DIFFEOLOGY AND ARITHMETIC OF IRRATIONAL TORI

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ABSTRACT. We present the classifying group $\mathbf{Fl}(X, \mathbf{R})$ for $(\mathbf{R}, +)$ principal bundles over a diffeological space X. As it is trivial for manifolds, this invariant is specific to diffeology. It reveals intricate structure within the diffeology of X. As an example, we compute $\mathbf{Fl}(T_\alpha, \mathbf{R})$ for the irrational torus. We show how it distinguishes whether α is Diophantine or not, demonstrating a direct link between the arithmetic properties of α and the geometry (specifically, the diffeology) of T_α. This result complements previous works connecting the algebraic nature of α to other diffeological invariants of the irrational torus.

1. Introduction

The geometry,[1] interpreted through diffeology, of the irrational torus $T_\alpha = T^2/\mathcal{S}_\alpha$ encapsulates algebraic and arithmetic properties of the slope α, despite its degenerate topology as the quotient of T^2 by a dense subgroup.[2] For instance, the structure of the group of connected components of the group of diffeomorphisms (a key diffeological invariant) is tied to the algebraic nature of α: it has a richer structure when α is a quadratic number. This sensitivity of diffeology to algebraic properties, initially observed for T_α, extends to quotients of n-dimensional tori by irrational hyperplanes, revealing the refined algebraic relationships captured by their diffeologies. The first part of this paper reviews this connection.

The second part is devoted to another connection, this time linking the quotient diffeology of the irrational torus to arithmetic properties of α, specifically Diophantine approximations. This is revealed by studying $(\mathbf{R}, +)$

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[1]If we understand the word geometry as Felix Klein conceived it, then a diffeological space has indeed a geometry with its group of diffeomorphisms and its invariants. That applies in particular to irrational tori as we shall see.

[2]T^2 is the usual 2-torus $(\mathbf{R}/\mathbf{Z})^2$ and $\mathcal{S}_\alpha$ is the 1-parameter subgroup generated by the translations $(1, \alpha)$, where $\alpha \in \mathbf{R} - \mathbf{Q}$.

principal bundles over T_α.[3] While in the classical category of differentiable manifold, every fiber bundle with a contractible fiber (like $\mathbf{R}$) admits a global smooth section, making all $(\mathbf{R}, +)$ principal bundles trivial, this is no longer the case in diffeology. The projection $\pi : \mathrm{T}^2 \to \mathrm{T}_\alpha$ serves as a key example of a non-trivial $(\mathbf{R}, +)$ principal fibration [PIZ85]. This observation motivates the introduction of a new class of invariants for diffeological spaces, nonexistent in ordinary differential geometry: the classifying group $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ of $(\mathbf{R}, +)$ principal bundles (modulo isomorphisms) over X. Such diffeological invariants are significant because they allow the quotient space X (often arising from dynamical systems or foliations) to be treated as a geometric object carrying its own structure, even when topologically trivial. Furthermore, invariants like $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ provide tools to measure this structure, potentially capturing and quantifying how properties of the underlying dynamics or foliation are reflected in the intrinsic geometry of the quotient.

The computation of $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$ presented in this paper exemplifies this potential. We compute the classifying group and establish the isomorphism $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R}) \simeq \mathbf{R} \times \mathrm{coker}(\Delta_\alpha)$. Here, the cokernel $\mathrm{coker}(\Delta_\alpha)$ relates to the solvability of the cohomological equation for circle rotations by angle α. Crucially, this cokernel vanishes if and only if α is Diophantine, thus directly linking the structure (and dimension) of the diffeological invariant $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$ to the arithmetic nature of α.

Moreover, the deep relationship between diffeological quasifolds and noncommutative geometry [IZL18, IZP21],[4] where K-theory plays an important role, motivates a deeper investigation into the classification of fiber bundles in diffeology, which could potentially lead to the development of a K-theory within diffeology, providing a geometric support to its noncommutative counterpart.

This paper builds upon the initial works in [DIZ83, PIZ85, PIZ86, IZL90] and examines the group structure of $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$, particularly in the context of irrational tori, providing full details and proofs for results announced originally.

