DIFFERENTIAL FORMS ON CORNERS

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We show that, for the subset diffeology, differential forms defined on half-spaces or corners of Euclidean spaces, are the restrictions of a differential forms defined on an open neighborhood of the corner in the ambient Euclidean space.

Usually, smooth maps from corners $K^n = \{(x_1, \ldots, x_n) \mid x_i \geq 0\}$ into the real line $\mathbf{R}$ are just defined as restrictions of smooth maps, defined on some open neighborhood of the corner [Cer61] [Dou62] etc. This heuristic becomes a theorem in diffeology where K^n is equipped with the subset diffeology. Indeed every map from K^n to $\mathbf{R}$ such that composed with a smooth parametrisation[1] $P\colon U \to \mathbf{R}^n$, taking its values in K^n, is smooth, is the restriction of a smooth maps defined on some open neighborhood of the corner [PIZ13, §4.16].

It is always a progress when a convention, based on mathematicians' intuition, becomes a theorem in a well defined axiomatic. Here the axiomatic is the theory of Diffeology. Noticing that $C^\infty(K^n, \mathbf{R})$ is just the space of differential 0-forms $\Omega^0(K^n)$, it is legitimate to ask about the behavior of differential k-forms on K^n, that is, $\Omega^k(K^n)$ a it is defined in [PIZ13, §6.28]. In this paper we prove the following theorem (art. 4):

Theorem. Every differential form on the corner K^n is the restriction of a smooth form on an open neighborhood of K^n in $\mathbf{R}^n$. Precisely, the pullback : $j^*\colon \Omega^k(\mathbf{R}^n) \to \Omega^k(K^n)$ is surjective, where j denotes the inclusion from K^n into $\mathbf{R}^n$.

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[1] A parametrisation is just a map defined on an open subset of an Euclidean space.

Let us just remind that a differential k-form on a diffeological space X is a mapping α that associates with each plot P in X, a smooth k-form $\alpha(\mathrm{P})$ on dom(P), such that the smooth compatibility condition $\alpha(\mathrm{F} \circ \mathrm{P}) = \mathrm{F}^*(\alpha(\mathrm{P}))$ is satisfied, where F is any smooth parametrisation in dom(P).

Smooth Structure on Corners

1. Corners as Diffeologies. — We denote by K^n the corner

$$\mathrm{K}^n = \{(x_i)_{i=1}^n \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\}.$$

And we equip it with the subset diffeology. A plot in K^n is just a regular smooth parametrization in $\mathbf{R}^n$ but taking its values in K^n.

(A) The corner K^n is the diffeological n-power of the half-line $\mathrm{K} = [0, \infty[\subset \mathbf{R}$, equipped with the subset diffeology.

(B) The corner K^n is embedded in $\mathbf{R}^n$, and closed. That is, the D-topology of the induction $\mathrm{K}^n \subset \mathbf{R}^n$ coincides with the induced topology[2] of $\mathbf{R}^n$, see [PIZ13, §2.13].

(C) Let $\mathrm{X}_0 = \{0\} \subset \mathrm{X}_1 \subset \cdots \subset \mathrm{X}_n = \mathrm{K}^n$ be the natural filtration of K^n, where the levels X_j are defined by

$$\mathrm{X}_j = \{(x_i)_{i=1}^n \in \mathrm{K}^n \mid \text{there exist } i_1 < \cdots < i_{n-j} \text{ such that } x_{i_\ell} = 0\}.$$

Then, the stratum

$$\mathrm{S}_j = \mathrm{X}_j - \mathrm{X}_{j-1}$$

is the subset of points in $\mathbf{R}^n$ that have j, and only j, coordinates strictly positive. The strata S_j are equipped with the subset diffeology[3].

$$\mathrm{S}_j = \left\{ (x_i)_{i=1}^n \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \text{There exist } i_1 < \cdots < i_j \text{ such that } x_{i_\ell} > 0, \\ \text{and } x_m = 0 \text{ for all } m \notin \{i_1, \ldots, i_j.\} \end{array} \right\}.$$

Then, S_j is D-open in X_j, $j \geq 1$. As a subset of X_j, S_j is the (diffeological) sum of $\binom{n}{j}$ connected components indexed by a string of j ones and $n-j$ zeros.

