SYMPLECTIC DIFFEOLOGY ON THE SPACE OF SMOOTH PERIODIC FUNCTIONS

PATRICK IGLESIAS-ZEMMOUR

ref.
http://math.huji.ac.il/~piz/documents/DBlog-Ex-SDOTSOSPF.pdf

In this exercise we consider a symplectic structure on the space of smooth periodic functions and the action of the infinite torus, see [PIZ14a, PIZ14b]. We study the action of the irrational solenoid and we propose a reduction process for its moment map level, extending a diffeological version of Elisa Prato's quasispheres construction [EP01] to infinite dimension[1], using the diffeology tools.

We consider the space $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$ of 1-periodic complex valued real functions equipped with the functional diffeology. We denote by $\mathcal{E}$ the space of Fourier coefficients of smooth periodic functions and by $j : \mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C}) \to \mathcal{E}$ the mapping

$$j(f) = (f_n)_{n\in\mathbf{Z}} \quad \text{with} \quad f_n = \int_0^1 f(x) e^{-2i\pi nx} dx,$$

and $\mathcal{E}$ is equipped with the pushforward by j of the functional diffeology on $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$, see [PIZ14a].

✎ Exercise (I). For all plots $\mathrm{P} : \mathrm{U} \to \mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$ let

$$\varepsilon(\mathrm{P})_r(\delta r) = \frac{1}{2i\pi} \int_0^1 \overline{f_r(x)}\, \frac{\partial f_r(x)}{\partial r}(\delta r)\, dx.$$

Q 1. Check that ε defines a differential 1-form on $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$.

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[1] This exercise has been inspired by a few and very interesting exchanges I have had with Elisa Prato and Fiametta Battaglia in January 2014 at Firenze where I stayed at the invitation of Elisa. Let me take this opportunity to thank them sincerely.

Q 2. Develop ε and $\omega = d\varepsilon$ also in terms of the real and imaginary parts $f(x) = a(x) + ib(x)$.

Q 3. Develop ε and ω in terms of Fourier coefficients.

Solution: Let us check first that ε is a well defined form. Let $\mathrm{P} : r \mapsto f_r$ be a plot in $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$, let $\mathrm{Q} : s \mapsto r$ be a plot in $\mathrm{dom}(\mathrm{P})$. We want to check that $\varepsilon(\mathrm{P} \circ \mathrm{Q}) = \mathrm{Q}^*(\varepsilon(\mathrm{P}))$, that is,

$$\varepsilon(\mathrm{P} \circ \mathrm{Q})_s(\delta s) = \varepsilon(\mathrm{P})_r(\delta r) \quad \text{with} \quad r = \mathrm{Q}(s) \quad \text{and} \quad \delta r = \frac{\partial r}{\partial s}(\delta s),$$

where δr is a tangent vector at r to $\mathrm{dom}(\mathrm{P})$. But

$$\begin{aligned}
\varepsilon(\mathrm{P} \circ \mathrm{Q})_s(\delta s) &= \frac{1}{2i\pi} \int_0^1 \overline{\mathrm{P} \circ \mathrm{Q}(s)(x)}\, \frac{\partial \mathrm{P} \circ \mathrm{Q}(s)(x)}{\partial s}(\delta s)\, dx \\
&= \frac{1}{2i\pi} \int_0^1 \overline{\mathrm{P}(r)(x)}\, \frac{\partial \mathrm{P}(r)(x)}{\partial r} \frac{\partial r}{\partial s}(\delta s)\, dx, \ \text{with } r = \mathrm{Q}(s) \\
&= \frac{1}{2i\pi} \int_0^1 \overline{f_r(x)}\, \frac{\partial f_r(x)}{\partial r}(\delta r)\, dx, \ \text{with } \delta r = \frac{\partial r}{\partial s}(\delta s) \\
&= \varepsilon(\mathrm{P})_r(\delta r), \ \text{with } r = \mathrm{Q}(s) \text{ and } \delta r = \frac{\partial r}{\partial s}(\delta s).
\end{aligned}$$

Therefore, $\varepsilon(\mathrm{P} \circ \mathrm{Q}) = \mathrm{Q}^*(\varepsilon(\mathrm{P}))$ and ε is a well defined differential 1-form on $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$.

