffeological space* X *is locally pathwise connected.*

Proof. Let us prove that D-open subsets $\mathcal{O} \subset X$ are embedded [PIZ13, §2.14]. That is, every D-open subset $\mathcal{U}$ of the subspace $\mathcal{O} \hookrightarrow X$ is the imprint of a D-open subset $\mathcal{U}'$ of X, *i.e.* $\mathcal{U} = \mathcal{U}' \cap \mathcal{O}$. On the one hand, since $\mathcal{U}$ is open for the D-topology of the subspace $\mathcal{O}$ of X, then for every plot P in $\mathcal{O}$, that is, for every plot P in X taking its values in $\mathcal{O}$, $P^{-1}(\mathcal{U})$ is open. On the other hand, let P be any plot in X. since $P^{-1}(\mathcal{U}) \subset P^{-1}(\mathcal{O})$, $P^{-1}(\mathcal{U}) = [P \restriction P^{-1}(\mathcal{O})]^{-1}(\mathcal{U})$. But, $P^{-1}(\mathcal{O})$ is open since $\mathcal{O}$ is D-open, then $P \restriction P^{-1}(\mathcal{O})$ is still a plot in X but taking its values in $\mathcal{O}$. That is, a plot in $\mathcal{O}$. Thus, $[P \restriction P^{-1}(\mathcal{O})]^{-1}(\mathcal{U})$ is open, and so is $P^{-1}(\mathcal{U})$. Thus, for all plots P in X, $P^{-1}(\mathcal{U})$ is open. Therefore, $\mathcal{U}$ is D-open in X, and since $\mathcal{U} = \mathcal{U} \cap \mathcal{O}$, $\mathcal{U}$ as D-open $\mathcal{O}$ is the imprint of $\mathcal{U}$ on $\mathcal{O}$ as D-open in X. In conclusion, every D-open subset of X is embedded.

Now, as a subspace of X, $\mathcal{O}$ is the sum of its component which are D-open subset in $\mathcal{O}$, and then, D-open subset of X. Thus, $\mathcal{O}$ is the sum of its

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components for the D-topology of X. Therefore, X is locally connected for the D-topology. □

References

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