SMOOTH FUNCTION ON PERIODIC FUNCTIONS

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Ex-SFOPF.pdf

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This exercise gives a simple example of a function on the space of rapidly decreasing sequences that is smooth for the functional diffeology, inherited by the smooth periodic functions, but not smooth for the ordinary product diffeology.

We consider the subspace $\mathcal{E}$ of rapidly decreasing complex sequences $(z_n)_{n\in\mathbf{Z}} \in \mathcal{E}$. We consider the diffeology (♣) on $\mathcal{E}$, inherited by the functional diffeology on the space of smooth periodic functions, defined in [FDOFC]. We denote by $\mathcal{C}\mathrm{an}$ the diffeology inherited by the product diffeology, (♣) is finer than $\mathcal{C}\mathrm{an}$.

✎ Exercise. Show that the linear map $\mathrm{F} : \mathcal{E} \to \mathbf{C}$,

$$\mathrm{F} : (z_n)_{n\in\mathbf{Z}} \mapsto \sum_{n\in\mathbf{Z}} z_n,$$

is smooth for the diffeology (♣), but not for the $\mathcal{C}\mathrm{an}$ diffeology. Hint: find a 1-plot $\gamma : t \mapsto (z_n(t))_{n\in\mathbf{Z}}$ for the diffeology $\mathcal{C}\mathrm{an}$ such that $\mathrm{F}\circ\gamma$ is not smooth.

Solution — We know that the map $j : f \mapsto (f_n)_{n\in\mathbf{Z}}$, from $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C})$ to $\mathcal{E}$, where the f_n are the Fourier coefficients of f, is a diffeomorphism when $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C})$ is equiped with the functional diffeology and $\mathcal{E}$ with the diffeology (♣) [FDOFC]. The inverse is given by $j^{-1} : (f_n)_{n\in\mathbf{Z}} \mapsto [x \mapsto \sum_{n\in\mathbf{Z}} f_n e^{2i\pi nx}]$. Let $\zeta = j^{-1}(z_n)_{n\in\mathbf{Z}}$, then $\mathrm{F}((z_n)_{n\in\mathbf{Z}}) = \zeta(0)$. Therefore $\mathrm{F} = \hat{0}\circ j^{-1}$ where $\hat{0}$ is the evaluation at the origin, $\hat{0}(f) = f(0)$. Since $\hat{0} : \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C}) \to \mathbf{C}$ is smooth for the functional diffeology and j^{-1} is a diffeomorphism, F is smooth.

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Next, consider the path

$$\gamma : t \mapsto (z_n(t))_{n\in\mathbf{Z}} \quad \text{with} \quad z_n(t) = e^{-|n|} e^{ie^{2|n|}t},$$

where $t \in \mathbf{R}$. Since every z_n is smooth, the path γ is smooth with $\mathcal{E} \subset \prod_{n\in\mathbf{Z}} \mathbf{C}$ equipped with the subset diffeology of the product diffeology. Since $|z_n(t)| = e^{-|n|}$ is rapidly decreasing in n, the partial sums $\sum_{n=-\mathrm{N}}^{\mathrm{N}} z_n(t)$ converge for all $t \in \mathbf{R}$. Let $f = \mathrm{F} \circ \gamma$, that is,

$$f(t) = \sum_{n\in\mathbf{Z}} e^{-|n|} e^{ie^{2|n|}t}.$$

We shall check now that f is not derivable at $t = 0$. Consider the variation

$$\begin{aligned} \Delta f(t,0) &= \frac{f(t) - f(0)}{t} \\ &= \sum_{n\in\mathbf{Z}} \frac{e^{-|n|} e^{ie^{2|n|}t} - e^{-|n|}}{t} \\ &= \sum_{n\in\mathbf{Z}} e^{-|n|} \, \frac{e^{ie^{2|n|}t} - 1}{t}. \end{aligned}$$

But,

$$\frac{e^{ie^{2|n|}t} - 1}{t} \xrightarrow[t\to 0]{} ie^{2|n|}.$$

Hence,

$$\Delta f(t,0) \xrightarrow[t\to 0]{} i \sum_{n\in\mathbf{Z}} e^{|n|},$$

but that is not convergent. Therefore f is not derivable in $t = 0$. □

References

[PIZ13] Patrick Iglesias-Zemmour, *Diffeology*. Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence RI, (2013).

[FDOFC] Patrick Iglesias-Zemmour, *Functional Diffeology on Fourier Coefficients* (2014). http://math.huji.ac.il/~piz/documents/DBlog-Ex-FDOFC.pdf

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