For Question 2. The exterior derivative of ε is given by an immediate application of the definition

$$d\varepsilon(\mathrm{P})_r(\delta_r, \delta' r) = \delta(\varepsilon(\mathrm{P})_r(\delta' r)) - \delta'(\varepsilon(\mathrm{P})_r(\delta r)),$$

for two independent variations δ and δ'. Then,

$$\begin{aligned}
\omega(\mathrm{P})_r(\delta_r, \delta' r) &= \frac{1}{2i\pi} \int_0^1 \overline{\frac{\partial f_r(x)}{\partial r}}(\delta r) \frac{\partial f_r(x)}{\partial r}(\delta' r) \\
&\quad - \overline{\frac{\partial f_r(x)}{\partial r}}(\delta' r) \frac{\partial f_r(x)}{\partial r}(\delta r)\, dx.
\end{aligned}$$

Now let $f_r(x) = \mathrm{a}_r(x) + ib_r(x)$ and let us denote simply

$$\mathrm{a} = \mathrm{a}_r(x), \ b = b_r(x), \ \delta \mathrm{a} = \frac{\partial \mathrm{a}}{\partial r}(\delta r), \ \text{and } \delta b = \frac{\partial b}{\partial r}(\delta r).$$

Then,

$$\begin{aligned}
\varepsilon(\mathrm{P})_r(\delta r) &= \frac{1}{2i\pi}\int_0^1 \overline{f_r(x)}\,\frac{\partial f_r(x)}{\partial r}(\delta r)\,dx \\
&= \frac{1}{2i\pi}\int_0^1 (a-ib)(\delta a+i\delta b)\,dx \\
&= \frac{1}{2\pi}\int_0^1 (a\delta b-b\delta a)\,dx+\frac{1}{2i\pi}\int_0^1 (a\delta a+b\delta b)\,dx \\
&= \frac{1}{2\pi}\int_0^1 (a\delta b-b\delta a)\,dx+\frac{1}{4i\pi}d\left[\int_0^1 (a^2+b^2)\,dx\right](\delta r)
\end{aligned}$$

Therefore, since $d\circ d=0$, the exterior derivative is

$$\omega(\mathrm{P})_r(\delta r,\delta' r)=\frac{1}{\pi}\int_0^1 (\delta a\,\delta' b-\delta' a\,\delta b)\,dx.$$

For Question 3. Since everything is smooth we can exchange limits and integrals. Let

$$f_n(r)=\int_0^1 f_r(x)e^{-2i\pi nx}dx,$$

we have then:

$$\begin{aligned}
\varepsilon(\mathrm{P})_r(\delta r) &= \frac{1}{2i\pi}\int_0^1 \overline{f_r(x)}\,\frac{\partial f_r(x)}{\partial r}(\delta r)\,dx. \\
&= \frac{1}{2i\pi}\int_0^1 \left(\sum_{n\in\mathbf{Z}}\overline{f_n(r)}e^{-2i\pi nx}\right)\left(\sum_{m\in\mathbf{Z}}\frac{\partial f_m(r)}{\partial r}(\delta r)e^{2i\pi mx}\right)\,dx \\
&= \frac{1}{2i\pi}\sum_{n,m\in\mathbf{Z}}\overline{f_n(r)}\,\frac{\partial f_m(r)}{\partial r}(\delta r)\int_0^1 e^{2i\pi(m-n)x}\,dx \\
&= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}}\overline{f_n(r)}\,\frac{\partial f_n(r)}{\partial r}(\delta r).
\end{aligned}$$

And therefore, for the exterior derivative,

$$\begin{aligned}
\omega(\mathrm{P})_r(\delta r,\delta' r) &= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}}\overline{\frac{\partial f_n(r)}{\partial r}}(\delta r)\frac{\partial f_n(r)}{\partial r}(\delta' r) \\
&\quad - \overline{\frac{\partial f_n(r)}{\partial r}}(\delta' r)\frac{\partial f_n(r)}{\partial r}(\delta r).
\end{aligned}$$

