THE DIFFEOMORPHISMS OF THE SQUARE

SERAP GÜRER AND PATRICK IGLESIAS-ZEMMOUR

ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-DOTS.pdf

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We show how diffeology, understood as the geometry of the group of diffeomorphisms in the sense of Felix Klein, fulfils its duty concerning the full square $\mathrm{Sq} = [0,1]^2 \subset \mathbf{R}^2$.

According to Klein's program [Kle72], a geometry must be understood as the action of a group (called the *principal group*) on some space. As an example, Euclidean geometry is the action of the Euclidean group on a Euclidean space, affine geometry, the action of the affine group etc. The example of the closed square discussed here, in the Diffeology framework, seems to support the point of view that "differential geometry" is the geometry of the group of diffeomorphisms, in the sense of Klein.

Proposition — Let $\mathrm{Sq} = [0,1]^2 \subset \mathbf{R}^2$ be equipped with the subset diffeology. The decomposition of the square under the action of its group of diffeomorphisms give the expected three orbits:

(1) The 4 corners: $\mathrm{NO} = (0,1)$, $\mathrm{NE} = (1,1)$, $\mathrm{SO} = (0,0)$ and $\mathrm{SE} = (1,0)$.
(2) The 4 vertices $\mathrm{B} =]0,1[\times\{0\}$, $\mathrm{T} =]0,1[\times\{1\}$, $\mathrm{L} = \{0\}\times]0,1[$ and $\mathrm{R} = \{1\}\times]0,1[$.
(3) The interior $\mathring{\mathrm{Sq}} =]0,1[^2$.

Note 1. The quotient diffeology on

$$\mathrm{Sq}/\mathrm{Diff}(\mathrm{Sq}) = \{\mathrm{Corners}, \mathrm{Vertices}, \mathrm{Interior}\}$$

is of course not the discrete diffeology, it captures the combinatorial structure of the orbits.

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Note 2. that under homeomorphisms there is only 2 orbits, the border and the interior. In our case, the differential structure is obviously indispensable.

Proof. 1) Let us show that, Sq is embedded in $\mathbf{R}^2$. That is, the D-topology of the induction $\mathrm{Sq} \subset \mathbf{R}^2$ coincides with the induced topology of $\mathbf{R}^2$. For any subset $\mathrm{U} \subset \mathrm{Sq}$ open for the induced topology, there exists an open $\mathcal{O} \subset \mathbf{R}^2$ such that $\mathrm{U} = \mathcal{O} \cap \mathrm{Sq}$. For all plots P in Sq, $\mathrm{P}^{-1}(\mathrm{U}) = \mathrm{P}^{-1}(\mathcal{O})$ is open, because plots are continuous. On the other hand, let $\mathrm{U} \subset \mathrm{Sq}$ be a D-open. $s^{-1}(\mathrm{U})$ is open, where $s\colon \mathbf{R}^2 \to \mathrm{K}^2$ is the map $s(x_1, x_2) = (x_1^2, x_2^2)$. $s^{-1}(\mathrm{U}) \upharpoonright \mathrm{Sq}$ is open for the induced topology of $\mathbf{R}^2$. Now, the map s restricted to Sq is an homeomorphism. Hence, since $\mathrm{U} = s(s^{-1}(\mathrm{U}) \upharpoonright \mathrm{Sq})$, U is open for the induced topology of $\mathbf{R}^2$. Therefore the D-topology of the induction coincides with the induced topology.

2) Now, let us show that a diffeomorphism of Sq cannot send a point of the border into the interior. Let $f\colon \mathrm{Sq} \to \mathrm{Sq}$ be a diffeomorphism for the subset diffeology. Hence f is a homeomorphism for the D-topology. Let us assume that f maps $x \in \mathrm{L}$ to $f(x) \in \mathring{\mathrm{Sq}}$. Obviously f induces a homeomorphism $\tilde{f}\colon \mathrm{Sq} \setminus \{x\} \to \mathrm{Sq} \setminus \{f(x)\}$. Since $x \in \mathrm{L}$, we have that $\mathrm{Sq} \setminus \{x\}$ is convex, therefore homotopy equivalent to a point. On the other hand, we can construct a homotopy equivalence $\mathrm{Sq} \setminus \{f(x)\} \simeq \partial\mathrm{Sq} \simeq \mathrm{S}^1$. Thus, we get $\{\mathrm{point}\} \simeq \mathrm{S}^1$, which is a contradiction.

3) A diffeomorphism from the square must send corners into corners. Let f be a diffeomorphism on Sq, such that $f \upharpoonright \mathrm{L}$ takes values in $\mathrm{L} \cup \mathrm{SO} \cup \mathrm{B}$ and the restriction of f to L is defined as follows,

$$f(te_2) = \begin{cases} \varphi(t)e_1, & \text{if } 0 < t < \frac{1}{2} \\ \mathrm{SO}, & \text{if } t = \frac{1}{2} \\ \psi(t)e_2, & \text{if } \frac{1}{2} < t < 1 \end{cases}$$

where e_1 and e_2 are the vectors of the canonical basis of $\mathbf{R}^2$. Thus, for all $p > 0$, $\lim_{t \to \frac{1}{2}^-} f^{(p)}(te_2) = \varphi^{(p)}(t)e_1$ and $\lim_{t \to \frac{1}{2}^+} f^{(p)}(te_2) = \psi^{(p)}(t)e_2$. Hence by continuity, for all $p > 0$, $\varphi^{(p)}\left(\frac{1}{2}\right) = \psi^{(p)}\left(\frac{1}{2}\right)$. Therefore, f is flat at $\frac{1}{2}$. The restriction of f^{-1} to $\mathrm{L} \cup \mathrm{SO}$ is a local diffeomorphism of half space with the subset diffeology. By [PIZ13, §4.14] (a consequence of [Whi43]), there exists $\mathrm{F} \in \mathcal{C}^\infty(]-\varepsilon, 1[, \mathbf{R})$ such that, $f^{-1} \upharpoonright \mathrm{L} \cup \mathrm{SO} = \mathrm{F} \upharpoonright \mathrm{L} \cup \mathrm{SO}$. We have $f^{-1} \circ f(t) = t$, for all

$t > \frac{1}{2}$. Thus, we have $(f^{-1})'(f(t))f'(t) = 1$. But, $(f^{-1})'(f(t))f'(t) = \lim_{t\to\frac{1}{2}} \mathrm{F}'(f(t))f'(t) = 0$ which is a contradiction. Therefore 4 corners of Sq is an orbit of Diff(Sq). ☐

References

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Serap Gürer, Galatasaray University, Istanbul, Turkey.

Patrick I-Z, CNRS France & The Hebrew University of Jerusalem, Israel.

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