1-FORMS ON THE HALF-LINE

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-1FOHL.pdf

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We give a direct proof that every differential 1-form on $[0,\infty[\subset \mathbf{R}$, for the subset diffeology, is the restriction of smooth 1-form defined on a neighborhood of the half-line.

This is a previous result obtained with Paul Donato in [DIZ14], that every 1-form on the half-line $[0,\infty[\subset \mathbf{R}$ is the restriction of a 1-form on $\mathbf{R}$. But we give here a direct proof, concocted with Serap Gürer, during a working cession.

Proposition — Let $\Delta = [0,\infty[\subset \mathbf{R}$ equipped with the subset diffeology. Let $\alpha \in \Omega^1(\Delta)$, then there exists a smooth 1-form $\bar\alpha$ defined on some neighborhood of Δ such that $\alpha = \bar\alpha \restriction \Delta$. In other words, there exists a smooth function $a \in \mathcal{C}^\infty(]-\varepsilon,\infty[)$, $\varepsilon > 0$, such that α is the restriction of $a(x)dx$ to $[0,\infty[$. In other words, for any n-plot $\mathrm{P} : r \mapsto x_r$ in Δ,

$$\alpha(\mathrm{P})_r(\delta r) = a(\mathrm{P}(r))\frac{\partial x_r}{\partial r}(\delta r),$$

where, $n \in \mathbf{N}$, r belongs to the domain of P and $\delta r \in \mathbf{R}^n$.

Proof. Since $]0,\infty[\subset [0,\infty[$ inherits the usual smooth diffeology, let $\alpha \restriction]0,\infty[= f(x)dx$, and $f \in \mathcal{C}^\infty(]0,\infty[,\mathbf{R})$.

Now, let $\mathrm{sq} : t \mapsto t^2$, then $\mathrm{sq}^*(\alpha)_t(\delta t) = \alpha(t \mapsto t^2)_t(\delta t) = \mathrm{F}(t)\delta t$, with $\mathrm{F} \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$. And for all $t \neq 0$, $\mathrm{F}(t) = f(t^2) \times 2t$.

Next, $\mathrm{sq}^*(\alpha)$ is invariant by $-1 : t \mapsto -t$, thus $(-1)^*(\mathrm{sq}^*(\alpha))_t(\delta t) = \mathrm{sq}^*(\alpha)_t(\delta t)$. That is, $-\mathrm{F}(-t) = \mathrm{F}(t)$ and then $\mathrm{F}(0) = 0$. Hence, there exists a smooth function $\varphi \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$ such that $\mathrm{F}(t) =$

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$2t\varphi(t)$, for all $t \in \mathbf{R}$, and then $f(t^2) = \varphi(t)$ for all $t \neq 0$. Therefore, the function φ is even, we can apply the Whitney Theorem:

Theorem [Whi43] *An even function $f(x) = f(-x)$, defined on a neighborhood of the origin, may be written as $g(x^2)$. If f is smooth, g may be made smooth.*

There exists then a smooth function g defined on a neighborhood of the origin such that $f(t^2) = g(t^2)$. That is, $f = g \restriction]0,\infty[$. Let us then define $\bar\alpha = g(x)dx$, $\bar\alpha$ is a smooth 1-form defined on an open neighborhood of $[0,\infty[$, and $\alpha \restriction]0,\infty[= \bar\alpha \restriction]0,\infty[$.

Now, let γ be any path in $[0,\infty[$. Let $\mathcal{O} = \gamma^{-1}(]0,\infty[)$, $\mathcal{O} \subset \mathbf{R}$ is open, and on this open subset $\bar\alpha(\gamma) = \alpha(\gamma)$. Hence, by continuity $\bar\alpha(\gamma) = \alpha(\gamma)$ on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (since $\bar\alpha(\gamma)$ and $\alpha(\gamma)$ are smooth). But γ is constant (and equal to 0) on $\mathbf{R} - \overline{\mathcal{O}}$, then $\bar\alpha(\gamma)$ and $\alpha(\gamma)$ both vanishe on $\mathbf{R} - \overline{\mathcal{O}}$. Thus $\alpha(\gamma) = \bar\alpha(\gamma)$ on the whole $\mathbf{R}$. Therefore, since $\bar\alpha$ and α coincide on the 1-plots, they coincide as 1-forms [PIZ13, §6.37]. □

References

[PIZ13] Patrick Iglesias-Zemmour. *Diffeology,* Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence, 2012. http://www.ams.org/bookstore-getitem/item=SURV-185

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