NON-SYMPLECTIC MANIFOLD WITH AN INJECTIVE UNIVERSAL MOMENT MAP

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Ex-NSIUMM.pdf

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This addendum is as an exercise, with a detailed solution, made with the Note 2 of the article 9.23, pp. 323-324 of the TextBook[1]. This example shows how the hypothesis of transitivity of the group of automorphisms is necessary in the statement 9.23.

✎ Exercise. **(Injective universal moment map for a non symplectic form)** Let us consider the real plane $\mathbf{R}^2$ equipped with the 2-form

$$\omega = (x^2 + y^2)\, dx \wedge dy.$$

Q1. Why is ω closed?

Q2. Describe the kernel of ω. We admit that the group of compact supported automorphisms of a symplectic manifold is transitive, deduce that $\mathrm{Diff}(\mathbf{R}^2, \omega)$, the group of automorphisms of ω, has 2 orbits: $\{0_{\mathbf{R}^2}\}$ and $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$.

Q3. Tell why every automorphism of $(\mathbf{R}^2, \omega)$ is Hamiltonian.

Q4. Exhibit the unique equivariant universal moment map μ_ω for $(\mathbf{R}^2, \omega)$ such that $\mu_\omega(0_{\mathbf{R}^2}) = 0_{\mathcal{G}^*}$. Why is it unique?

Q5. Show that if $z = (x, y) \neq (0, 0)$, then $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$. Conclude that μ_ω is injective.

☞ Solution: Q1. ω is closed because it is a 2-form on a 2-dimensional space, see [TextBook, 6.39].

Q2. Let $z = (x, y) \in \mathbf{R}^2$, by definition

$$\ker(\omega_z) = \{u \in \mathbf{R}^2 \mid \omega_z(u)(v) = 0 \text{ for all } v \in \mathbf{R}^2\}.$$

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[1]The word TextBook always refers to the book *Diffeology* [TextBook].

But, $\omega_z(u)(v) = (x^2 + y^2)\det(u\,v)$. Thus, since $\det(\cdot\,\cdot)$ is nondegenerate, if $z \neq (0,0)$ then $\ker(\omega_z) = \{0_{\mathbf{R}^2}\}$, else $\ker(\omega_0) = \mathbf{R}^2$. It follows that, since an automorphism of ω induces an isomorphism on the kernel, the origin $0_{\mathbf{R}^2}$ must be mapped onto itself by any automorphism of ω. Indeed, $0_{\mathbf{R}^2}$ is the only point with kernel $\mathbf{R}^2$. Then, for all $\varphi \in \mathrm{Diff}(\mathbf{R}^2, \omega)$, $\mathbf{R}^2 - \{0\}$ is invariant by φ, thus $\varphi \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is a symplectomorphism of $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$. Now, $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is open and $\omega \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is symplectic, according to the assumption: for every two points z and z' in $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ there exists a compact supported automorphism φ of $(\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}, \omega \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\})$ mapping z to z'. But since a compact in $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is a closed and bounded subset, the complement of the support contains an open neighborhood of $0_{\mathbf{R}^2}$ on which φ is the identity, thus φ can be smoothly extended by $\varphi(0_{\mathbf{R}^2}) = 0_{\mathbf{R}^2}$. This extension, still mapping z to z', satisfies obiously $\varphi^*(\omega) = \omega$ on $\mathbf{R}^2$ and then belongs to $\mathrm{Diff}(\mathbf{R}^2, \omega)$. Therefore, $\mathrm{Diff}(\mathbf{R}^2, \omega)$ is transitive on $\mathbf{R}^2 - \{0\}$, that is, the group $\mathrm{Diff}(\mathbf{R}^2, \omega)$ has two orbits in $\mathbf{R}^2$: $\{0_{\mathbf{R}^2}\}$ and $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$.

Q3. Because $\mathbf{R}^2$ is contractible (precisely, has a vanishing first homology), the automorphisms $\mathrm{Diff}(\mathbf{R}^2, \omega)$ are Hamiltonian, see [TextBook, 9.7, 9.15].

