FUNCTIONAL DIFFEOLOGY ON FOURIER COEFFICIENTS

PATRICK IGLESIAS-ZEMMOUR

ref. http://math.huji.ac.il/~piz/documents/DBlog-Ex-FDOFC.pdf

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In this exercise we transfer the functional diffeology defined on smooth complex periodic functions to the space of Fourier coefficients of smooth complex periodic functions.

We denote by $\mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C})$ the space of 1-periodic complex valued real functions,

$$\mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C}) = \{f \in \mathcal{C}^\infty(\mathbf{R},\mathbf{C}) \mid f(x+1) = f(x)\}.$$

This space is then equipped with the functional diffeology. Let $f \in \mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C})$, and $(f_n)_{n\in\mathbf{Z}}$ be its sequence of Fourier coefficients

$$f_n = \int_0^1 f(x) e^{-2i\pi nx}\,dx,\ \ n \in \mathbf{Z}.$$

We know that the Fourier series converges to f, uniformly on $[0,1]$ (see for example [Vil68, §2 Thm. 1]). We note

$$f(x) = \lim_{\mathrm{N}\to\infty} \sum_{n=-\mathrm{N}}^{+\mathrm{N}} f_n\, e^{2i\pi nx} \quad \text{or} \quad f(x) = \sum_{n\in\mathbf{Z}} f_n\, e^{2i\pi nx}.$$

The set of Fourier coefficients of the smooth periodic real functions with values in $\mathbf{C}$ is a vector subspace $\mathcal{E}$ of $\mathrm{Maps}(\mathbf{Z},\mathbf{C})$. Let

$$j : \mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C}) \to \mathrm{Maps}(\mathbf{Z},\mathbf{C}) \quad \text{with} \quad j(f) = (f_n)_{n\in\mathbf{Z}}.$$

The map j is injective. We denote by $\mathcal{E}$ the subspace of $\mathrm{Maps}(\mathbf{Z},\mathbf{C})$ made of the Fourier coefficients of the smooth periodic functions, that is,

$$\mathcal{E} = \Big\{(f_n)_{n\in\mathbf{Z}} = j(f) \mid f \in \mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C})\Big\}.$$

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We know that the subspace $\mathcal{E}$ is made exactly of all rapidly decreasing sequences of complex numbers (op. cit.),

$$\text{for all } p \in \mathbf{N}, \quad n^p f_n \xrightarrow[|n|\to\infty]{} 0.$$

✎ **Exercise**. We consider the vector space $\mathcal{E}$ of rapidly decreasing sequences of complex numbers defined above.

Q1. Show that the parametrizations $\mathrm{P} : r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ in $\mathcal{E}$ satisfying the following conditions define a diffeology.

(1) The functions $f_n : \mathrm{dom}(\mathrm{P}) \to \mathbf{C}$ are smooth.
(2) For all ball $\bar{\mathcal{B}} \subset \mathrm{dom}(\mathrm{P})$, for all $k, p \in \mathbf{N}$, there exists a positive number $\mathrm{M}_{k,p}$ such that for all nonzero integer n

$$\left| \frac{\partial^k f_n(r)}{\partial r^k} \right| \leq \frac{\mathrm{M}_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}. \tag{$\clubsuit$}$$

Q2. Show that the diffeology defined by ($\clubsuit$) is the pushforward on $\mathcal{E}$, by j, of the functional diffeology on $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$

Note 1. In other words, the parametrization $r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ is a plot of the pushed forward functional diffeology if the functions f_n are smooth and their derivatives are uniformly rapidly decreasing, what we denote (rather imprecisely) by

$$n^p \frac{\partial^k f_n(r)}{\partial r^k} \xrightarrow[|n|\to\infty]{} 0.$$

Note 2. Thanks to paracompacity, it is enough that, for every point $r_0 \in \mathrm{dom}(\mathrm{P})$, there exists a ball $\mathcal{B}'$ centered at r_0 such that ($\clubsuit$) holds to ensure that ($\clubsuit$) holds on every ball $\bar{\mathcal{B}} \subset \mathrm{dom}(\mathrm{P})$.

Solution: We verify, first of all, that the condition ($\clubsuit$) defines a diffeology on the space $\mathcal{E}$ of rapidly decreasing sequences of complex numbers.

A1 (Covering axiom). If $f_n(r) = f_n$ is constant in r, for all n, then the condition ($\clubsuit$) is trivially satisfied for $k > 0$, and for $k = 0$ it means that the series is rapidly decreasing, what it is.

A2 (Locality axiom). According to Note 2, the condition ($\clubsuit$) is local in r.