2. Irrational Tori and Algebraic Number Fields

Early investigations into the geometry, i.e. diffeology, of the irrational torus $\mathrm{T}_\alpha = \mathrm{T}^2/\mathcal{S}_\alpha$[5] reveal interesting arithmetic properties that connect it to

[3]The notions of fiber bundles in diffeology, and principal fiber bundles, were introduced in [PIZ85], and taken up again in the textbook [PIZ13, §8.10, 8.11, 8.12].

[4]The irrational torus T_α is a quasifold to which the construction of an associated C*-algebra applies, according to noncommutative geometry rules.

[5]$\mathcal{S}_\alpha = \{(x, \alpha x) \mod 1 \mid x \in \mathbf{R}\}$ is the irrational 1-parameter subgroup of irrational slope α. I chose the notation $\mathcal{S}_\alpha$ to evoke a *solenoid*, by analogy with a toroidal solenoid in

algebraic number theory [DIZ83]. For instance, the computation of the group of components of the group of diffeomorphisms of T_α, $\mathrm{Diff}(\mathrm{T}_\alpha)$, reveals that:

$$\pi_0(\mathrm{Diff}(\mathrm{T}_\alpha)) = \begin{cases} \{\pm 1\} \times \mathbf{Z} & \text{if } \alpha \text{ is a quadratic number} \\ \{\pm 1\} & \text{otherwise.} \end{cases}$$

We recall that a quadratic number is a solution of a quadratic equation with integer coefficients. This relation has been extended to the general case of a codimension 1 linear foliation of a torus, that is, the quotient T_H of a torus T^n by an *irrational hyperplane* H [IZL90]. Precisely, let T^n be the quotient of $\mathbf{R}^n$ by the lattice $\mathbf{Z}^n$, and $\mathrm{H} \subset \mathbf{R}^n$ be the hyperplane

$$\mathrm{H} = \ker \boldsymbol{\alpha}, \quad \text{with} \quad \boldsymbol{\alpha} = (\alpha_1 = 1, \alpha_2, \ldots, \alpha_n),$$

where $\boldsymbol{\alpha}$ is a linear 1-form on $\mathbf{R}^n$, and the α_i are real numbers independent over $\mathbf{Q}$:

$$\sum_{i=1}^{n} q_i \alpha_i = 0, \quad \text{with } q_i \in \mathbf{Q} \quad \Rightarrow \quad q_i = 0.$$

We also denote by $\mathrm{H} \subset \mathrm{T}^n$ the image of H by the standard projection $\mathrm{pr} : \mathbf{R}^n \to \mathrm{T}^n$, i.e., $\mathrm{pr}(x) = \left(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}\right)$, and let

$$\mathrm{T}_{\boldsymbol{\alpha}} = \mathrm{T}^n/\mathrm{H}.$$

The irrationality of the hyperplane H (so-called because its director vector $\boldsymbol{\alpha}$ has rationally independent components: $\mathrm{H} \cap \mathbf{Z}^n = \{0\}$.) ensures that its image is dense in T^n, leading to a non-Hausdorff quotient space $\mathrm{T}_{\boldsymbol{\alpha}}$ with a trivial D-topology, i.e., its only open subsets are $\varnothing$ and $\mathrm{T}_{\boldsymbol{\alpha}}$. But its diffeology is not trivial and one of its non-trivial aspect is revealed by the computation of the group of components of its group of diffeomorphisms. Here are the objects involved in its computation,

$$\mathrm{E}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(\mathbf{Q}^n) = \left\{ \sum_{i=1}^{n} q_i \alpha_i \mid q_i \in \mathbf{Q} \right\} \quad \text{and} \quad \mathbf{K}_{\boldsymbol{\alpha}} = \{\lambda \in \mathbf{R} \mid \lambda \mathrm{E} \subset \mathrm{E}\}.$$

The set $\mathrm{E}_{\boldsymbol{\alpha}}$ is a $\mathbf{Q}$-vector subspace of dimension n in $\mathbf{R}$, it captures the "frequencies" of the irrational foliation. The set $\mathbf{K}_{\boldsymbol{\alpha}}$ is the set of stabilizers of $\mathrm{E}_{\boldsymbol{\alpha}}$, by multiplication, representing the scaling symmetries of the frequencies $\mathrm{E}_{\boldsymbol{\alpha}}$.

