1-FORMS ON HALF-LINES

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In this note we characterize the differential 1-forms, defined on the various half-lines Δ, Δ_n, Δ_∞, that share the same underlying set $[0,\infty[$.

Diffeologies on the half-line

We consider a series of diffeologies on the set $[0,\infty[$:

A) Equipped with the subset diffeology, it is a *manifold with boundary* $\{0\}$, [DBook, 4.12, 4.16]. It is denote by Δ.

B) Equipped with the pushforward of the standard diffeology of $\mathbf{R}^n$, $n > 0$, by the norm-square map:

$$\mathrm{Sq} : \mathbf{R}^n \to [0,\infty[\quad \text{with} \quad \mathrm{Sq} : x \mapsto \|x\|^2,$$

it represents the quotients $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ [DBook, 1.50, Ex. 50]. We denote $\Delta_\infty = \lim_{n\to\infty} \Delta_n$, and when we write Δ_n, we allow n to represent also ∞.

Note 1. There are no two different half-lines above that are diffeomorphic. We recall that $\dim_0(\Delta_n) = n$, $\dim_0(\Delta_\infty) = \infty$ and $\dim_0(\Delta) = \infty$, [DBook, Ex. 50,51] and [DBlog].

Note 2. The choice of the function Sq to characterize the quotient $\mathbf{R}^n/\mathrm{O}(n)$ is irrelevant. Every other bijection with the space of orbits could have been used to push forward the standard diffeology of $\mathbf{R}^n$, as explained in [DBook, 1.52]. For example we could have chosen equivalently $\mathrm{X} \mapsto \|\mathrm{X}\|$, but then the injection $j : [0,\infty[\to \mathbf{R}$ would have not been smooth.

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1-forms on the half-lines

1. The 1-forms are closed. Every differential 1-form defined on Δ_n, Δ_∞, or $\Delta \subset \mathbf{R}$, is closed. Moreover, every half-line is contractible. Therefore, every 1-form is exact.
Note. In the case $n = 1$, the 1-forms are closed simply because the dimension of Δ_1 is 1 [DBook, 6.39]. But because the dimension of Δ_n and Δ at the origin are n or ∞, the argument of the dimension doesn't apply so simply.

Proof. Any of these half-lines has $[0, \infty[$ as underlying space, only the diffeology change. In each case, the subset $]0, \infty[$ is D-open. In each case, the diffeology induced on $]0, \infty[$ is the standard diffeology. The difference of behavior in the diffeology happens only on the neighborhood of $\{0\}$. Now, let α be a 1-form on a half-line, let P be a m-plot and $\mathrm{U} \subset \mathbf{R}^m$ be its domain. The subset $\mathrm{V} = \mathrm{P}^{-1}(]0, \infty[)$ is open in U and because $]0, \infty[$ is 1-dimensional $d[\alpha(\mathrm{P} \restriction \mathrm{V})] = 0$. The complementary W of V in U is closed. On its interior $\mathring{\mathrm{W}}$ the plot is constant, therefore $\alpha(\mathrm{P} \restriction \mathring{\mathrm{W}}) = 0$, and then $d[\alpha(\mathrm{P} \restriction \mathring{\mathrm{W}})] = 0$. Now, on the boundary $\partial \mathrm{V} = \bar{\mathrm{V}} - \mathrm{V} = \mathrm{W} - \mathring{\mathrm{W}}$, every point r is a limit $\lim_{n\to\infty} r_n$, with $r_n \in \mathrm{V}$. Thus, by continuity, for all $\xi, \xi' \in \mathbf{R}^m$, $d[\alpha(\mathrm{P})]_r(\xi, \xi') = \lim_{n\to\infty} d[\alpha(\mathrm{P} \restriction \mathrm{V})]_{r_n}(\xi, \xi') = 0$. Hence, $d\alpha(\mathrm{P}) = 0$ everywhere, that is, $d\alpha = 0$.

Next, about contractibility. The radial retraction $x \mapsto sx$ in $\mathbf{R}^n$ is equivariant under the action of $\mathrm{O}(n)$. Thus, the quotients $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ are contractible, and also the limit Δ_∞. For Δ we have the retraction $\rho_s : t \mapsto s^2 t$. The map $(s, t) \mapsto s^2 t$, defined on $\mathbf{R} \times \Delta$ takes its values in Δ and is smooth. Thus, Δ is contractible. According to [DBook, 6.90] every closed differential form on a contractible diffeological space is exact. Therefore, every 1-form on these half-lines is the differential of a smooth function, this function can be normalized by zero at the origin, and then is unique. □

2. The case of Δ. Every differential 1-form on the embedded half-line $\Delta \subset \mathbf{R}$ is the restriction of a differential 1-form defined on $\mathbf{R}$. In other words, the natural induction $j : [0, \infty[\to \mathbf{R}$ induces a surjective pullback $j^* : \Omega^1(\mathbf{R}) \to \Omega^1(\Delta)$. That is, for all $\alpha \in \Omega^1(\Delta)$, there exists $a \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ such that $\alpha_x = a(x)\,dx$, for all $x \geq 0$.

Proof. Thanks to [DBook, 4.13] we know that a smooth function from Δ to $\mathbf{R}$ is the restriction of a smooth function from $\mathbf{R}$ to $\mathbf{R}$. Together with the proposition 1, that gives the result. □

3. The case of Δ_n. The set $[0,\infty[$ is equipped with the pushforward of the standard diffeology of $\mathbf{R}^n$ by the norm-square map Sq. That identifies Δ_n with $\mathbf{R}^n/\mathrm{O}(n)$ by $\mathrm{class}(\mathrm{X}) \simeq \|\mathrm{X}\|^2$. The injection $j : \Delta_n \simeq [0,\infty[\to \mathbf{R}$ is smooth. The pullback $j^* : \Omega^1(\mathbf{R}) \to \Omega^1(\Delta_n)$ is, here again, surjective[1].

