1-FORMS ON HALF-SPACES

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-1FOHS.pdf

We prove that every differential 1-form on the half-space $\mathrm{H}^n = [0,\infty[\times \mathbf{R}^{n-1}$ is the restriction of a smooth 1-form on $\mathbf{R}^n$.

This is a (almost) straightforward generalisation of the proposition of blog post [PIZ16]. And with a little bit of rewriting, this proof applies to all differential k-forms on half-spaces.

Proposition — Let $\mathrm{H}^n = [0,\infty[\times \mathbf{R}^{n-1}$ be the half n-space, equipped with the subset diffeology. Let $\alpha \in \Omega^1(\mathrm{H}^n)$ be a differential 1-form on H^n. Then, there exists a smooth 1-form $\bar\alpha$ defined on some neighborhood of $\mathrm{H}^n \subset \mathbf{R}^n$ such that $\alpha = \bar\alpha \restriction \mathrm{H}^n$.

Proof. Since $]0,\infty[\times \mathbf{R}^{n-1} \subset [0,\infty[\times \mathbf{R}^{n-1}$ inherits the usual smooth diffeology,

$$\alpha \restriction\,]0,\infty[\times \mathbf{R}^{n-1} = a(x,y)dx + \sum_{i=1}^{n-1} b_i(x,y)dy_i$$

where $(x,y) \in]0,\infty[\times \mathbf{R}^{n-1}$ and $a, b_i \in \mathcal{C}^\infty(]0,\infty[\times \mathbf{R}^{n-1}, \mathbf{R})$.

Now, let

$$\mathrm{sq}_1 : (t,y) \mapsto (t^2,y),$$

then

$$\begin{aligned} \mathrm{sq}_1^*(\alpha)_{(t,y)}(\delta t,\delta y)) &= \alpha((t,y)\mapsto (t^2,y))_{(t,y)}(\delta t,\delta y) \\ &= \mathrm{A}(t,y)\delta t + \sum_{i=1}^{n-1} \mathrm{B}_i(t,y)\delta y_i, \end{aligned}$$

with $\mathrm{A}, \mathrm{B}_i \in \mathcal{C}^\infty(\mathbf{R}^n,\mathbf{R})$. And for all $t \neq 0$,

$$\mathrm{A}(t,y) = 2t\,\mathrm{a}(t^2,y) \text{ and } \mathrm{B}_i(t,y) = b_i(t^2,y).$$

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Next, $\mathrm{sq}_1^*(\alpha)$ is invariant by $(-1,1) : (t,y) \mapsto (-t,y)$, thus

$$\begin{aligned}
\mathrm{sq}_1^*(\alpha)_{(t,y)}(\delta t, \delta y) &= (-1,1)^*(\mathrm{sq}_1^*(\alpha))_{(t,y)}(\delta t, \delta y) \\
&= \mathrm{sq}_1^*(\alpha)_{(-t,y)}(-\delta t, \delta y) \\
&= \mathrm{A}(-t,y)(-\delta t) + \sum_{i=1}^{n-1} \mathrm{B}_i(-t,y)\delta y.
\end{aligned}$$

Thus, $-\mathrm{A}(-t,y) = \mathrm{A}(t,y)$ and $\mathrm{B}_i(t,y) = \mathrm{B}_i(-t,y)$. In particular, $\mathrm{A}(0,y) = 0$. Hence, there exists a smooth function $\underline{\mathrm{A}} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$ such that $\mathrm{A}(t,y) = 2t\underline{\mathrm{A}}(t,y)$, for all $t \in \mathbf{R}$. Thus, $a(t^2,y) = \underline{\mathrm{A}}(t,y)$. Now, $\underline{\mathrm{A}}$ is even in t, as well as the B_i. We can then apply the Hassler Whitney Theorem [Whi43, Theorem 1 and final remark], stated as follows:

Theorem [H. Whitney] *If a smooth function $f(t,x)$ is even in t, $f(t,x) = f(-t,x)$, then there exists a smooth function $g(t,x)$ such that $f(t,x) = g(t^2,x)$.*

Hence, there exists a smooth function $\underline{a}(t,y)$ such that $\underline{\mathrm{A}}(t,y) = \underline{a}(t^2,y)$, and there exists $(n-1)$ smooth functions $\underline{b}_i$ such that $\mathrm{B}_i(t,y) = \underline{b}_i(t^2,y)$. We have then, for all $t > 0$, $a(t,y) = \underline{a}(t,y)$ and $b_i(t,y) = \underline{b}_i(t,y)$.