Let us notice that if $\hat{x} : \mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C}) \to \mathbf{C}$ denotes the *evaluation map* $\hat{x}(f) = f(x)$, then the 2-form ω is the mean value of the pullbacks

$$\omega = \frac{1}{\pi}\int_0^1 \hat{x}^*(\omega_0)\,dx,$$

where ω_0 is the canonical symplectic form on $\mathbf{C}$. ☐

✎ Exercise (II). We consider the 2-form $\omega = d\varepsilon$ defined on the space of Fourier coefficients $\mathcal{E}$, equipped with the pushforward of the functional diffeology. We consider then the action, described in [PIZ14b], of the infinite torus T^∞ on $\mathcal{E}$,

$$(z_n)_{n\in\mathbf{Z}} \cdot (\mathrm{Z}_n)_{n\in\mathbf{Z}} = (z_n \mathrm{Z}_n)_{n\in\mathbf{Z}}.$$

Q 1. Verify that the action of T^∞ on $\mathcal{E}$ is Hamiltonian and exact[2].

Q 2. Show that the moment maps of the action of T^∞ on $\mathcal{E}$ are given by

$$\mu(\mathrm{Z}) = \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} |\mathrm{Z}_n|^2\,\pi_n^*(\theta) + \sigma,$$

where $\mathrm{Z} = (\mathrm{Z}_n)_{n\in\mathbf{Z}} \in \mathcal{E}$, $\pi_n : \mathrm{T}^\infty \to \mathrm{U}(1)$ is the n-th projection $\pi_n(\mathrm{Z}) = \mathrm{Z}_n$, θ is the canonical invariant 1-form on $\mathrm{U}(1)$, and σ is a constant momentum of T^∞, that is, a constant invariant 1-form.

Q 3. Let $(\alpha_n)_{n\in\mathbf{Z}}$ be a sequence of irrational numbers independent over Q, see [PIZ14b, Ex. II]. Consider the induction

$$\iota : \mathbf{R} \to \mathrm{T}^\infty \quad \text{with} \quad \iota(t) = \left(e^{2i\pi\alpha_n t}\right)_{n\in\mathbf{Z}},$$

and the induced action of $\mathbf{R}$ on $\mathcal{E}$

$$\underline{t}(\mathrm{Z}_n)_{n\in\mathbf{Z}} = \left(e^{2i\pi\alpha_n t}\mathrm{Z}_n\right)_{n\in\mathbf{Z}}.$$

Show that the 1-point moment maps are given by

$$\nu(\mathrm{Z}) = h(\mathrm{Z})\,dt \quad \text{with} \quad h(\mathrm{Z}) = \sum_{n\in\mathbf{Z}} \alpha_n |\mathrm{Z}_n|^2 + c,$$

where c is some constant.

☞ Solution: For Question 1. The primitive ε is invariant by the action of T^∞. Let us recall that

$$\varepsilon(\mathrm{P})_r(\delta r) = \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{\mathrm{Z}_n(r)}\,\frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r),$$

[2] See [PIZ13, Chapter 9]

for all plots $\mathrm{P} : r \mapsto (\mathrm{Z}_n(r))_{n\in\mathbf{Z}}$ in $\mathcal{E}$. Then, let $z = (z_n)_{n\in\mathbf{Z}} \in \mathrm{T}^\infty$, we have

$$\begin{aligned}
\underline{z}^*(\varepsilon)(\mathrm{P})_r(\delta r) &= \varepsilon(\underline{z} \circ \mathrm{P})_r(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n\in\mathbf{Z}} \overline{z_n\, \mathrm{Z}_n(r)}\, \frac{\partial z_n\, \mathrm{Z}_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n\in\mathbf{Z}} \overline{\mathrm{Z}_n(r)}\, \bar{z}_n z_n\, \frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n\in\mathbf{Z}} \overline{\mathrm{Z}_n(r)}\, \frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r) \\
&= \varepsilon(\mathrm{P})_r(\delta r).
\end{aligned}$$

Therefore, the action is Hamiltonian and equivariant [PIZ13, §9.11].

