A FEW HALF-LINES

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-AFHL.pdf

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In this note we shall talk about a few diffeologies that appeared, equipping the half-line $[0,\infty[$.

As you may know, the half-line $[0,\infty[\subset \mathbf{R}$ can be equipped with the subset diffeology [TextBook, 1.33], that is, a plot in $[0,\infty[$ is just a smooth parametrization in $\mathbf{R}$ taking its values in $[0,\infty[$. Let us denote this space by Δ. Actually, Δ is a *manifold with boundary* according to [TextBook, 4.12, 4.16], the boundary being the point $\{0\}$. Now the set $[0,\infty[$ appears in many other places, as the underlying set for the quotients $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ [TextBook, 1.50, Ex. 50]. Indeed, the quotient space[1] Δ_n can be realized as the set $[0,\infty[$ equipped with the pushforward of the usual diffeology of $\mathbf{R}^n$ by the norm-square map $\mathrm{sq}_n : x \mapsto \|x\|^2$. Now, for every integer n, thanks to the inclusion

$$\mathrm{J}_n^{n+1} : \mathbf{R}^n \to \mathbf{R}^{n+1}, \quad \text{defined by} \quad \mathrm{J}_n^{n+1}(x) = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

we get a family of smooth injections on the quotient spaces, denoted by j_n^{n+1}.

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{\mathrm{J}_n^{n+1}} & \mathbf{R}^{n+1} \\ \pi_n \downarrow & & \downarrow \pi_{n+1} \\ \Delta_n & \xrightarrow[j_n^{n+1}]{} & \Delta_{n+1} \end{array}$$

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[1] Actually $\Delta_1 = \mathbf{R}/\{\pm 1\}$ is an orbifold [TextBook, 4.17].

These definitions give a *direct system* $\{\Delta_n, j_n^m\}_{n,m\in\mathrm{N}}$ indexed by the integers, where the j_n^m, $m = n+k$, are defined by

$$j_n^{n+k} : \Delta_n \to \Delta_{n+k}, \quad \text{and} \quad j_n^{n+k} = j_{n+k-1}^{n+k} \circ j_{n+k-2}^{n+k-1} \circ \cdots \circ j_{n+1}^{n}.$$

Let us recall that since the category {Diffeology} is stable by the operations of sum (disjoint union or coproduct) [TextBook, 1.39] and quotient [TextBook, 1.50], it is possible to define the direct limit (or inductive limit or colimit) of a direct system $\{\mathrm{X}_i, f_i^j\}_{i,j\in\mathrm{I}}$, where I is an up-directed set of indices[2], the X_i are diffeological spaces, and the $f_i^j : \mathrm{X}_i \to \mathrm{X}_j$ are smooth maps such that $f_k^j \circ f_i^k = f_i^j$ and $f_i^i = \mathbf{1}_{\mathrm{X}_i}$. By definition

$$\varinjlim \mathrm{X}_i = \left(\coprod_{i\in\mathrm{I}} \mathrm{X}_i\right) / \sim,$$

where the equivalence relation is defined by

$$(m,x) \sim (n,y) \Leftrightarrow \exists k, k \geq m, k \geq n \text{ and } j_m^k(x) = j_n^k(y).$$

Then, a plot in $\varinjlim \mathrm{X}_i$ is any parametrization $\mathrm{P} : \mathrm{U} \mapsto \varinjlim \mathrm{X}_i$ such that there exists everywhere in U, locally, a plot $\mathrm{Q} : \mathrm{V} \to \coprod_{i\in\mathrm{I}} \mathrm{X}_i$ satisfying $\mathrm{class}(\mathrm{Q}(r)) = \mathrm{P}(r)$ for all $r \in \mathrm{V}$, where class is the projection from $\coprod_{i\in\mathrm{I}} \mathrm{X}_i$ onto its quotient $\varinjlim \mathrm{X}_i$. Now, thanks to the definition of the sum of diffeological spaces, that means that everywhere in V, there exist an index i and a domain $\mathrm{W} \subset \mathrm{V}$ such that $\mathrm{val}(\mathrm{Q} \restriction \mathrm{W}) \subset \mathrm{X}_i$. In other words, there exist everywhere in U, an index i, a domain $\mathrm{W} \subset \mathrm{U}$, and a plot $\mathrm{Q} : \mathrm{W} \to \mathrm{X}_i$, such that $\mathrm{P}(r) = \mathrm{class}_i(\mathrm{Q}(r))$, where $\mathrm{class}_i = \mathrm{class} \restriction \mathrm{X}_i$ and $r \in \mathrm{W}$. So, that is the natural diffeological construction of limit we inherit from the standard definition of sums and quotients[3].

