DIFFERENTIAL FORMS ON THE CROSS

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-DFOTC.pdf

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We describe the differential forms on the diffeological space consisting of the union of the two axes $ox \cup oy$ in $\mathbf{R}^2$.

This question has been asked by Professor Jedrzej Sniatycki: describe the differential forms on the space X, union of the two axes in $\mathbf{R}^2$

$$X = \{(x,0) \mid x \in \mathbf{R}\} \cup \{(0,y) \mid y \in \mathbf{R}\},$$

equipped with the subset diffeology.

Proposition. — *Every differential* 1*-form* α *on* X*, is the restriction of a* 1*-form on* $\mathbf{R}^2$ *of type* $A = a(x)dx + b(y)dy$*. In other words, for all plot* $P\colon r \mapsto (x_r, y_r)$ *in* X*,*

$$\alpha(P)_r(\delta r) = a(x_r)\frac{\partial x_r}{\partial r}(\delta r) + b(y_r)\frac{\partial y_r}{\partial r}(\delta r).$$

The 1*-forms* a *and* b *are the restrictions of* α *on the axes:*

$$a(x)dx = \alpha \restriction ox \quad \text{and} \quad b(y)dy = \alpha \restriction oy.$$

Note. — *According to the definition* [PIZ13, §6.40]*, the value of* α *at* $(0,0)$ *is zero if and only if* $a(0) = b(0) = 0$*. Indeed evaluated on the plots* $t \mapsto (t,0)$*, and* $t \mapsto (0,t)$ *at* $t = 0$*,* α *is equal to* $a(0)dx$ *or* $b(0)dy$*, which are not zero, except if both are* 0*.*

Proof. First of all, notice that the four semi-axes: ox^-, ox^+, oy^- and oy^+, are D-open in X. That is, open for the D-topology [PIZ13, §2.8]. Indeed, for example, for any plot P in X, $P^{-1}(ox^+) = P^{-1}\{(x,y) \mid x > 0\}$, which is open because it is the pullback of an open subset by a continuous map. And *mutatis mutandis* for ox^-, oy^- and oy^+.

Let P be defined on $U \subset \mathbf{R}^n$, and let

$$\mathcal{O} = P^{-1}(ox^- \cup ox^+ \cup oy^- \cup oy^+).$$

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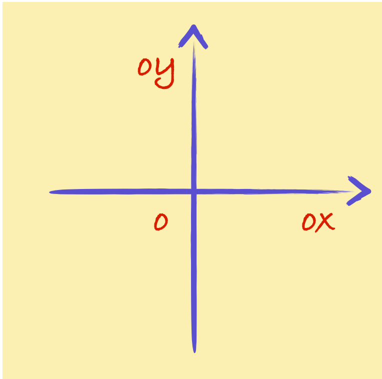

The Cross

The subset $\mathcal{O}$ is open. Let us prove first that,

$$\alpha - \mathrm{A} \upharpoonright \mathcal{O} = 0.$$

That is, for all $r \in \mathcal{O}$ and all $\delta r \in \mathbf{R}^n$

$$\alpha(\mathrm{P})_r(\delta r) - a(x_r)\delta x_r - b(y_r)\delta y_r = 0.$$

We denoted

$$\delta x_r = \frac{\partial x_r}{\partial r}(\delta r) \quad \text{and} \quad \delta y_r = \frac{\partial y_r}{\partial r}(\delta r).$$

Let $r \in \mathcal{O}$ and $\mathrm{P}(r) \in ox^+$, for example. Then, there is an open neighbourhood V of r such that $\mathrm{P} \upharpoonright \mathrm{V}$ takes its values in ox^+. Thus, on V, by definition $\alpha(\mathrm{P})_r(\delta r) = a(x_r)\delta x_r$. And since $y_r = 0$ on V, $\alpha(\mathrm{P})_r(\delta r) = a(x_r)\delta x_r + b(y_r)\delta y_r$. *Mutatis mutandis* for all other values of $\mathrm{P}(r)$ when $r \in \mathcal{O}$. We get eventually that on $\mathcal{O}$, $(\alpha - \mathrm{A})(\mathrm{P}) \upharpoonright \mathcal{O} = 0$.

Now, for all $\delta r \in \mathbf{R}^n$, $(\alpha - \mathrm{A})(\mathrm{P})_r(\delta r)$ is a smmoth function in r. This function vanishes on $\mathcal{O}$, by continuity it vanishes on the closure $\overline{\mathcal{O}}$. Let $\mathrm{U}' = \mathrm{U} - \overline{\mathcal{O}}$, U' is an open subset of $\mathbf{R}^n$ and then $\mathrm{P} \upharpoonright \mathrm{U}'$ is a plot in X. Then, it makes sense to evaluate $(\alpha - \mathrm{A})(\mathrm{P})_r(\delta r)$ on U'. But since, for all $r \in \mathrm{U}'$, $\mathrm{P}(r) = (0,0)$, $(\alpha - \mathrm{A})(\mathrm{P})_r(\delta r) = 0$. Indeed, a differential form evaluated on a constant plot is zero (since it factorizes through $\mathbf{R}^0$). Thus, for all $r \in \mathrm{U}$, $(\alpha - \mathrm{A})(\mathrm{P})_r(\delta r) = 0$. Hence $\alpha(\mathrm{P})_r(\delta r) = a(x_r)\delta x_r + b(y_r)\delta y_r$, and $\alpha = \mathrm{A} \upharpoonright \mathrm{X}$. ☐

References

[PIZ13] Patrick Iglesias-Zemmour. Diffeology. Mathematical Surveys and Monographs. The American Mathematical Society, vol. 185, RI USA, 2013.

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