Q4. Since the action of $\mathrm{Diff}(\mathbf{R}^2, \omega)$ has a fixed point, $0_{\mathbf{R}^2}$, the moment map is exact and there exists an invariant primitive μ_ω, see [TextBook, 9.10, Note 2]. Actually, applying the expressions ($\diamondsuit$) and ($\heartsuit$) of [TextBook, 9.20], to a path p, connecting $0_{\mathbf{R}^2}$ to $z = (x, y)$, an equivariant primitive μ_ω [TextBook, 9.9] of the 2-points moment map [TextBook, 9.8] is given, for every plot F of $\mathrm{Diff}(\mathbf{R}^2, \omega)$, by

$$\mu_\omega(z)(\mathrm{F})_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot{p}(t))(\delta p(t))\,dt,$$

with

$$\delta p(t) = [\mathrm{D}(\mathrm{F}(r))(p(t))]^{-1} \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r).$$

This moment is unique because two moment maps differ only by a constant in $\mathcal{G}^*$ ($\mathbf{R}^2$ is connected), see [TextBook, 9.9], and the constant is fixed by $\mu_\omega(0_{\mathbf{R}^2}) = 0_{\mathcal{G}^*}$.

Q5. The proof that the moment map μ_ω restricted to $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is injective is contained in the proof of [TextBook, 9.23, B] by chosing a real smooth function f on $\mathbf{R}^2$ vanishing outside a small ball centered at $z' \neq z$ not containing z nor $0_{\mathbf{R}^2}$. Now we have just to show that for all $z \in \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$, $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$. For that we will apply the previous formula to

$$p(t) = tz \quad \text{and} \quad \mathrm{F}(r) = \begin{pmatrix} \cos(2\pi r) & -\sin(2\pi r) \\ \sin(2\pi r) & cos(2\pi r) \end{pmatrix}$$

with

$$t \in \mathbf{R}, \quad z = (x, y) \in \mathbf{R}^2, \quad r \in \mathbf{R} \quad \text{and} \quad \delta r = 1.$$

By linearity, $\mathrm{D}(\mathrm{F}(r))(p(t)) = \mathrm{F}(r)$, and then

$$\begin{aligned}
\delta p(t) &= \mathrm{F}(r)^{-1} \frac{\partial \mathrm{F}(r)}{\partial r}(p(t)) \\
&= \begin{pmatrix} \cos(2\pi r) & \sin(2\pi r) \\ -\sin(2\pi r) & cos(2\pi r) \end{pmatrix} \times 2\pi \times \begin{pmatrix} -\sin(2\pi r) & -\cos(2\pi r) \\ \cos(2\pi r) & -\sin(2\pi r) \end{pmatrix} \begin{pmatrix} tx \\ ty \end{pmatrix} \\
&= 2\pi \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \begin{pmatrix} tx \\ ty \end{pmatrix} \\
&= 2\pi t \begin{pmatrix} -y \\ x \end{pmatrix}.
\end{aligned}$$

On the other hand, $\dot{p}(t) = \frac{d(tz)}{dt} = z$. Hence,

$$\dot{p}(t) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad \delta p(t) = 2\pi t \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\mu_\omega(z)(\mathrm{F})_r(\delta r) &= \int_0^1 \omega_{p(t)}(\dot{p}(t))(\delta p(t))\, dt, \\
&= \int_0^1 \omega_{tz} \begin{pmatrix} x \\ y \end{pmatrix} \left(2\pi t \begin{pmatrix} -y \\ x \end{pmatrix} \right) dt, \\
&= \int_0^1 ((tx)^2 + (ty)^2) \times 2\pi t \times \det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} dt \\
&= \int_0^1 t^2(x^2 + y^2) \times 2\pi t \times (x^2 + y^2) dt \\
&= 2\pi(x^2 + y^2)^2 \int_0^1 t^3 dt \\
&= \frac{2\pi}{4}(x^2 + y^2)^2.
\end{aligned}$$

Hence, if $z = (x, y) \neq (0, 0)$ the value of the moment map above, computed on the 1-path F, is not zero, which implies $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$.

In conclusion, $\mu_\omega(0_{\mathbf{R}^2}) = 0_{\mathcal{G}^*}$, $\mu_\omega \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is injective and if $z \neq 0_{\mathbf{R}^2}$ then $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$, therefore μ_ω is injective. ☐

So, what is this exercise all about? The paragraph 9.23 states that a closed 2-form ω on a manifold is symplectic, that is, non-degenerate, if and only if its group of automorphisms is transitive and the universal moment map is injective. This exercise shows that the injectivity of the universal moment map is not sufficient (and the condition of transitivity is necessary), since it exhibits a non symplectic closed 2-form on $\mathbf{R}^2$ with an injective universal moment map.

References

[TextBook] Patrick Iglesias-Zemmour *Diffeology,* Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence, 2012. http://www.ams.org/bookstore-getitem/item=SURV-185

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