Now, applied to our system above, and after identifying each Δ_n with $[0,\infty[$ equipped with the pushforward of the smooth diffeology of $\mathbf{R}^n$ by the square map $\mathrm{sq} : x \mapsto \|x\|^2$, the maps j_n^m reduce to $\mathbf{1}_{[0,\infty[}$, and the plots in $\Delta_\infty = \varinjlim(\Delta_n)$ are the parametrizations $\mathrm{P} : \mathrm{U} \to [0,\infty[$ such that everywhere in U, there exist an integer

[2] In french: un ensemble filtrant croissant d'indices.

[3] I (PIZ) was hesitant to include or not this definition in the TextBook. I eventually renounced because I nowhere use this construction in the book, and moreover it is something flowing naturally from the definition of sums and quotients. But thanks to this note I could include now a paragraph on the definition of limits, inductive and projective.

N, a domain $V \subset U$, and a smooth map $Q : V \to \mathbf{R}^N$ such that $P(r) = \|Q(r)\|^2$ for all $r \in V$. In other words, there exist N smooth real functions Q_i, defined on V, such that

$$P(r) = \sum_{i=1}^{N} Q_i(r)^2.$$

And that's all for the description of the diffeology of Δ_∞. The plots are the non-negative parametrizations of $\mathbf{R}$ which write locally as a finite sum of squares of smooth real functions.

Now: one day of June, last year, during the Conference in honor of Souriau, I was drinking a Coke on the Cours Mirabeau in Aix-en-Provence with Enxin when he asked me if Δ_∞ and Δ could coincide as diffeological spaces? In other words, if any non-negative parametrization of $\mathbf{R}$ could be locally written as a sum of squares of smooth real functions? I had no idea...

Then, I took my iPhone and googled this: "non negative function as sums of squares", magically the first link appeared on the screen was the paper of Bony and al. [BBCP], which states in its very abstract that:

> "*For $n \geq 4$, there are $\mathcal{C}^\infty$ nonnegative functions f of n variables (...) which are not a finite sum of squares of C^2 functions.*"

We were done, since — said with our words — that means that there exist 4-plots in Δ which cannot be locally lifted smoothly in $\coprod_{n \in \mathrm{N}} \Delta_n$, *i.e.*, there are plots in Δ which are not plots in Δ_∞. Therefore, if clearly the diffeology of the limit Δ_∞ is finer than the diffeology of Δ, the converse is not true, and these two diffeologies on $[0, \infty[$ do not coincide[4]. We have however the chain of strictly ordered diffeological spaces on the same underlying set $[0, \infty[$,

$$\Delta_1 \prec \Delta_2 \prec \cdots \prec \Delta_\infty \prec \Delta.$$

References

[TextBook] Patrick Iglesias-Zemmour *Diffeology,* Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence, 2012. http://www.ams.org/bookstore-getitem/item=SURV-185

[4]Recently Enxin told me that he found a way to avoid using the theorem of Bony and al. thanks to some simpler Hilbert result.

[BBCP] Jean-Michel Bony, Fabrizio Broglia, Ferruccio Colombini, Ludovico Pernazza. *Nonnegative functions as squares or sums of squares.* Journal of functional analysis, 232, pp. 137–147. Elsevier, 2006.

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