ON RIEMANNIAN METRIC IN DIFFEOLOGY

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Ex-ORMID.pdf

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We propose a formal framework for Riemannian diffeology, with a definition for Riemannian metric that coincides with the standard definition for manifolds.

Pointed or pointwise diffeology

The notion of pointed or pointwise diffeology has been used at a few places already: for the dimension in diffeology [PIZ13, §2.19/20], or for the construction of differential p-forms bundles and p-vectors bundles [PIZ13, §6.45]. We can detach the definition of pointwise diffeological objects from an ambient diffeology, and make it depending only of a pointed diffeology. Let us remind first that:

Definition (pointed parametrization). *A* pointed parametrization *in a set* X *at a point* x *is any parametrization* $\mathrm{P} : \mathrm{U} \to \mathrm{X}$ *such that:* $0 \in \mathrm{U}$ *and* $\mathrm{P}(0) = x$*. We denote by* $\mathrm{Param}_x(\mathrm{X})$ *the set of all parametrizations pointed at* x*.*

Definition (pointed diffeology). *Let* X *a set and* $x \in \mathrm{X}$*, we call a* pointed diffeology *at* x *any set* $\mathcal{D}_x \subset \mathrm{Param}_x(\mathrm{X})$ *that satisfies the two axioms:*

1. *The constant parametrization* $0 \mapsto x$ *belongs to* $\mathcal{D}_x$*.*
2. *For all parametrization* $\mathrm{P} : \mathrm{U} \to \mathrm{X}$ *belonging to* $\mathcal{D}_x$ *and for all smooth parametrization* $\mathrm{F} : \mathrm{V} \to \mathrm{U}$*, pointed at* 0*,* $\mathrm{P} \circ \mathrm{F}$ *belongs to* $\mathcal{D}_x$*.*

Note that a pointed diffeology at x is always the germ at the point x of the diffeology it generates [PIZ13, §1.6]. And, for any

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diffeology $\mathcal{D}$, the subset $\mathcal{D}_x \subset \mathcal{D}$ of plots pointed at x is a pointed diffeology.

Definition (pointed path). *A pointed path at $x \in X$ is a pointed smooth parametrization at x, defined on $\mathbf{R}$ or some interval $]a, b[$.*

Definition (A pointed p-form). A pointed p-form *at x is a map α_x that associates to every pointed plot P at x a linear p-form $\alpha_x(P) \in \Lambda^p(\mathbf{R}^n)$ at $0 \in \mathrm{dom}(P) \subset \mathbf{R}^n$, such that $\alpha_x(F \circ P) = F^*(\alpha_x(P))_0$, that is:*

$$\alpha_x(F \circ P)(v_1, \ldots, v_p) = \alpha_x(P)(Mv_1, \ldots, Mv_p), \quad \text{with } M = D(F)(0),$$

for all smooth parametrisation F in $\mathrm{dom}(P)$ pointed at 0. We shall denote the space of pointed p-forms at x by $\lambda^p_x(X)$.

Note that, the value at x of a (global) p-form is a pointed p-form, but maybe not all pointed p-forms are values of a (global) p-form. We have denoted by $\Lambda^p_x(X)$ the space of values of p-forms at x [PIZ13, §6.45]. Actually, according to our notations:

$$\Lambda^p_x(X) \subset \lambda^p_x(X).$$

Smooth covariant tensor

We recall [PIZ13, §6.20 Note, 6.21] that a *smooth covariant tensor* on a diffeological space is a map ε that associates to every plot P in X a smooth covariant tensor $\varepsilon(P)$ on $U = \mathrm{dom}(P)$, such that

$$\varepsilon(P \circ F) = F^*(\varepsilon(P))$$

for all smooth parametrization $F : V \to U$. The tensor ε is symetric if $\varepsilon(P)$is symetric for all P. In the following we deal with symetric 2-tensor and we denotes their space by

$$\Sigma^2(X),$$

and the space of symetric 2-tensor on the Euclidean subset U by $\Sigma^2(U)$ with the identification $\varepsilon \sim \varepsilon(\mathbf{1}_U)$.