Proof. For the first item, it's immediately by definition. Considering the second item: for any subset $\mathrm{U} \subset \mathrm{K}^n$ open for the induced topology, there exists (by definition) an open subset $\mathcal{O} \in \mathbf{R}^n$ such that $\mathrm{U} = \mathcal{O} \cap \mathrm{K}^n$. Then, for all plots P in K^n, $\mathrm{P}^{-1}(\mathrm{U}) = \mathrm{P}^{-1}(\mathcal{O})$ is open, because plots are continuous. On the other hand, let $\mathrm{U} \subset \mathrm{K}^n$ be D-open. Then, $\mathrm{sq}^{-1}(\mathrm{U}) \subset \mathbf{R}^n$ is open, where $\mathrm{sq} : \mathbf{R}^n \to \mathrm{K}^n$ is the map $\mathrm{sq}\,(x_i)_{i=1}^n = (x_i^2)_{i=1}^n$. And

[2] The standard topology of $\mathbf{R}^n$ is the D-topology of its standard smooth structure.

[3] Recall that, by transitivity of subset diffeology, to be a subspace of S_ℓ or K^n or of $\mathbf{R}^n$ is identical.

$\mathrm{sq}^{-1}(\mathrm{U}) \restriction \mathrm{K}^n$ is open for the induced topology of $\mathbf{R}^n$. Now, the map sq restricted to K^n is an homeomorphism. Hence, since $\mathrm{U} = \mathrm{sq}(\mathrm{sq}^{-1}(\mathrm{U}) \restriction \mathrm{K}^n)$, U is open for the induced topology of $\mathbf{R}^n$. Therefore the D-topology of the induction coincides with the induced topology, as we claimed.

For the third item: let $x \in \mathrm{X}_j$, then the number ν of coordinates of x that are 0 is at least $n-j$, *i.e.* $\nu \geq n-1$. Next, if $x \in \mathrm{X}_j$ and $x \notin \mathrm{X}_{j-1}$, then $\nu \geq n-j$ and $\nu < n-j+1$, thus, $\nu = n-j$. Therefore, $\mathrm{X}_j - \mathrm{X}_{j-1}$ is the subset of points in $\mathbf{R}^n$ that have exactly $n-j$ coordinates equal to 0 and the other j strictly positive:

Consider now a point $x = (x_1, \ldots, x_n) \in \mathrm{S}_j - \mathrm{S}_{j-1}$. Since the j non-zero coordinates of x are strictly positive, there exists $\varepsilon > 0$ such that $x_i - \varepsilon > 0$, for all non-zero coordinate of x. The open n-parallelepiped $\mathrm{C}_x =]x_1 - \varepsilon, x_1 + \varepsilon[\times \cdots \times]x_n - \varepsilon, x_n + \varepsilon[\subset \mathbf{R}^n$ contains x, and $\mathrm{C}_x \cap \mathrm{S}_j \subset \mathrm{S}_j - \mathrm{S}_{j-1}$. Thus,

$$\mathrm{S}_j - \mathrm{S}_{j-1} = \bigcup_{x \in \mathrm{S}_j - \mathrm{S}_{j-1}} \mathrm{C}_x \cap \mathrm{S}_j.$$

Now, let $\mathrm{P} : \mathrm{U} \to \mathrm{S}_j$ be a plot for the subset diffeology. Then, $\mathrm{P}^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1}) = \cup_{x \in \mathrm{S}_j - \mathrm{S}_{j-1}} \mathrm{P}^{-1}(\mathrm{C}_x \cap \mathrm{S}_j)$, but $\mathrm{P}^{-1}(\mathrm{C}_x \cap \mathrm{S}_j) = \mathrm{P}^{-1}(\mathrm{C}_x)$ since $\mathrm{val}(\mathrm{P}) \subset \mathrm{S}_j$. Next, since P is smooth as a map into $\mathbf{R}^n$ and C_x is open, $\mathrm{P}^{-1}(\mathrm{C}_x)$ is open and then $\mathrm{P}^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$ is open. Therefore, $\mathrm{S}_j - \mathrm{S}_{j-1}$ is D-open in S_j. ☐