For Question 2. Let θ be the canonical 1-form on U(1) defined by

$$\theta(\zeta)_r(\delta r) = \bar{\zeta}(r)\frac{\partial \zeta(r)}{\partial r}(\delta r),$$

for all plots ζ in U(1). Let $\pi_n : \mathrm{T}^\infty \to \mathrm{U}(1)$ be the n-th projection $\mathrm{Z} \mapsto \mathrm{Z}_n$, for all $\mathrm{Z} = (\mathrm{Z}_n)_{n\in\mathbf{Z}} \in \mathcal{E}$. Let $\zeta : r \mapsto (\zeta_n(r))_{n\in\mathbf{Z}}$ be a plot in T^∞, the 1-point moment map μ of the action of T^∞ on $\mathcal{E}$ is given, up to a constant, by

$$\begin{aligned}
\mu(\mathrm{Z})(\zeta)_r(\delta r) &= \hat{\mathrm{Z}}^*(\varepsilon)(\zeta)_r(\delta r) \\
&= \varepsilon(\hat{\mathrm{Z}} \circ \zeta)_r(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n\in\mathbf{Z}} \overline{\mathrm{Z}_n \zeta_n(r)}\, \frac{\partial \mathrm{Z}_n \zeta_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n\in\mathbf{Z}} |\mathrm{Z}_n|^2\, \bar{\zeta}_n(r)\, \frac{\partial \zeta_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n\in\mathbf{Z}} |\mathrm{Z}_n|^2\, \pi_n^*(\theta)(\zeta)_r(\delta r).
\end{aligned}$$

Therefore, a general 1-point moment map writes

$$\mu(\mathrm{Z}) = \frac{1}{2i\pi} \sum_{n\in\mathbf{Z}} |\mathrm{Z}_n|^2\, \pi_n^*(\theta) + \sigma.$$

Remark that on any ball, for all $n \in \mathbf{Z}$, $\zeta_n(r) = \exp(2i\pi\tau_n(r))$ for some smooth real functions τ_n. Then, the moment map is also given, modulo a constant, by

$$\mu(Z)(\zeta)_r(\delta r) = \sum_{n\in\mathbf{Z}} |Z_n|^2 \frac{\partial \tau_n(r)}{\partial r}(\delta r).$$

Note also that, since ζ is a plot in T^∞ for the tempered diffeology, the norm of the derivatives $\partial\zeta_n(r)/\partial r$, that is, $\partial\tau_n(r)/\partial r$, are dominated by a polynomial in n what insures the convergence of the series defining the moment map just above.

For Question 3. The induction $\iota : \mathbf{R} \to T^\infty$

$$\iota : t \mapsto \left(e^{2i\pi\alpha_n t}\right)_{n\in\mathbf{Z}}$$

induces a projection $\iota^* : \mathcal{T}^{\infty *} \to \mathbf{R}^*$, where $\mathcal{T}^{\infty *}$ is the space of momenta of T^∞. The moment maps with respect to the group $\mathbf{R}$ are then the composites $\nu = \iota^* \circ \mu$, that is,

$$\begin{aligned} \nu &= \iota^*\Big\{\frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} |Z_n|^2\, \pi_n^*(\theta) + \sigma\Big\} \\ &= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} |Z_n|^2\, (\pi_n\circ\iota)^*(\theta) + \iota^*(\sigma). \end{aligned}$$

But

$$\pi_n \circ \iota : t \mapsto \exp(2i\pi\alpha_n t), \quad \text{then} \quad (\pi_n\circ\iota)^*(\theta) = 2i\pi\alpha_n\, dt.$$

Thus,

$$\nu(Z) = h(Z)\, dt \quad \text{with} \quad h(Z) = \sum_{n\in\mathbf{Z}} \alpha_n |Z_n|^2 + c,$$

where $\iota^*(\sigma) = c\, dt$, $c \in \mathbf{R}$. □

✎ Exercise (III). We continue with the data of previous exercise. Let Y be a level of the moment map ν of the action of $\mathbf{R}$, that is,

$$Y = \Big\{Z = (Z_n)_{n\in\mathbf{Z}} \in \mathcal{E} \mid \sum_{n\in\mathbf{Z}} \alpha_n |Z_n|^2 = c\Big\} \quad \text{with } c > 0.$$

Q 1. Verify that, for all $Z \in Y$, if there exist $Z_n \neq 0$ and $Z_m \neq 0$, then the stabilizer of Z is reduced to $\{0\}$ and the orbit of Z by $\mathbf{R}$, equipped with the subset diffeology, is diffeomorphic to $\mathbf{R}$. Such orbits are called *principal orbits*.

Q 2. Verify that the non principal orbits, that is, the *singular orbits*, are the subspaces

$$S^1_n = \{Z \in Y \mid Z_m = 0 \text{ if } m \neq n\}, \quad \text{with} \quad n \in \mathbf{N},$$

each diffeomorphic to the circle S^1.