About the notations, if ε is a smooth covariant k-tensor on a domain $U \subset \mathbf{R}^n$, we denote by $\varepsilon_r(v_1)\cdots(v_k)$ the evaluation of ε at the point $r \in U$, applied to the k-uple of vectors $v_1, \cdots, v_k \in \mathbf{R}^n$. For $n = 1$, a vector is just a number and 1 is the canonical basis vector.

Riemannian metric on a diffeological space

Definition. (Riemannian metric) *We shall define, for now, a Riemannian metric on a diffeological space* X *a covariant* 2*-tensor that satifies the following conditions:*

- *(Symetric) The tensor g is symetric:*
$$g \in \Sigma^2(\mathrm{X}).$$
- *(Positive) For all path* $\gamma \in \mathrm{Paths}(\mathrm{X})$*,* $g(\gamma) \geq 0$*, that is*
$$g(\gamma)_t(1)(1) \geq 0 \quad \textit{for all } t \in \mathbf{R}.$$
Actually we can restrict the case to paths defined on $\mathbf{R}$ *or on intervals* $]a, b[$*, in that case* $g(\gamma)_t(1)(1) \geq 0$ *for all* $t \in \mathrm{dom}(\gamma)$ *obviously.*
- *(Definite) The tensor g is positive definite:*
$$g(\gamma)_t(1)(1) = 0 \quad \Rightarrow \quad \forall \alpha \in \Omega^1(\mathrm{X}),\ \alpha(\gamma)_t(1) = 0.$$

The last condition can be weakened by considering pointed differential forms, as defined above. Considering the space $\lambda^k_x(\mathrm{X})$ of pointed k-forms at x by, the positivity condition becomes:

- *(Definite') The tensor g is positive definite if for all point* $x \in \mathrm{X}$*, for all path* γ *pointed at* x *:*
$$g(\gamma)_0(1)(1) = 0 \quad \Rightarrow \quad \forall \alpha_x \in \lambda^1_x(\mathrm{X}),\ \alpha_x(\gamma)_0(1) = 0.$$

It is not clear what definition is the best, for many examples built with manifolds and spaces of smooth maps they do coincide. But they may differ in general and, depending on the problem, one must choose one or the other.

Definition. (Length and energy of a path) *Let g be a Riemannian metric on a diffeological space* X*. For all path* γ *in* X*, we define its* length *and its* energy *by:*

$$\mathrm{length}(\gamma) = \int_0^1 \sqrt{g(\gamma)_t(1)(1)}\, dt, \ \textit{and}\ \ \mathrm{E}(\gamma) = \frac{1}{2}\int_0^1 g(\gamma)_t(1)(1)\, dt\,.$$

Exercises

✎ Exercise (1). For all $x \in \mathrm{X}$ we say that a path γ is centered at x if $\gamma(0) = x$. Let g be a symmetric 2-tensor on X. Show that:

- g is positive if for all $x \in \mathrm{X}$, for all path γ centered at x, $g(\gamma)_0(1)(1) \geq 0$.

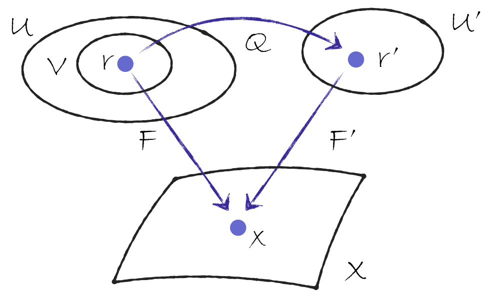

Figure 1. The transition function.

- g is definite if for all $x \in X$, for all path γ centered at x, $g(\gamma)_0(1)(1) = 0$ implies that for all 1-form α on X, pointed or not, $\alpha(\gamma)_0(1) = 0$.