Theorem (1-IZL). *The set* $\mathbf{K}_{\boldsymbol{\alpha}}$ *is an algebraic extension of* $\mathbf{Q}$ *and* $\mathrm{E}_{\boldsymbol{\alpha}}$ *is a* $\mathbf{K}_{\boldsymbol{\alpha}}$*-vector space. In particular, the dimension* d *of* $\mathbf{K}_{\boldsymbol{\alpha}}$ *divides* n*. The extension* $\mathbf{K}_{\boldsymbol{\alpha}}$ *is called the* characteristic field *of the space* $\mathrm{T}_{\boldsymbol{\alpha}}$.

That is, $\mathbf{K}_{\boldsymbol{\alpha}} = \mathbf{Q}(\lambda)$ for some algebraic number λ of degree d. And we get the following theorem that highlights how diffeology captures arithmetic invariants

electrical engineering, where a wire wound around a toroid traces a similar dense, uniform path.

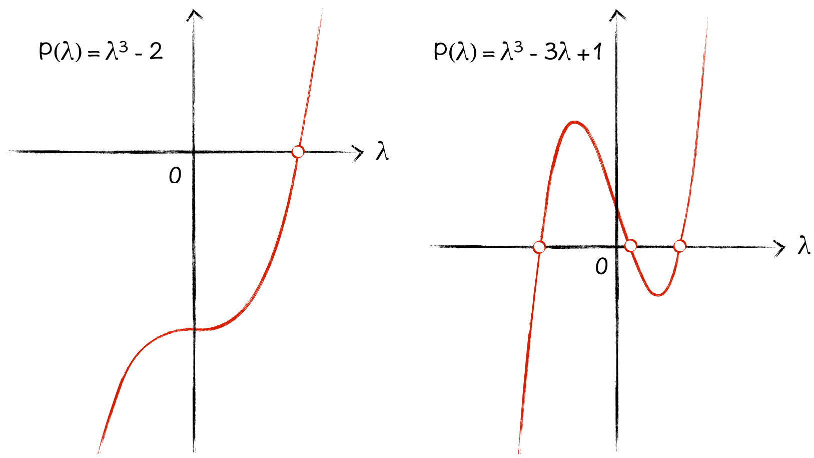


FIGURE 1. The two cases of characteristic field $\mathbf{K}_{\boldsymbol{\alpha}}$ when $n = 3$.

within the smooth structure of singular spaces arising from irrational foliations, Establishing a fundamental algebraic structure underlying the geometry of irrational tori.

Theorem (2-IZL). *The group $\pi_0(\mathrm{Diff}(\mathrm{T}_{\boldsymbol{\alpha}}))$, of components of the group of diffeomorphisms of $\mathrm{T}_{\boldsymbol{\alpha}}$ is isomorphic to the group of units of an order in the characteristic field $\mathbf{K}_{\boldsymbol{\alpha}}$. Thanks to Dirichlet's theorem,[6] it is isomorphic to*

$$\pi_0(\mathrm{Diff}(\mathrm{T}_{\boldsymbol{\alpha}})) \simeq \{\pm 1\} \times \mathbf{Z}^{r+s-1},$$

where r is the number of real roots of the characteristic polynomial $\mathrm{P}(\lambda)$ of the field $\mathbf{K}_{\boldsymbol{\alpha}}$ and $2s$ the number of complex non real solutions.

As an example the case of $\boldsymbol{\alpha} = (1, \alpha)$, the ordinary irrational torus: $n = 2$, d divides n, thus $d = 1$ or $d = 2$. If $d = 1$ then $\mathbf{K}_\alpha = \mathbf{Q}$, $r = 1, s = 0$ and $\pi_0(\mathrm{Diff}(\mathrm{T}_\alpha)) \simeq \{\pm 1\}$, otherwise $d = 2$ then $\mathbf{K}_{\boldsymbol{\alpha}} = \mathbf{Q}(\sqrt{\alpha})$, $r = 2, s = 0$ and $\pi_0(\mathrm{Diff}(\mathrm{T}_\alpha)) \simeq \{\pm 1\} \times \mathbf{Z}$. The factor $\mathbf{Z}$ represents the subgroup of matrices in $\mathrm{GL}(2, \mathbf{Z})$ that stabilize the line[7] in $\mathbf{R}^2$. That is:

$$\begin{aligned} \begin{pmatrix} 1 & \alpha \end{pmatrix} \begin{pmatrix} c & a \\ d & b \end{pmatrix} &= \begin{pmatrix} c + \alpha d & a + \alpha b \end{pmatrix} \\ &= (c + \alpha d) \begin{pmatrix} 1 & \dfrac{a + \alpha b}{c + \alpha d} \end{pmatrix} \\ &= \lambda \begin{pmatrix} 1 & \alpha \end{pmatrix} \quad \text{with} \quad \lambda = c + \alpha d. \end{aligned}$$

This is an Abelian group made of the powers of its generator.

For an hyperplane $\mathrm{H} \subset \mathbf{R}^3$, except the trivial case we have two possible situations, depending on the characteristic polynomial represented in Figure

[6]For a reference to Dirichlet's theorem see [BC67].

[7]Actually the kernel of the linear 1-form $\boldsymbol{\alpha}$ is the line $y = -x/\alpha$, which is equivalent.

1. On left: Field corresponding to the characteristic polynomial $\mathrm{P}(\lambda) = \lambda^3 - 2$ (one real root, $r = 1, s = 1$), leading to $\pi_0(\mathrm{Diff}(\mathrm{T}_{\boldsymbol{\alpha}})) \simeq \{\pm 1\} \times \mathbf{Z}^{1+1-1} = \{\pm 1\} \times \mathbf{Z}$. On right: Field corresponding to $\mathrm{P}(\lambda) = \lambda^3 - 3\lambda + 1$ (three real roots, $r = 3, s = 0$), leading to $\pi_0(\mathrm{Diff}(\mathrm{T}_{\boldsymbol{\alpha}})) \simeq \{\pm 1\} \times \mathbf{Z}^{3+0-1} = \{\pm 1\} \times \mathbf{Z}^2$. Here again, It's an Abelian group made of the products of its generators. There can be more than one depending on the roots of the characteristic polynomial.

3. $(\mathbf{R}, +)$ Principal Bundles over Diffeological Spaces

In the classical category of differentiable manifold, every fiber bundle with contractible fiber has a global smooth section; see for example [Die70]. Therefore, every principal bundle over a manifold, with structure group the additive group of real numbers $(\mathbf{R}, +)$, is trivial. However, this does not hold in diffeology. For instance, the irrational torus fibration $\pi : \mathrm{T}^2 \to \mathrm{T}_\alpha$ is a nontrivial $(\mathbf{R}, +)$ principal fibration. See [PIZ85] and [PIZ13, §8.11] for the formal definition of fibration, and principal fibration, in diffeology.

This observation justifies the definition of a new invariant for any diffeological space X: the set $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ of $(\mathbf{R}, +)$ principal bundles, modulo isomorphisms, with base X. Such fiber bundles are also called *flows* over X, since they are defined by a smooth action of $\mathbf{R}$. By construction, $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ is a uniquely diffeological invariant, as $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ contains only the trivial bundle when X is a manifold. It reveals an internal complexity that is not revealed by previously introduced diffeological invariants.

This construction, initially introduced in [PIZ85] and [PIZ86], is summarized below.

3.1. The Group $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$.

Let $\pi : \mathrm{Y} \to \mathrm{X}$ and $\pi' : \mathrm{Y}' \to \mathrm{X}$ be two $(\mathbf{R}, +)$-principal bundle. Let us define the following operation, described in two steps:

1. Let $\pi \otimes \pi' : \mathrm{Y} \otimes \mathrm{Y}' \to \mathrm{X}$ be the fiber product defined by

$$\mathrm{Y} \otimes \mathrm{Y}' = \{(y, y') \in \mathrm{Y} \times \mathrm{Y}' \mid \pi(y) = \pi'(y')\},$$

with $\pi \otimes \pi'(y, y') = \pi(y) = \pi'(y')$. Note that the total space $\mathrm{Y} \otimes \mathrm{Y}'$ is identical to the pullback $\pi^*(\mathrm{Y}')$. The result bundle $\pi \otimes \pi'$ is an $(\mathbf{R}^2, +)$ principal bundle for the product of the action on each factor, i.e.,

$$(t, t')_{\mathrm{Y} \otimes \mathrm{Y}'}(y, y') = (t_{\mathrm{Y}}(y), t'_{\mathrm{Y}'}(y')).$$

Let $\overline{\mathbf{R}}$ be the antidiagonal:

$$\overline{\mathbf{R}} = \{(t, -t) \mid t \in \mathbf{R}\} \subset \mathbf{R}^2,$$

It is a subgroup of $(\mathbf{R}^2, +)$.