Q 3. Show that the union

$$\mathcal{S} = \bigcup_{n\in\mathbf{Z}} S^1_n \subset Y,$$

equipped with the subset diffeology is actually the sum of the S^1_n [PIZ13, 1.39], that is,

$$\mathcal{S} = \coprod_{n\in\mathbf{Z}} S^1_n \quad \text{and then} \quad \dim(\mathcal{S}) = 1.$$

Q 4. Show that $Y - \mathcal{S}$ is D-open, that is, open for the D-topology [PIZ13, 2.8].

Solution: For Question 1. Let $Z \in Y$ with $Z_n \neq 0$ and $Z_m \neq 0$, the map

$$t \mapsto (e^{2i\pi\alpha_n t} z_n, e^{2i\pi\alpha_m t} z_m) \quad \text{with} \quad z_n = \frac{Z_n}{|Z_n|} \quad \text{and} \quad z_m = \frac{Z_m}{|Z_m|}$$

is an induction from $\mathbf{R}$ into T^2, because α_n and α_m are independent over Q, see [PIZ13, Exercise 31]. Therefore, the orbit map $t \mapsto \underline{t}(Z)$ is an induction.

For Question 2. There exists n such that for all $m \neq n$, $Z_m = 0$ but $Z_n \neq 0$, since $\sum_{n\in\mathbf{Z}} \alpha_n |Z_n|^2 > 0$. The orbit map is a covering onto the circle S^1_n induced in Y.

For Question 3. Let $P : U \to \mathcal{S}$ be a plot. For every $n \in \mathbf{Z}$ let $\mathcal{O}_n = (\pi_n \circ P)^{-1}(\mathbf{C} - \{0\})$, where $\pi_n : Y \to \mathbf{C}$ is the projection $\pi_n((Z_m)_{m\in\mathbf{Z}}) = Z_n$. Since $\pi_n \circ P$ is smooth, then continuous, every $\mathcal{O}_n \subset U$ is open. Moreover, let $n \neq m$, assume $r \in \mathcal{O}_n \cap \mathcal{O}_m$, that is, $Z_n(r) \neq 0$ and $Z_m(r) \neq 0$, but P takes its values in the union of the S^1_m, $m \in \mathbf{Z}$, hence $Z_n(r) \neq 0$ implies $Z_m(r) = 0$ for all $m \neq n$ thus $\mathcal{O}_n \cap \mathcal{O}_m = \varnothing$. Therefore,

$$U = \bigcup_{n\in\mathbf{Z}} \mathcal{O}_n \quad \text{and} \quad \mathcal{O}_n \cap \mathcal{O}_m = \varnothing \text{ for all } n \neq m.$$

That means that the $\mathcal{O}_n$ are the connected components of U. Thus, P takes locally its values in the S^1_n, $n \in \mathbf{Z}$, that means that $\mathcal{S}$ is the diffeological sum of the circles S^1_n, $n \in \mathbf{Z}$, see [PIZ13, §1.39].

For Question 4. Let $P : U \to Y$ be a plot, $P(r) = (Z_n(r))_{n\in\mathbf{Z}}$. For all $r_0 \in P^{-1}(Y - \mathcal{S})$ there exist at least two different indices n and m such that $Z_n(r_0) \neq 0$ and $Z_m(r_0) \neq 0$. Since Z_n and Z_m are smooth there exists an open neighborhood V of r_0 such that

$Z_n(r) \neq 0$ and $Z_m(r) \neq 0$, for all $r \in V$, that is, $V \subset P^{-1}(Y - \mathcal{S})$. Thus, $P^{-1}(Y - \mathcal{S})$ is a union of open domains, it is then an open domain, and consequently $Y - \mathcal{S}$ is D-open. □

✎ Exercise (IV). We denote by X the space of orbits of the action of $\mathbf{R}$ on Y. We equip Y with the quotient diffeology, and denote by $\mathrm{pr} : Y \to X$ the projection.

Q 1. Why could we call the space X, an *infinite quasiprojective space*?

Q 2. Show that there exists a closed 2-form ϖ on X such that

$$\omega \restriction Y = \mathrm{pr}^*(\varpi).$$

We shall say that ϖ is *quasi-symplectic* without giving it a precise meaning, just by commodity, for now.