Solution — It is a part of the definition that if $g(\gamma)_t(1)(1) \geq 0$ for all path γ in X and all $t \in \mathrm{dom}(\gamma)$, then, for all $x \in X$, for all path γ centered in x, $g(\gamma)_0(1)(1) \geq 0$. Conversely, assume that for all $x \in X$, for all path γ centered in x, $g(\gamma)_0(1)(1) \geq 0$. Let γ' be a path in X and $t \in \mathrm{dom}(\gamma')$, let $x = \gamma'(t)$. Let $T_t(t') = t' + t$ be the translation by t in $\mathbf{R}$ and $\gamma = \gamma' \circ T_t$ Then,

$$\begin{aligned} g(\gamma)_0(1)(1) &= g(\gamma' \circ T_t)_0(1)(1) \\ &= T_t^*(g(\gamma'))_0(1)(1) \\ &= g(\gamma')_t(1)(1) \qquad \text{because } D(T_t)_0(1) = 1. \end{aligned}$$

Thus, for all $t \in \mathrm{dom}(\gamma')$, $g(\gamma')_t(1)(1) \geq 0$.

The same use of translation by t proves the second proposition. □

✎ Exercise (2). Show that for $X = M$ be a manifold, this definition coincide with the standard definition. We choose the definition of positive definite metric with pointed forms. Say why on manifold the two conditions coincide.

Solution — Let $\mathcal{A}$ be an atlas of M and let $\dim(M) = n$. By definition, for all chart $F \in \mathcal{A}$, $g(F)$ is a symetric 2-tensor on dom(F), since a chart is a particular plot. Let $x \in M$ and two charts $F, F' \in \mathcal{A}$ such that $F(r) = F(r') = x$. Since $\mathcal{A}$ is a generating family, there exist an open neighborhood $V \subset \mathrm{dom}(F)$ of r and a plot $Q : V \to \mathrm{dom}(F')$ such that $F' \circ Q = F \upharpoonright V$,

$Q(r) = r'$, and we can choose V such that $Q(V) \subset \mathrm{dom}(F')$. Thus, $g(F \restriction V) = g(F \circ Q) = Q^*(g(F))$, but $Q = F'^{-1} \circ F \restriction V$ is the transition diffeomorphism, hence: $g(F) = (F'^{-1} \circ F)^*(g(F'))$ which is the definition of a 2-tensor on a manifold.

Now, let F be a chart of M. As we said $g(F)$ is a symmetric 2-tensor on $\mathrm{dom}(F) \subset \mathbf{R}^n$. Let $g(F)_r$ be its value in r, and $x = F(r)$. Let $v \in \mathbf{R}^n$ and $\gamma_v : t \mapsto r + tv$, γ_v is a smooth path in dom(F), defined on some interval in $\mathbf{R}$ and centered at r. Let $\gamma^v = F \circ \gamma$, then $\gamma^v(0) = F(r) = x$. Then:

$$\begin{aligned} g(\gamma^v)_0(1)(1) &= g(F \circ \gamma_v)_0(1)(1) \\ &= \gamma_v^*(g(F))_0(1)(1) \\ &= g(F)_{\gamma_v(0)}(\dot\gamma_v(0))(\dot\gamma_v(0)), \text{with } \dot\gamma_v(t) = \frac{d\gamma_v(t)}{dt} \\ &= g(F)_r(v)(v) \end{aligned}$$

Since $g(\gamma)_0(1)(1) \geq 0$ for all γ, then, for $\gamma = \gamma^v$, $g(F)_r(v)(v) \geq 0$ for all $r \in \mathrm{dom}(F)$ and $v \in \mathbf{R}^n$. Thus, $g(F)$ is a non-negative symmetric 2-tensor.

Now, let F be a chart and let us check that $g(F)$ is positive definite. Let $r \in \mathrm{dom}(F)$ and $v \in \mathbf{R}^n$. Let $x = F(r)$. Assume that $g(F)_r(v, v) = 0$. Let $\gamma_v(t) = r + tv$ and $\gamma^v = F \circ \gamma_v$, then:

$$\begin{aligned} g(F)_r(v)(v) &= g(F)_{\gamma_v(0)}(\dot\gamma_v(0))(\dot\gamma_v(0)) \\ &= \gamma_v^*(g(F))_0(1)(1) \\ &= g(F \circ \gamma_v)_0(1)(1) \\ &= g(\gamma^v)_0(1)(1) \end{aligned}$$