2. Let Y'' be the quotient $\mathrm{Y}'' = (\mathrm{Y} \otimes \mathrm{Y}')/\overline{\mathbf{R}}$ with $\pi'' : \mathrm{Y}'' \to \mathrm{X}$, such that, for all $y'' = [y, y']$, $\pi''(y'') = \pi(y) = \pi'(y')$, where $[y, y']$ denotes the orbit of the pair (y, y') by the action of $\overline{\mathbf{R}}$. That is:
$$\left[y, y'\right] = \{(t_{\mathrm{Y}}(y), -t_{\mathrm{Y}'}(y')) \mid t \in \mathbf{R}\}.$$

This construction leads to the following propositions, on the properties of this operation.

Proposition (A). *The projection $\pi'' : \mathrm{Y}'' \to \mathrm{X}$ is again a $(\mathbf{R}, +)$-principal bundle for the additive action of $\mathbf{R}$, defined by:*
$$s_{\mathrm{Y}''}\left[y, y'\right] = \left[s_{\mathrm{Y}}(y), y'\right] = \left[y, s_{\mathrm{Y}'}(y')\right].$$

Proposition (B). *The operation $(\pi, \pi') \mapsto \pi''$ passes to the equivalence classes of $(\mathbf{R}, +)$-principal bundle over X. This operation is denoted additively:*
$$\mathrm{class}(\pi) + \mathrm{class}(\pi') = \mathrm{class}(\pi'').$$
Let $p = \mathrm{class}(\pi)$, $p' = \mathrm{class}(\pi')$, and $p'' = \mathrm{class}(\pi'')$: the operation $(p, p') \mapsto p'' = p + p'$ is an Abelian group operation on $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$:

(a) *This operation is Abelian $p + p' = p' + p$.*
(b) *This operation is associative $(p + p') + p'' = p + (p' + p'')$.*
(c) *The class of the trivial bundle $\mathrm{pr}_1 : \mathrm{X} \times \mathbf{R} \to \mathrm{X}$ is the identity element.*
(d) *The inverse $-p$ is the class of the same principal fiber bundle $\pi : \mathrm{Y} \to \mathbf{R}$, but with the inverse $(\mathbf{R}, +)$-action $\bar{t}_{\mathrm{Y}}(y) = (-t)_{\mathrm{Y}}(y)$.*

Note. The group of flows $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ appears also, independently, in the *Čech-de-Rham bi-complex in Diffeology*, as the bi-graded cohomology groups $\mathsf{H}^{1,0}_{\delta}(\mathrm{X})$, see [PIZ24, §21] and here [PIZ88].

Proof. (a) First of all, this operation is Abelian: $\mathrm{Y} \otimes \mathrm{Y}'$ is equivalent to $\mathrm{Y}' \otimes \mathrm{Y}$ and the action of $(t, -t) \in \overline{\mathbf{R}}$ becomes $(-t, t) \in \overline{\mathbf{R}}$, and $[y, y'] \simeq [y', y]$, and $p + p' = p' + p$.

