DIFFERENTIAL OF A LIE-GROUP VALUED FUNCTION

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ref.
http://math.huji.ac.il/~piz/documents/DBlog-Rmk-DOALGVF.pdf

This note gives a meaning of the differential of a smooth function defined on a diffeological space, with values in a Lie group.

The General Question

This question has been raised by Cheyne Miller[1], as a byproduct of his reflexion on the differential of the Holonomy function, in the case of a general fiber bundle, or a more general situation. I'm sure he will not see any objection that I share my reflexion on his question in my blog.

Precisely, a part of his question translates into:

Question *Let $h : \mathrm{X} \to \mathrm{G}$ be a smooth function defined on a general diffeological space, with values into an ordinary Lie group. What would be the meaning of dh ?*

When G is the abelian group $(\mathbf{R}, +)$, we know the answer:

$$dh = h^*(dt),$$

where dt is the standard 1-form on $\mathbf{R}$. More recently we have seen, in the blog post on *Differential of Holonomy For Torus Bundles* [PIZ15], that we can define a "differential" $d_{\mathrm{T}}h$ by:

$$d_{\mathrm{T}}h = h^*(\theta),$$

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[1] Private email exchange of the February 27, 2016.

where θ is the standard 1-form on the (maybe irrational) torus $T = \mathbf{R}/\Gamma$, pushforward of dt by the projection $\pi : \mathbf{R} \to T$. In other words, satisfying $\pi^*(\theta) = dt$.

We remark that, in these two cases, the differential of the function is the pullback of some fundamental form defined on the group $\mathbf{R}$ or T. This consideration will be the guide to define more generally the differential of a function with values in any Lie group.

The Maurer-Cartan Form

There is a canonical 1-form defined on any Lie group G, with values on the tangent space $\mathcal{G} = T_1G$, that is, the Maurer-Cartan form $\Theta \in \Omega^1(G, \mathcal{G})$. With the ordinary notations of differential calculus on a lie group, let $g \in G$, let $L(g)$ be the left-multiplication by g, that is, $L(g)(g') = gg'$. Let $\delta g \in T_gG$, be any tangent vector at the point g, then the Maurer-Cartan form Θ is defined by[2]

$$\Theta_g(\delta g) = [D(L(g))(1)]^{-1}(\delta g).$$

You note that this Maurer-Cartan form is left-invariant, that is, $L(k)^*(\Theta) = \Theta$ for all $k \in G$. We have an analogous right-invariant Maurer-Cartan form, replacing $L(g)$ by $R(g)$ in the definition above. The choice depends on our need or preferences.

Note that the Maurer-cartan form has a diffeology-compliant definition. Let $P : r \mapsto g_r$ be a n-plot of G, let $v \in \mathbf{R}^n$, then:

$$\Theta(P)_r(v) = D(s \mapsto g_r^{-1} g_{r+s})(s = 0)(v).$$

The Definition of the Differential

Now we have the tools to define the differential of the smooth function $h : X \to G$. I suggest this:

Definition *Let X be a diffeological space, let G be an ordinary Lie group, let $h : X \to G$ be a smooth function. We define the* differential *of h as the pullback by h of the Maurer-Cartan form. And we shall denote,*

$$d_G h = h^*(\Theta), \quad d_G h \in \Omega^1(X, \mathcal{G}).$$

[2]The notation $D(f)(x)$ means: the tangent linear map of f at the point x.

This is a well defined diffeological 1-form with values in a vector space, and that corresponds exactly to what we did in the particular abelian case of $\mathbf{R}$ or T. In that cases the Maurer-Cartan form is just the identity.

Now, how is that a good replacement for the differential in the general case? If X is a manifold, we should have on the one hand, for all $x \in \mathrm{X}$ and $\delta x \in \mathrm{T}_x\mathrm{X}$:

$$dh_x(\delta x) \in \mathrm{T}_{h(x)}\mathrm{G},$$

where dh_x, also denoted by $\mathrm{D}(h)(x)$, is the tangent linear map of h at the point x. The Maurer-Cartan form is just the tool for trivialising the tangent fiber bundle. The map $(g, v) \mapsto (g, \theta_g(v))$ is the canonical isomorphism from TG to $\mathrm{G} \times \mathcal{G}$. Therefore $\theta_{h(x)}(dh_x(\delta x))$ is the image of δx by dh_x after trivialisation.

On the other hand, $\theta_{h(x)}(dh_x(\delta x))$ is, by definition, $h^*(\theta)_x(\delta x)$. Therefore, the definition given above $d_{\mathrm{G}}h = h^*(\theta)$ seems to be a good replacement, in diffeology, of the differential of a Lie-group valuated smooth function.

Note that in the 1-dimension case $\mathrm{G} = \mathbf{R}$ or T the differential $d_{\mathrm{G}}h$ is closed (because 2-forms on 1-dimensional spaces vanishe), which is no more true in the general case. Indeed, $d[d_{\mathrm{G}}h] = h^*(d\theta) \in \Omega^2(\mathrm{X}, \mathcal{G})$. The differential d_{G} is a replacement for the tangent linear map, not for the differential operator of exterior calculus. But this is what is useful in mathematical physics when one deals with principal connexions, or more exotic structures like gerbes.

References

[DBook] Patrick Iglesias-Zemmour. *Diffeology,* Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence, 2013. http://www.ams.org/bookstore-getitem/item=SURV-185

[PIZ15] Patrick Iglesias-Zemmour. *Differential of Holonomy For Torus Bundles.* Blog Post November 26, 2015. http://math.huji.ac.il/~piz/documents/DBlog-EX-DOHFTB.pdf

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