Proof. The smoothness of the injection comes from the smoothness of the square $\mathrm{Sq} : \mathbf{R}^n \to [0,\infty[$, with $\mathrm{Sq}(\mathrm{X}) = \|\mathrm{X}\|^2$. Now, for the same reason than previously, every 1-form α on Δ_n is exact, that is, there exists a function $f \in \mathcal{C}^\infty(\Delta_n, \mathbf{R})$ such that $\alpha = df$. Pulled back on $\mathbf{R}^n$, we have $\mathrm{Sq}^*(\alpha) = \mathrm{Sq}^*(df) = d(f \circ \mathrm{Sq})$. The function $\mathrm{F} = f \circ \mathrm{Sq}$ is smooth and invariant by $\mathrm{O}(n)$. Conversely every smooth function $\mathrm{F} : \mathbf{R}^n \to \mathbf{R}$ that is $\mathrm{O}(n)$-invariant is the pullback, by Sq, of a smooth function f on Δ_n. So, every 1-form on Δ_n is the pushforward of a differential $d\mathrm{F}$, where F is smooth and $\mathrm{O}(n)$-invariant. Let us restrict F to the subspace of the vectors $(x, 0)$, $x \in \mathbf{R}$, and let $\mathrm{F}(x)$ for $\mathrm{F}(x, 0)$. We have $\mathrm{F}(x) = f(x^2)$, that is, $\mathrm{F}(+x) = \mathrm{F}(-x)$. Thanks to Whitney theorem [Whi43], there exists a smooth function $g \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ such that $\mathrm{F}(x) = g(x^2)$. Thus, $f(x^2) = g(x^2)$, in other words: $f = g \restriction [0,\infty[$, f is the restriction of a smooth function to the interval $[0,\infty[$. Thus $\alpha = df = d(g \circ j) = j^*(dg)$; written differently, $\alpha_t = a(t)\,dt$, for all $t \in [0,\infty[$. On $\mathbf{R}^n$, the pullback of α writes,

$$\mathrm{Sq}^*(\alpha)_\mathrm{X} = 2a\left(\|\mathrm{X}\|^2\right)\mathrm{X}\cdot d\mathrm{X} = 2a\left(\|\mathrm{X}\|^2\right)\sum_{i=1}^{n} \mathrm{X}^i d\mathrm{X}^i,$$

where a is a smooth function on $\mathbf{R}$. □

4. The 1-forms vanish at the origin. Every differential 1-form α, defined on any half-line Δ or Δ_n or Δ_∞, vanishes at the origin [DBook, 6.40]. That is, for every 1-plot γ pointed at the origin, $\gamma(0) = 0$, we have $\alpha(\gamma)_0 = 0$.

In other words, the *cotangent space* reduces to $\{0\}$ at the origin, [DBook, 6.48]; it is equal to $\mathbf{R}$ everywhere else. An interesting question would be to describe the diffeology of the cotangent space,

[1] Note that the inclusion j is smooth injective but not an induction.

and to study its parasymplectic structure[2], that is, the struture defined by the differential of its Liouville form [DBook, 6.49].

Proof. According to what comes before, in every case the injection j from the half-line into $\mathbf{R}$ is smooth, and the form α is the pullback, by j, of some smooth 1-form $A \in \Omega^1(\mathbf{R})$. Thus, $\alpha(\gamma)_0 = j^*(A)(\gamma)_0 = A(j \circ \gamma)_0$. But $j \circ \gamma(0) = 0$ and $j \circ \gamma(t) \geq 0$ imply $d\gamma(t)/dt \restriction_{t=0} = 0$. Therefore, $A(j \circ \gamma)_0 = A_0(d\gamma(t)/dt \restriction_{t=0}) = 0$, and $\alpha(\gamma)_0 = 0$. $\square$

5. The 1**-forms as a** 1**-dimensional module.** From what precedes we conclude that, in every case: $\Delta_\star = \Delta$ or Δ_n or Δ_∞, the space of differential 1-forms $\Omega^1(\Delta_\star)$ is a 1-dimensional module on $\Omega^0(\Delta_\star)$, with $dt \restriction [0, \infty[$ as a generator.

6. Gauges on diffeological spaces. There is a notion of volume for diffeological spaces of finite constant dimension, in [DBook, 6.44] that almost applies to the half-lines but not completely. First of all, in our case the dimension is not constant (except for the case $n = 1$), but more importantly, the 1-form vanishes at the origin, and volumes are assumed to be nowhere vanishing. Nevertheless, in every case above, the space of 1-forms is a 1-dimensional module on the space of smooth functions, and that is an important remark. That leads to the introduction of the concept of *k-jauge* on a diffeological space, which is slightly different from the concept of volume, but pursues the same idea:

Definition. We call a *k-gauge* on a diffeological space X, any k-form generating $\Omega^k(X)$ as a 1-dimensional module on $\Omega^0(X)$.

In our case, for every half-line X, the pullback $j^*(dt)$, where j is the smooth injection of $[0, \infty[$ into $\mathbf{R}$, is a generator of $\Omega^1(X)$. The concept of k-gauge on diffeological spaces worth being studied. There are a few questions around it that need to be answered.

References

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