2. Smooth Maps on Corners. — It has been proved that a map $f : \mathrm{K}^n \to \mathbf{R}$, is smooth in the sense of diffeology, if and only if it is the restriction of a smooth map F defined on some open neighborhood $\mathcal{O}$ of K^n into $\mathbf{R}$ [PIZ13, §4.16]. That is, $f \in \mathrm{C}^\infty(\mathrm{K}^n, \mathbf{R})$ if and only if, $f = \mathrm{F} \restriction \mathrm{K}^n$ and $\mathrm{F} \in \mathrm{C}^\infty(\mathcal{O}, \mathbf{R})$.

3. The Square Function Lemma. — Let $\mathrm{sq} : \mathbf{R}^n \to \mathrm{K}^n$ be the smooth parametrisation:

$$\mathrm{sq}(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2).$$

Then $\mathrm{sq}^* : \Omega^k(\mathrm{K}^n) \to \Omega^k(\mathbf{R}^n)$ is injective. That is, for all $\alpha \in \Omega^k(\mathrm{K}^n)$, if $\mathrm{sq}^*(\alpha) = 0$, then $\alpha = 0$.

Proof. Note that each component of $\mathrm{S}_j - \mathrm{S}_{j-1}$ is isomorphic to $\mathbf{R}^j$. Hence, if $\mathrm{sq}^*(\alpha) = 0$, since $\mathrm{sq} \restriction \mathrm{sq}^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$ is a 2-folds covering over $\mathrm{S}_j - \mathrm{S}_{j-1}$, $\alpha \restriction \mathrm{S}_j - \mathrm{S}_{j-1} = 0$. that is, for all plot Q in $\mathrm{S}_j - \mathrm{S}_{j-1}$, $\alpha(\mathrm{Q}) = 0$. Let then, for some $j \geq 1$, $\mathrm{P}_j : \mathrm{U}_j \to \mathrm{S}_j$ be a plot. In view of what precedes, the subset $\mathcal{O}_j = \mathrm{P}_j^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$ is open, and $\alpha(\mathrm{P}_j \restriction \mathcal{O}_j) = \alpha(\mathrm{P}_j) \restriction \mathcal{O}_j = 0$. By continuity, $\alpha(\mathrm{P}_j) \restriction \overline{\mathcal{O}}_j = 0$, where $\overline{\mathcal{O}}_j$ is the closure of $\mathcal{O}_j$. Let then $\mathrm{U}_{j-1} = \mathrm{U}_j - \overline{\mathcal{O}}_j$ and $\mathrm{P}_{j-1} = \mathrm{P}_j \restriction \mathrm{U}_{j-1}$. Then, U_{j-1} is open and $\mathrm{P}_{j-1} : \mathrm{U}_{j-1} \to \mathrm{S}_{j-1}$ is a plot. This construction gives a descending recursion, starting with any plot $P: U \to K^n$, by initializing $P_n = P$, $U_n = U$ and $S_n = K^n$. One has $P_j = P \restriction U_j$, $U_{j-1} \subset U_j$, the recursion ends with a plot P_0 with values in $S_0 = \{0\}$, and $\alpha(P_0) = 0$ since P_0 is constant. Therefore $\alpha = 0$. ☐

4. Differential Forms On Corners. — The previous article (art. 2) deals with smooth real functions on corners, that is, $\Omega^0(K^n)$. It is a particular case of the more general theorem:

Theorem. *Any differential k-form on the corner K^n, equipped with the subset diffeology of $\mathbf{R}^n$, is the restriction of a smooth differential k-form defined on some open neighborhood of the corner.* Precisely, the pullback : $j^*: \Omega^k(\mathbf{R}^n) \to \Omega^k(K^n)$ is surjective, where j denotes the inclusion from K^n to $\mathbf{R}^n$.