(b) Next, let us prove first that this operation is associative: Consider $[y, [y', y'']]$ on the one hand and $[[y, y'], y'']$ on the other hand:
$$\begin{aligned}
\left[y, \left[y', y''\right]\right] &= \left\{\left(t_{\mathrm{Y}}(y), -t_{\mathrm{Y}'''}\left[y', y''\right]\right) \mid t \in \mathbf{R}\right\} \quad \text{with} \quad \mathrm{Y}''' = (\mathrm{Y}' \otimes \mathrm{Y}'')/\overline{\mathbf{R}} \\
&= \left\{\left(t_{\mathrm{Y}}(y), \left[-t_{\mathrm{Y}'}(y'), y''\right]\right) \mid t \in \mathbf{R}\right\} \\
&= \left\{\left(t_{\mathrm{Y}}(y), \left\{\left((s-t)_{\mathrm{Y}'}(y'), -s_{\mathrm{Y}''}(y'')\right) \mid s \in \mathbf{R}\right\}\right) \mid t \in \mathbf{R}\right\} \\
&\simeq \left\{\left(t_{\mathrm{Y}}(y), (s-t)_{\mathrm{Y}'}(y'), -s_{\mathrm{Y}''}(y'')\right) \mid s, t \in \mathbf{R}\right\} \\
&\simeq \left\{\left(s_{\mathrm{Y}''''}\left[y, y'\right], -s_{\mathrm{Y}''}(y'')\right) \mid s \in \mathbf{R}\right\} \quad \text{with} \quad \mathrm{Y}'''' = (\mathrm{Y} \otimes \mathrm{Y}')/\overline{\mathbf{R}} \\
&= \left[\left[y, y'\right], y''\right].
\end{aligned}$$

Thus,

$$\begin{aligned} p + (p' + p'') &= \operatorname{class}\left(\left\{\left[y, \left[y', y''\right]\right] \mid y \in \mathrm{Y}, y' \in \mathrm{Y}' \text{ and } y'' \in \mathrm{Y}''\right\}\right) \\ &= \operatorname{class}\left(\left\{\left[\left[y, y'\right], y''\right] \mid y \in \mathrm{Y}, y' \in \mathrm{Y}' \text{ and } y'' \in \mathrm{Y}''\right\}\right) \\ &= (p + p') + p''. \end{aligned}$$

(c) Next, let us show that the class of the trivial bundle is the identity element. Let us denote $\mathbf{0}_{\mathrm{X}}$ the class of the trivial principal bundle $\mathrm{pr}_1 : \mathrm{X} \times \mathbf{R} \to \mathrm{X}$. For all element $(y, (x, t)) \in \mathrm{Y} \otimes (\mathrm{X} \times \mathbf{R})$, we have $[y, (x, t)] = [-t_{\mathrm{Y}}(y), (x, 0)]$ with $\pi(y) = x$. Thus, $\{[(y, (x, t))] \mid y \in \mathrm{Y}, \pi(y) = x, \text{ and } t \in \mathbf{R}\} \simeq \mathrm{Y}$, and $p + \mathbf{0}_{\mathrm{X}} = p$, where $p = \operatorname{class}(\pi)$.

(d) Finally, let us construct the inverse of a class of a bundle: Let $\bar{\pi} : \bar{\mathrm{Y}} \to \mathrm{X}$ be the same fiber bundle but with the inverse action, denoted by $t_{\bar{\mathrm{Y}}}(y) = -t_{\mathrm{Y}}(y)$. Consider the diagonal map $\Delta : y \mapsto (y, y) \mapsto [y, y]$, from Y to $(\mathrm{Y} \otimes \bar{\mathrm{Y}})/\overline{\mathbf{R}}$. The pair (y, y) is equivalent to $t_{\mathrm{Y} \otimes \bar{\mathrm{Y}}}(y, y) = (t_{\mathrm{Y}}(y), -t_{\bar{\mathrm{Y}}}(y) = (t_{\mathrm{Y}}(y), t_{\mathrm{Y}}(y))$. Thus, $\Delta(y) = \Delta(t_{\mathrm{Y}}(y))$. Therefore there exists a smooth map $\sigma : \mathrm{X} \to (\mathrm{Y} \otimes \bar{\mathrm{Y}})/\overline{\mathbf{R}}$ such that $\Delta = \sigma \circ \pi$, and $\pi' \circ \sigma = \mathbf{1}_{\mathrm{X}}$.

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{\ \Delta\ } & (\mathrm{Y} \otimes \bar{\mathrm{Y}})/\overline{\mathbf{R}} \\ {\scriptstyle \pi}\downarrow & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle \pi'} \\ \mathrm{X} & \xrightarrow[\ \mathbf{1}_{\mathrm{X}}\ ]{} & \mathrm{X} \end{array}$$

The map σ is a smooth section of the $(\mathbf{R}, +)$ principal fiber bundle $\pi' : (\mathrm{Y} \otimes \bar{\mathrm{Y}})/\overline{\mathbf{R}} \to \mathrm{X}$, hence it is trivial [PIZ13, §8.12], and $\bar{p} = \operatorname{class}(\bar{\pi})$ is the inverse of p: $p + \bar{p} = \mathbf{0}_{\mathrm{X}}$.