A3 (Smooth compatibility axiom). Let $\mathrm{P} : (r \mapsto f_n(r))_{n\in\mathbf{Z}}$ satisfying ($\clubsuit$) and $\mathrm{F} : s \mapsto r$ be a smooth parametrization in the domain of P. We have, for all $k > 0$,

$$\frac{\partial^k f_n(s)}{\partial s^k} = \sum_{\ell=1}^{k} \frac{\partial^\ell f_n(r)}{\partial r^\ell} \cdot \mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right),$$

where the $\mathrm{Q}_{k,\ell}$ are polynomials. Now, since the function $s \mapsto r$ is smooth, the operators $\mathrm{Q}_{k,\ell}$ are bounded on every ball. Let then $s_0 \in \mathrm{dom}(\mathrm{F})$, $r_0 = \mathrm{F}(s_0)$ and $\mathcal{B}$ be a ball centered at r_0 such that the condition (♣) is satisfied. Let $\mathcal{B}' \subset \mathrm{F}^{-1}(\mathcal{B})$ be a ball centered at s_0, we have for all $s \in \mathcal{B}'$,

$$\begin{aligned}\left|\frac{\partial^k f_n(s)}{\partial s^k}\right| &\leq \sum_{\ell=1}^{k} \left|\frac{\partial^\ell f_n(r)}{\partial r^\ell}\right| \left|\mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right| \\ &\leq \sum_{\ell=1}^{k} \frac{\mathrm{M}_{\ell,p}}{|n|^p} \sup_{s\in\mathcal{B}'} \left|\mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right|,\end{aligned}$$

where the $\mathrm{M}_{\ell,p}$ are the constants of the inequality (♣) for the ball $\mathcal{B}$. Let then

$$m_{k,\ell} = \sup_{s\in\mathcal{B}'} \left|\mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right| \text{ and } \mathrm{M}'_{k,p} = \sum_{\ell=1}^{k} m_{k,\ell}\, \mathrm{M}_{\ell,p},$$

we get, for all $s \in \mathcal{B}'$,

$$\left|\frac{\partial^k f_n(s)}{\partial s^k}\right| \leq \frac{\mathrm{M}'_{k,p}}{|n|^p}.$$

Hence, thanks to Note 2, $\mathrm{P} \circ \mathrm{F}$ still satisfies the condition (♣). Therefore, this condition defines a diffeology on the set $\mathcal{E}$ of rapidly decreasing sequences of complex numbers.

Let us show now that the diffeology defined on $\mathcal{E}$ by (♣) is the pushforward by j of the functional diffeology of $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$.

1.— Let us prove, first of all, that j is smooth, where $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ is equipped with the functional diffeology and $\mathcal{E}$ with the diffeology defined by (♣). Let $\mathrm{P} : r \rightarrow f_r$ be a plot of $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ defined on some real domain U, the composite $j \circ \mathrm{P}$ writes

$$j \circ \mathrm{P} : r \mapsto (f_n(r))_{n\in\mathbf{Z}} \quad \text{with} \quad f_n(r) = \int_0^1 f_r(x) e^{-2i\pi n x}\, dx.$$

Clearly, since $(r, x) \mapsto f_r(x)$ is smooth, the $f_n : r \mapsto f_n(r)$ are smooth, and we have

$$\frac{\partial^k f_n(r)}{\partial r^k} = \int_0^1 \frac{\partial^k f_r(x)}{\partial r^k} e^{-2i\pi nx}\, dx.$$

For all nonzero integer n, after p integrations by parts, we get

$$\frac{\partial^k f_n(r)}{\partial r^k} = \frac{1}{(2i\pi n)^p} \int_0^1 \frac{\partial^p}{\partial x^p} \left(\frac{\partial^k f_r(x)}{\partial r^k} \right) e^{-2i\pi nx}\, dx.$$

Therefore, defining for every ball $\mathcal{B} \subset \operatorname{dom}(\mathrm{P})$ the number

$$\mathrm{M}_{k,p} = \frac{1}{(2\pi)^p} \sup_{r\in\mathcal{B}} \sup_{x\in[0,1]} \left| \frac{\partial^p}{\partial x^p} \frac{\partial^k}{\partial r^k} f_r(x) \right|,$$

we have

$$\left| \frac{\partial^k f_n(r)}{\partial r^k} \right| \leq \frac{\mathrm{M}_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}.$$

Hence, $j \circ \mathrm{P}$ is a plot of $\mathcal{E}$. Therefore j is smooth.