Proof. Let $\omega \in \Omega^k(K^n)$ and $\mathring{K}^n = \{(x_i)_{i=1}^n \mid x_i > 0, i = 1, \dots, n\}$. One has

$$\omega \restriction \mathring{K}^n = \sum_{i_1 < \cdots < i_k} a_{i_1 \dots i_k}(x_1, \dots, x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

with $i_j = 1, \dots, n$ and $a_{i_1 \dots i_k} \in C^\infty(\mathring{K}^n, \mathbf{R})$. Recall that $\mathrm{sq} : (x_i)_{i=1}^n \mapsto (x_i^2)_{i=1}^n$, then

$$\mathrm{sq}^*(\omega) = \sum_{i_1 < \cdots < i_k} A_{i_1 \dots i_k}(x_1, \dots, x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $A_{i_1 \dots i_k} \in C^\infty(\mathbf{R}^n, \mathbf{R})$. Let $\varepsilon_j : (x_1, \dots, x_j, \dots, x_n) \mapsto (x_1, \dots, -x_j, \dots, x_n)$, then $\mathrm{sq} \circ \varepsilon_j = \mathrm{sq}$ and $(\mathrm{sq} \circ \varepsilon_j)^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$, that is, $\mathrm{sq}^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$. Hence,

$$\begin{aligned} \varepsilon_j^*(\mathrm{sq}^*(\omega)) &= \sum_{\substack{i_1 < \cdots < i_k \\ i_\ell \neq j}} A_{i_1 \dots i_k}(x_1, \dots, -x_j, \dots, x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &\quad - \sum_{i_1 < \cdots \leq j \leq \cdots < i_k} A_{i_1 \dots j \dots i_k}(x_1, \dots, -x_j, \dots, x_n)\ dx_{i_1} \wedge \dots dx_j \cdots \wedge dx_{i_k}. \end{aligned}$$

Then,

$$\begin{aligned} A_{\substack{i_1 \dots i_k \\ i_\ell \neq j}}(x_1, \dots, -x_j, \dots, x_n) &= A_{i_1 \dots i_k}(x_1, \dots, x_j, \dots, x_n), \\ A_{i_1 \dots j \dots i_k}(x_1, \dots, -x_j, \dots, x_n) &= -A_{i_1 \dots j \dots i_k}(x_1, \dots, x_j, \dots, x_n). \end{aligned}$$

Hence,

$$A_{i_1 \dots j \dots i_k}(x_1, \dots, x_j = 0, \dots, x_n) = 0.$$

Thus,

$$A_{i_1 \dots j \dots i_k}(x_1, \dots, x_j, \dots, x_n) = 2x_j \underline{A}_{i_1 \dots j \dots i_k}(x_1, \dots, x_j, \dots, x_n),$$

with $\underline{A}_{i_1 \dots j \dots i_k} \in C^\infty(\mathbf{R}^n, \mathbf{R})$. Therefore, there are real smooth functions $\hat{A}_{i_1 \dots i_k}$ defined on $\mathbf{R}^n$ such that

$$A_{i_1 \dots i_k}(x_1, \dots, x_n) = 2^k x_{i_1} \dots x_{i_k} \hat{A}_{i_1 \dots i_k}(x_1, \dots, x_n).$$

Now,

$$\mathrm{sq}^*(\omega \upharpoonright \mathring{K}^n) = \mathrm{sq}^*(\omega) \upharpoonright \{x_i \neq 0\}$$

implies

$$\begin{aligned} & \sum_{i_1<\cdots<i_k} 2^k x_{i_1} \dots x_{i_k} a_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ = & \sum_{i_1<\cdots<i_k} 2^k x_{i_1} \dots x_{i_k} \hat{A}_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