This concludes the proof that $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$, equipped with this operation, is an Abelian group. ▶

3.2. The Group $\mathsf{Fl}(\mathrm{T}, \mathbf{R})$

Let us illustrate this construction by detailing the group $\mathsf{Fl}(\mathrm{T}, \mathbf{R})$ for $\mathrm{T} = \mathbf{R}/\mathrm{K}$, where $\mathrm{K} \subset \mathbf{R}$ is a strict subgroup. We refer to T as a *general* 1-*dimensional torus*, noting that it is irrational when K has more than one generator.

(A) The cohomology of K with values in $\mathcal{C}^\infty(\mathbf{R})$

The purpose of this section is to identify the group $\mathsf{Fl}(\mathrm{T}, \mathbf{R})$, of equivalence classes of $(\mathbf{R}, +)$ principal bundles over T, with a first group of cohomology of K with values in the Abelian group $\mathcal{C}^\infty(\mathbf{R})$ of smooth real functions on $\mathbf{R}$.

Definition (The Cocycles τ). *Consider the subgroup* $\mathrm{K} \subset \mathbf{R}$ *acting on* $\mathcal{C}^\infty(\mathbf{R})$ *by translations*

$$(k, f) \mapsto \mathrm{T}_k^*(f) = f \circ \mathrm{T}_k \quad \textit{with} \quad (k, f) \in \mathrm{K} \times \mathcal{C}^\infty(\mathbf{R}) \quad \textit{and} \quad \mathrm{T}_k(x) = x + k.$$

According to the standard definition,[8] *a* 1*-cocycle of the group* K*, with values in* $\mathcal{C}^\infty(\mathbf{R})$*, twisted by the action of* K *on* $\mathcal{C}^\infty(\mathbf{R})$*, is a map*

$$\tau : \mathrm{K} \to \mathcal{C}^\infty(\mathbf{R}) \quad \textit{such that} \quad \tau(k + k') = \mathrm{T}_{k'}^*(\tau(k)) + \tau(k'),$$

for all $k, k' \in \mathrm{K}$*. Explicitly, for all* $x \in \mathbf{R}$*,*

$$\tau(k + k')(x) = \tau(k)(x + k') + \tau(k')(x).$$

We can write also $\tau(k, x)$ *instead of* $\tau(k)(x)$*. We shall denote by*

$$\mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) = \{\tau \in \mathrm{Maps}(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) \mid \tau(k + k') = \mathrm{T}_{k'}^*(\tau(k)) + \tau(k'), \\ \textit{for all } k, k' \in \mathrm{K}\},$$

the space of cocycles τ*, where* $\mathrm{Maps}(\mathrm{A}, \mathrm{B})$ *denotes the set of all maps from* A *to* B.

Definition (The Coboundaries $\Delta\sigma$). *A cocycle* τ *is a coboundary if there exists a function* $\sigma \in \mathcal{C}^\infty(\mathbf{R})$ *such that:*

$$\tau = \Delta\sigma \quad \textit{with} \quad \Delta\sigma(k) = \mathrm{T}_k^*(\sigma) - \sigma.$$

In other words if $\tau(k)(x) = \sigma(x + k) - \sigma(x)$*. We denote,*

$$\mathbf{B}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) = \{\Delta\sigma \mid \sigma \in \mathcal{C}^\infty(\mathbf{R})\}.$$

Definition (The First Cohomology Group). *As usual, the first cohomology group of* K *with values in* $\mathcal{C}^\infty(\mathbf{R})$*, twisted by the action on* $\mathcal{C}^\infty(\mathbf{R})$ *by translations, is the quotient:*

$$\mathbf{H}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) = \mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R}))/\mathbf{B}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})).$$

Actually, with the vector space structure on $\mathcal{C}^\infty(\mathbf{R})$*, these groups are also real vector spaces.*

(B) Lifting on $\mathbf{R} \times \mathbf{R}$ the action of K on $\mathbf{R}$.