Solution: For Question 1. Let $Z = (Z_n)_{n \in \mathbf{Z}} \in Y$, by the change $Z_n \mapsto \sqrt{\alpha_n} Z_n / \sqrt{c}$ the subspace Y is mapped into $S^\infty \in \mathcal{E}$, the unit sphere in $\mathcal{E}$. Now, if all α_n would be equal to 1 then the action of $\mathbf{R}$ would be the action of S^1 and X would be diffeomorphic to $\mathbf{CP}^\infty$, the infinite projective space, and the projection $\mathrm{pr} : Y \to X$ would be the infinite Hopf fibration, see [PIZ13, §4.11] for the same infinite projective space with another diffeology. That explains the choice of vocabulary.

For Question 2. We shall apply the general criterion for a differential form to be the pullback of another one. Let $P : U \to Y$ and $P' : U \to Y$ be two plots

$$\begin{array}{ccc} U & \overset{P}{\underset{P'}{\rightrightarrows}} & Y \\ & & \big\downarrow \mathrm{pr} \\ & & X \end{array} \qquad \text{such that} \quad \mathrm{pr} \circ P = \mathrm{pr} \circ P'.$$

We want to check if, in these conditions, $\omega(P) = \omega(P')$. That would insure the existence of ϖ, a (necessarily closed) 2-form on X such that $\omega = \mathrm{pr}^*(\varpi)$ [PIZ13, §6.38]. We consider first of all what happens on the open subset

$$U_0 = P^{-1}(Y - \mathcal{S}).$$

Since $\mathrm{pr} \circ P = \mathrm{pr} \circ P'$, $P^{-1}(Y - \mathcal{S}) = P'^{-1}(Y - \mathcal{S}) = U_0$. Now, the restrictions of P and P' on U_0 take their values in the subset of Y made of principal orbits of $\mathbf{R}$, for which the stabilizer of the action of $\mathbf{R}$ is $\{0\}$. Therefore, for each $r \in \mathrm{U}_0$ there is a unique $\tau(r) \in \mathbf{R}$ such that, for all n, $\mathrm{Z}'_n(r) = e^{2i\pi\alpha_n\tau(r)}\mathrm{Z}_n(r)$. Now, $\omega = d\varepsilon$, and

$$\begin{aligned}
\varepsilon(\mathrm{P}')_r(\delta r) &= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \bar{\mathrm{Z}}'_n(r)\frac{\partial \mathrm{Z}'_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \bar{\mathrm{Z}}_n(r)\frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r) \\
&+ \Big(\sum_{n\in\mathbf{Z}} \alpha_n \bar{\mathrm{Z}}_n(r)\mathrm{Z}_n(r)\Big)\frac{\partial \tau(r)}{\partial r}(\delta r) \\
&= \varepsilon(\mathrm{P})_r(\delta r) + c\,\tau^*(dt).
\end{aligned}$$

Therefore, $[\omega(\mathrm{P}') - \omega(\mathrm{P})] \upharpoonright \mathrm{U}_0 = 0$. Thus, by continuity, $[\omega(\mathrm{P}') - \omega(\mathrm{P})] \upharpoonright \bar{\mathrm{U}}_0 = 0$, where $\bar{\mathrm{U}}_0$ is the closure of U_0. It remains to check what happens on the complementary $\mathrm{V} = \mathrm{U} - \bar{\mathrm{U}}_0$. The subset V is open, thus $\mathrm{P} \upharpoonright \mathrm{V}$ and $\mathrm{P}' \upharpoonright \mathrm{V}$ are two plots of Y but with values in the subset of singular orbits $\mathcal{S}$. Since $\mathcal{S}$ has dimension 1 and ω is a 2-form, $\omega(\mathrm{P} \upharpoonright \mathrm{V}) = \omega(\mathrm{P}' \upharpoonright \mathrm{V}) = 0$. In conclusion $\omega(\mathrm{P}') = \omega(\mathrm{P})$ everywhere on U. That proves that there exists a 2-form ϖ on $\mathrm{X} = \mathrm{Y}/\mathbf{R}$ such that $\mathrm{pr}^*(\varpi) = \omega$. □

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E-mail address: piz@math.huji.ac.il
URL: http://math.huji.ac.il/~piz/