Hence,

$$\hat{A}_{i_1\dots i_k}(x_1,\dots,x_n) = a_{i_1\dots i_k}(x_1^2,\dots,x_n^2) \quad \text{for } x_i \neq 0, i = 1,\dots,n.$$

Thus $(x_1,\dots,x_n) \mapsto \hat{A}_{i_1\dots i_k}(x_1,\dots,x_n)$, which belongs to $C^\infty(\mathbf{R}^n,\mathbf{R})$, is even in each variable. Therefore, according to Schwartz Theorem [Sch75][4], there exist

$$\underline{a}_{i_1\dots i_k} \in C^\infty(\mathbf{R}^n,\mathbf{R}),$$

such that

$$\hat{A}_{i_1\dots i_k}(x_1,\dots,x_n) = \underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2).$$

One deduces:

$$\underline{a}_{i_1\dots i_k}(x_1,\dots,x_n) = a_{i_1\dots i_k}(x_1,\dots,x_n), \text{ for all } (x_1,\dots,x_n) \in \mathring{K}^n.$$

Then, defining the k-form $\underline{\omega}$ on $\mathbf{R}^n$ by

$$\underline{\omega} = \sum_{i_1<\cdots<i_k} \underline{a}_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

one has already

$$\underline{\omega} \upharpoonright \mathring{K}^n = \omega \upharpoonright \mathring{K}^n.$$

Let us show that $\underline{\omega} \upharpoonright K^n = \omega$. That is, let us check that for all plot $P\colon U \to \mathbf{R}^n$, $P^*(\underline{\omega}) = \omega(P)$. Actually, one has

$$\mathrm{sq}^*(\omega) = \mathrm{sq}^*(\underline{\omega} \upharpoonright K^n).$$

Indeed:

$$\begin{aligned} \mathrm{sq}^*(\omega) &= \sum_{i_1\dots i_k} A_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1\dots i_k} 2^k x_{i_1} \dots x_{i_k} \hat{A}_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1\dots i_k} 2^k x_{i_1} \dots x_{i_k} \underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

And, on the other hand:

$$\mathrm{sq}^*(\underline{\omega} \upharpoonright K^n) = \sum_{i_1\dots i_k} 2^k x_{i_1} \dots x_{i_k} \underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

[4] Which is a generalisation of a famous Whitney Theoreme [Whi43]

Thus, $\mathrm{sq}^*(\omega - \underline{\omega} \restriction \mathrm{K}^n) = 0$. Therefore, according to the previous lemma (art. 3), $\omega - \underline{\omega} \restriction \mathrm{K}^n = 0$. And indeed, ω is the restriction of the smooth k-form $\underline{\omega}$ on K^n. ☐

5. An Exemple Of Application. — Among the possible applications of the theorems above (art. 3) and (art. 4), there is already one worthy of mention. It is about the description of closed 2-form, invariant with respect to the action of a Lie group. As it has been showed in particular in the classification of SO(3)-symplectic manifolds [Igl84, Igl91], any closed 2-form form ω on a manifold M, invariant by a compact group[5] G, is characterized by its moment map $\mu \colon \mathrm{M} \to \mathcal{G}^*$ (we assume the action Hamiltonian), and for each moment map, a closed 2-form $\varepsilon \in \mathrm{Z}^2(\mathrm{M}/\mathrm{G})$. Let us be precise: the space of closed 2-forms $\mathrm{Z}^2(\mathrm{M})$ is a vector space, the space of G-equivariant maps from M to $\mathcal{G}^*$ is also a vector space. Then, the map associating its moment map[6] μ with each invariant closed 2-form ω is linear. What we claim is that the kernel of this map is exactly $\mathrm{Z}^2(\mathrm{M}/\mathrm{G})$, where M/G is equipped with the quotient diffeology.

Now, if an equivariant map is easy to conceive, it is more problematic for a differential form on the space of orbits, which is generally not a manifold. This is where the above theorem can help, because it happens that M/G is not far to be a manifold with boundary or corners, as show the following example.