2.— Conversely, remember that j is injective, then let us show that $j^{-1} : \mathcal{E} \to \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ is smooth. Let $\mathrm{P} : r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ be a parametrization in $\mathcal{E}$ satisfying the condition (♣), and let

$$j^{-1} \circ \mathrm{P} : r \mapsto [x \mapsto f_r(x)].$$

The parametrization $r \mapsto f_r$ is given by

$$f_r(x) = \lim_{\mathrm{N}\to\infty} \mathrm{S_N}(r, x) \quad \text{with} \quad \mathrm{S_N}(r, x) = \sum_{n=-\mathrm{N}}^{+\mathrm{N}} f_n(r) e^{2i\pi nx}.$$

The $\mathrm{S_N}$ are smooth for all N, we want to show that the limit $(r, x) \mapsto f_r(x)$ is smooth too. For all $k, p, \mathrm{N} \in \mathbf{N}$, let us introduce the series of partial derivatives

$$\mathrm{S}^{k,p}_{\mathrm{N}}(r, x) = \frac{\partial^p}{\partial x^p} \frac{\partial^k}{\partial r^k} \mathrm{S_N}(r, x) = \sum_{-\mathrm{N}}^{+\mathrm{N}} (2i\pi n)^p \frac{\partial^k f_n(r)}{\partial r^k} e^{2i\pi nx}.$$

The functions $\mathrm{S}^{k,p}_{\mathrm{N}}$ are smooth, with respect to the pair (r, x), for all N, k, p. Let us denote by $|\mathrm{S}|^{k,p}_{\mathrm{N}}(r, x)$ the series of moduli of the terms of $\mathrm{S}^{k,p}_{\mathrm{N}}$

$$|\mathrm{S}|^{k,p}_{\mathrm{N}}(r, x) = \sum_{-\mathrm{N}}^{+\mathrm{N}} |2\pi n|^p \left| \frac{\partial^k f_n(r)}{\partial r^k} \right|.$$

Then, let $\bar{\mathcal{B}} \subset \mathrm{dom}(\mathrm{P})$ be some ball. Thanks to the hypothesis, applying (2) of Q1 to $p+2$, we have, for all $r \in \mathcal{B}$,

$$\sum_{n=-\mathrm{N}}^{+\mathrm{N}} |2\pi n|^p \left|\frac{\partial^k f_n(r)}{\partial r^k}\right| \quad \leq \quad c + 2\sum_{n=1}^{\mathrm{N}} (2\pi)^p \frac{\mathrm{M}_{k,p+2}}{n^2},$$

where c, corresponding to $n=0$, is some constant. Hence, for all $r \in \mathcal{B}$, $x \in [0,1]$ and $\mathrm{N} \in \mathbf{N}$

$$|\mathrm{S}|_{\mathrm{N}}^{k,p}(r,x) \leq \mathrm{K}$$

where $\mathrm{K} = c+2(2\pi)^p\mathrm{M}_{k,p+2}(\pi^2/6)$. Next, since the series $|\mathrm{S}|_{\mathrm{N}}^{k,p}(r,x)$ is increasing and upper bounded, it is convergent, and therefore the series $\mathrm{S}_{\mathrm{N}}^{k,p}(r,x)$ is also convergent. Moreover, according to what preceded: the sequence of smooth functions $\mathrm{S}_{\mathrm{N}}^{k,p}$ converges uniformly on $\mathcal{B} \times [0,1]$ to some function $\mathrm{S}_{\infty}^{k,p}$ when $\mathrm{N} \to \infty$. Now, according to (an obvious improvement of) [Don00, Thm. 3.10] the map $(r,x) \mapsto f_r(x) = \lim_{\mathrm{N}\to\infty} \mathrm{S}_{\mathrm{N}}(r,x)$ is smooth, and the $\mathrm{S}_{\infty}^{k,p}$ are the partial derivatives

$$\mathrm{S}_{\infty}^{k,p}(r,x) = \frac{\partial^p}{\partial x^p}\frac{\partial^k}{\partial r^k} f_r(x).$$

Thus, the parametrization $r \mapsto [x \mapsto f_r(x)]$ is a plot of $\mathcal{C}^{\infty}_{\mathrm{per}}(\mathbf{R},\mathbf{C})$ such that $j(r \mapsto f_r) = (f_n(r))_{n\in\mathbf{Z}}$. Therefore, j is a diffeomorphism from $\mathcal{C}^{\infty}_{\mathrm{per}}(\mathbf{R},\mathbf{C})$, equipped with the functional diffeology, to $\mathcal{E}$, equipped with the diffeology defined by ($\clubsuit$). In other words, the diffeology defined on $\mathcal{E}$ by ($\clubsuit$) is the pushforward of the functional diffeology on $\mathcal{C}^{\infty}_{\mathrm{per}}(\mathbf{R},\mathbf{C})$. $\square$

References

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E-mail address: piz@math.huji.ac.il
URL: http://math.huji.ac.il/~piz/