So, if $g(F)_r(v, v) = 0$ then $g(\gamma^v)_0(1)(1) = 0$, which implies, by hypothesis, that for all 1-form α_x pointed at x, $\alpha_x(\gamma^v) = 0$. Consider now the coordinate 1-forms $e_i^* : v \mapsto v^i$, for all $v = \sum_i v^i e_i$, where $(e_i)_{i=1}^n$ is the canonical basis of $\mathbf{R}^n$. Push the form e_i^* onto M by the chart F: Let ε_i^x defined as follow: for all plot $P : U \to X$ pointed at x, there exists a smooth parametrization Q pointed at r, with $F(r) = x$, defined on neighborhood V of $0 \in U$ such that $P \restriction V = F \circ Q$. Then, let $\varepsilon_i^x(P) = Q^*(e_i^*)$, this is a 1-form centered at x. Indeed, for $P' = P \circ F$, $Q' = Q \circ F$ and $\varepsilon_i^x(P \circ F) = (Q \circ F)^*(e_I^*) = F^*(Q^*(e_i^*)) = F^*(\varepsilon_i^x(P))$. Now, since g is assumed to be positive definite: $\varepsilon_i^x(\gamma^v) = 0$, but $\gamma^v = F \circ \gamma_v$, thus $\varepsilon_i^x(\gamma^v) = \gamma_v^*(e_i^*) = (e_i^*)_r(\dot\gamma_v(0)) = e_i^*(v) = v^i$. Hence, for all

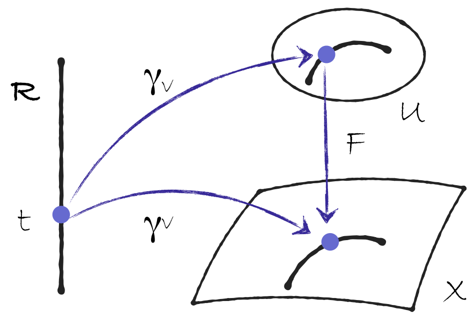

Figure 2. A path in a chart.

i, $v_i = 0$ and then $v = 0$. The 2-tensor $g(\mathrm{F})$ defined on $\mathrm{dom}(\mathrm{F})$ is a positive definite metric. ☐

✎ Exercise (3). For two vectors $v, v' \in \mathbf{R}^3$, denote by $\langle v, v' \rangle$ their ordinary scalar product. Let $\gamma \in \mathrm{Paths}(\mathbf{R}^3)$, call a *variation* of γ a path $t \mapsto (x_t, v_t)$ such that $x_t = \gamma(t)$ and $v_t \in \mathbf{R}^3$. For two variation $\nu = [t \mapsto (x_t, v_t)]$ and $\nu' = [t \mapsto (x_t, v'_t)]$ of γ, define the product

$$\langle \nu, \nu' \rangle = \int_0^1 \langle v_t, v'_t \rangle \, dt$$

Q1: Considers this product to define a formal Riemannian metric on $\mathrm{Paths}(\mathbf{R}^3)$.

Q2: Explicit the energy of a path $[s \mapsto \gamma_s]$ in $\mathrm{Paths}(\mathbf{R}^3)$.

Solution — Let $\mathrm{P} : \mathrm{U} \to \mathrm{Paths}(\mathrm{X})$ be a n-plot. Let us define $g(\mathrm{P})$ a 2-tensor on U by: for all $r \in \mathrm{U}$ and $v, v' \in \mathbf{R}^3$,

$$g(\mathrm{P})_r(v)(v') = \int_0^1 \left\langle \frac{\partial \gamma_r(t)}{\partial r}(v), \frac{\partial \gamma_r(t)}{\partial r}(v') \right\rangle \, dt\,.$$