Let us consider the simple case of $\mathrm{M} = \mathbf{R}^{2n}$ equipped with the standard symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. It is invariant by the group $\mathrm{SO}(2,\mathbf{R})^n$ acting naturally, each factor on its respective copie of $\mathbf{R}^2$. The (diffeological) quotient space $\mathcal{Q}^n = \mathbf{R}^{2n}/\,\mathrm{SO}(2,\mathbf{R})^n$ is the n-th power of $\mathcal{Q} = \mathbf{R}^2/\,\mathrm{SO}(2,\mathbf{R})$. Let $\mathrm{X} = (\mathrm{X}_i)_{i=1}^n$ with $\mathrm{X}_i = (q_i, p_i)$. There is a natural smooth bijection $j_{2n} \colon \mathcal{Q}^n \to \mathrm{K}^n$, given by $j_{2n} \colon \mathrm{class}(\mathrm{X}) \mapsto (\|\mathrm{X}_i\|^2)_{i=1}^n$. It turns out that this smooth bijection[7] induces, by pullback, an injection j_{2n}^* from $\Omega^k(\mathrm{K}^n)$ into $\Omega^k(\mathcal{Q}_n)$. Thus, thanks to (art. 4), for each 2-form ε on the quotient $\mathcal{Q}_n$ there exists a 2-form $\underline{\varepsilon}$ on $\mathbf{R}^n$, such that $\varepsilon = j_{2n}^*(\underline{\varepsilon})$. And the 2-form ω is characterized by μ and $\underline{\varepsilon} \restriction \mathrm{K}^n$, with $\underline{\varepsilon} \in \Omega^k(\mathbf{R}^n)$.

Proof. Let us prove that j_{2n}^* is injective. Let $x = (x_i)_{i=1}^n \in \mathbf{R}^n$ and $\iota_n \colon x \mapsto (x_i, 0)_{i=1}^n$ from $\mathbf{R}^n$ into $\mathbf{R}^{2n}$. Let $j_n \colon \mathbf{R}^n/\{\pm 1\}^n \to \mathrm{K}^n$ be defined by $j_n \colon \mathrm{class}(x) \mapsto \mathrm{sq}\,(x) = (x_i^2)_{i=1}^n$. Then, $j_n = j_{2n} \circ \underline{\iota}$, where $\underline{\iota}$ is the

[5] There could a diffeological generalisation possible here to non compact group.

[6] The manifold M is supposed to be connected. To have a unicity of the moment maps we decide to fix their value to 0 at some base point $m_0 \in \mathrm{M}$, for example.

[7] Which is not a diffeomorphism [PIZ07].

projection of ι, from $\mathbf{R}^n/\{\pm 1\}^n$ to $\mathcal{Q}^n$.

$$\begin{array}{ccccc}
\mathbf{R}^n & \xrightarrow{\quad\iota\quad} & \mathbf{R}^{2n} & & \\
\downarrow{\scriptstyle \text{class}} & & \downarrow{\scriptstyle \text{class}} & & \\
\mathbf{R}^n/\{\pm 1\}^n & \xrightarrow[\underline{\iota}]{} & \mathcal{Q}_n = \mathbf{R}^{2n}/\operatorname{SO}(2,\mathbf{R})^n & \xrightarrow[j_{2n}]{} & \mathrm{K}^n = [0,\infty[^n
\end{array}$$

But $\mathrm{sq} = j_n \circ \mathrm{class}$ and we know that $\mathrm{sq}^* = \mathrm{class}^* \circ j_n^* \colon \Omega^k(\mathrm{K}^n) \to \Omega^k(\mathbf{R}^n)$ is injective (art. 3), thus $j_n^* \colon \Omega^k(\mathrm{K}^n) \to \Omega^k(\mathbf{R}^n/\{\pm 1\})$ is injective. On the other hand, $j_n = j_{2n} \circ \underline{\iota}$, then $j_n^* = \underline{\iota}^* \circ j_{2n}^*$. Since j_n is injective, j_{2n}^* is necessarilly injective too. $\square$

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