Let us then define $\bar{\alpha}$ on $\mathbf{R}^n$,

$$\bar{\alpha} = \underline{a}(x,y)dx + \sum_{i=1}^{n-1} \underline{b}_i(t,y)dy_i.$$

The form $\bar{\alpha}$ is a smooth 1-form defined on an open neighborhood of H^n, and $\alpha \restriction]0,\infty[\times \mathbf{R}^{n-1} = \bar{\alpha} \restriction]0,\infty[\times \mathbf{R}^{n-1}$. Let us prove now that α and $\bar{\alpha}$ coincide on the whole H^n. Since α and $\bar{\alpha} \restriction \mathrm{H}^n$ are two differential 1-forms on H^n, it is enough to show that they take the same value on any smooth path.

Let γ be any path in $[0,\infty[\times \mathbf{R}^{n-1}$. Let $\mathcal{O} = \gamma^{-1}(]0,\infty[\times \mathbf{R}^{n-1})$, $\mathcal{O} \subset \mathbf{R}$ is open, and on this open subset $\bar{\alpha}(\gamma) = \alpha(\gamma)$. Hence, by continuity $\bar{\alpha}(\gamma) = \alpha(\gamma)$ on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (since $\bar{\alpha}(\gamma)$ and $\alpha(\gamma)$ are smooth). But on the open subset $\mathbf{R} - \overline{\mathcal{O}}$, γ takes its values in $\partial\mathrm{H}^n = \{0\} \times \mathbf{R}^{n-1}$; $\gamma \restriction \mathbf{R} - \overline{\mathcal{O}}$ is a plot of the boundary $\partial\mathrm{H}^n$. Let $i_2 : \mathbf{R}^{n-1} \to \partial\mathrm{H}^n$, $i_2(y) = (0,y)$. Then, $i_2^*(\alpha)$ and $i_2^*(\bar{\alpha})$ are both 1-forms on $\mathbf{R}^{n-1}$. Let us prove that they coincide. On the one hand

$$i_2^*(\bar{\alpha})_y(\delta y) = \sum_{i=1}^{n-1} \underline{b}_i(0, y)\delta y.$$

On the other hand, let us notice that

$$i_2 = \mathrm{sq}_1 \circ i_2 : y \mapsto (0, y) \mapsto (0^2, y).$$

Thus, $i_2^*(\alpha) = i_2^*(\mathrm{sq}_1^*(\alpha))$ and then $i_2^*(\alpha)_y(\delta y) = \mathrm{sq}_1^*(\alpha)_{(0,y)}(0, \delta y)$. But,

$$\begin{aligned} \mathrm{sq}_1^*(\alpha)_{(0,y)}(0, \delta y) &= \mathrm{A}(0, y) \times 0 + \sum_{i=1}^{n-1} \mathrm{B}_i(0, y)\delta y_i \\ &= \sum_{i=1}^{n-1} \underline{b}_i(0, y)\delta y_i, \end{aligned}$$

since $\mathrm{B}_i(t, y) = \underline{b}_i(t^2, y)$. Hence α and $\bar{\alpha}$ coincide on $\partial\mathrm{H}^n$ and then $\bar{\alpha}(\gamma)$ and $\alpha(\gamma)$ coincide everywhere. Therefore, since $\bar{\alpha}$ and α coincide on the 1-plots in H^n, they coincide as 1-forms [PIZ13, §6.37], and then, $\alpha = \bar{\alpha} \restriction \mathrm{H}^n$. $\square$

References

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