Let us prove that g is a Riemannian metric on $\mathrm{Paths}(\mathrm{X})$. Consider $g(\mathrm{P} \circ \mathrm{F})$, with F a smooth paramerization in U. Let us denote $\mathrm{F} : s \mapsto r$, $\mathrm{P} : r \mapsto \gamma$ and then $\mathrm{P} \circ \mathrm{F} : s \mapsto r \mapsto \gamma$. We have:

$$\begin{aligned} g(\mathrm{P} \circ \mathrm{F})_s(w)(w') &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial s}(w), \frac{\partial \gamma(t)}{\partial s}(w') \right\rangle \, dt \\ &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial r} \frac{\partial r}{\partial s}(w), \frac{\partial \gamma(t)}{\partial r} \frac{\partial r}{\partial s}(w') \right\rangle \, dt \end{aligned}$$

We remind that for a smooth parametrization $f : x \mapsto y$, where x and y are two real variables, we use indiferently the notations

$$\mathrm{D}(f) \quad \text{or} \quad \mathrm{D}(x \mapsto y) \quad \text{or} \quad \frac{\partial y}{\partial x}.$$

Then for $g \circ f : x \mapsto y \mapsto z$, the chain-rule writes:

$$\mathrm{D}(x \mapsto z) = \mathrm{D}(x \mapsto y \mapsto z) = \mathrm{D}(y \mapsto z) \circ \mathrm{D}(x \mapsto x),$$

or:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \circ \frac{\partial y}{\partial x}.$$

Therefore, by denoting

$$v = \frac{\partial r}{\partial s}(w) \quad \text{and} \quad v' = \frac{\partial r}{\partial s}(w')$$

we get:

$$\begin{aligned} g(\mathrm{P} \circ \mathrm{F})_s(w)(w') &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial r} \frac{\partial r}{\partial s}(w), \frac{\partial \gamma(t)}{\partial r} \frac{\partial r}{\partial s}(w') \right\rangle \, dt \\ &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial r}(v), \frac{\partial \gamma(t)}{\partial r}(v') \right\rangle \, dt \\ &= g(\mathrm{P})_{r=\mathrm{Q}(s)} \left(\frac{\partial \mathrm{Q}(s)}{\partial s}(w) \right) \left(\frac{\partial \mathrm{Q}(s)}{\partial s}(w') \right) \\ &= \mathrm{Q}^*(g(\mathrm{P}))_s(w)(w') \end{aligned}$$

Hence, g is a covariant 2-tensor on Paths($\mathbf{R}^3$). It is symmetric because the scalar product is symetric. Now, let $s \mapsto \gamma_s$ be a path in Paths($\mathbf{R}^3$),

$$\begin{aligned} g(s \mapsto \gamma_s)_s(1)(1) &= \int_0^1 \left\langle \frac{\partial \gamma_s(t)}{\partial s}, \frac{\partial \gamma_s(t)}{\partial s} \right\rangle \, dt \\ &= \int_0^1 \left\| \frac{\partial \gamma_s(t)}{\partial s} \right\|^2 \, dt \end{aligned}$$

Obviously $g(s \mapsto \gamma_s)_s(1)(1)$ is positive. Now

$$g(s \mapsto \gamma_s)_s(1)(1) = 0 \quad \Rightarrow \quad \left\| \frac{\partial \gamma_s(t)}{\partial s} \right\|^2 = 0.$$

then

$$\frac{\partial \gamma_s(t)}{\partial s} = 0 \quad \Rightarrow \quad \gamma_s = \gamma.$$

The path $s \mapsto \gamma_s$ is constant. Thus, for all 1-form α on $\mathrm{Paths}(\mathbf{R}^3)$, $\alpha(s \mapsto \gamma) = 0$, forms vanishe on constant plots [PIZ13, Ex. 96]. Therefore, the tensor g defined on $\mathrm{Paths}(\mathbf{R}^3)$ is positive and definite, it is a diffeological Riemannian metric according to the definition above. Hence,

$$\mathrm{E}(s \mapsto \gamma_s) = \frac{1}{2}\int_0^1 ds \int_0^1 dt \left\| \frac{\partial \gamma_s(t)}{\partial s} \right\|^2$$

is the energy of the path $s \mapsto \gamma_s$ in $\mathrm{Paths}(\mathbf{R}^3)$. □

References

[PIZ13] Patrick Iglesias-Zemmour, *Diffeology*. Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence RI, (2013).

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