The cocycles $\tau \in \mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R}))$ are used to lift, on the product $\mathbf{R} \times \mathbf{R}$, the action of K on $\mathbf{R}$ by:

$$k : \begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} x + k \\ t + \tau(k, x) \end{pmatrix}, \tag{E}$$

for all $(x, t) \in \mathbf{R} \times \mathbf{R}$. Indeed, denote this action by $k_{\mathbf{R}\times\mathbf{R}}$, we have:

$$\begin{aligned} k_{\mathbf{R}\times\mathbf{R}} \circ k'_{\mathbf{R}\times\mathbf{R}}(x, t) &= k_{\mathbf{R}\times\mathbf{R}}\big(k'_{\mathbf{R}\times\mathbf{R}}(x, t)\big) = k_{\mathbf{R}\times\mathbf{R}}(x + k', t + \tau(k', x)) \\ &= (x + k' + k, t + \tau(k', x) + \tau(k, x + k')) \end{aligned}$$

[8] For example in [Kir74].

$$= (x + k + k', t + \tau(k + k', x)) = (k + k')_{\mathbf{R}\times\mathbf{R}}(x, t).$$

Note also that

$$\tau(0, x) = 0 \quad \text{and} \quad \tau(-k, x) = \tau(k, x - k).$$

(C) $(\mathbf{R}, +)$ principal fiber bundle over T associated with a cocycle τ. Let $\tau \in \mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R}))$. Denote by

$$\mathbf{R} \times_\tau \mathbf{R} = (\mathbf{R} \times \mathbf{R})/\mathrm{K}$$

the quotient of $\mathbf{R} \times \mathbf{R}$ by the action of K lifted by the cocycle τ, as defined in (E). Let

$$\mathrm{pr} : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times_\tau \mathbf{R} \quad \text{with} \quad \mathrm{pr}(x, t) = [x, t]\,,$$

the canonical projection. We denote also by $[x]$ the orbit of $x \in \mathbf{R}$ by K, and by

$$\pi : \mathbf{R} \times_\tau \mathbf{R} \to \mathrm{T} = \mathbf{R}/\mathrm{K}, \quad \text{the projection} \quad [x, t] \mapsto [x]\,.$$

Theorem (Construction). *Let us denote by* Y *the space* $\mathbf{R} \times_\tau \mathbf{R}$*. For all* $s \in \mathbf{R}$*, let*

$$s_{\mathrm{Y}}[x, t] = [x, t + s].$$

This defines a smooth free action of $(\mathbf{R}, +)$ *on* Y *that makes* $\pi : \mathrm{Y} \to \mathrm{T}$*, the projection* $[x, t] \to [x]$*, a principal fiber bundle.*

Then, let us denote by H this association:

$$\mathrm{H} : \mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) \to \mathbf{Fl}(\mathrm{T}, \mathbf{R}) \quad \text{with} \quad \mathrm{H}(\tau) = \text{class}\left(\pi : \mathrm{Y} = \mathbf{R}\times_\tau\mathbf{R} \to \mathrm{T} = \mathbf{R}/\mathrm{K}\right)$$

We conclude by the main theorem:

Theorem (Classification). *The association* $\mathrm{H} : \mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) \to \mathbf{Fl}(\mathrm{T}, \mathbf{R})$ *is a homomorphism,*

$$\mathrm{H}(\tau + \tau') = \mathrm{H}(\tau) + \mathrm{H}(\tau').$$

It is surjective (epimorphism) and its kernel is the subgroup of coboundaries:

$$\ker(\mathrm{H}) = \mathbf{B}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})).$$

Therefore, its projection (still denoted by H*) is an isomorphism:*

$$\mathbf{Fl}(\mathrm{T}, \mathbf{R}) \underset{\mathrm{H}}{\simeq} \mathbf{H}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})).$$

Note 1. In the construction above, the projection $\mathrm{pr} : (x, t) \mapsto [x, t]$ is the universal covering on $\mathrm{Y} = \mathbf{R} \times_\tau \mathbf{R}$, and $\pi_1(\mathrm{Y}) = \mathrm{K}$. We recover this result by applying the long exact sequence of homotopy to the fiber bundle $\pi : \mathrm{Y} \to \mathrm{T}$, since the fiber $\mathbf{R}$ is contractible.

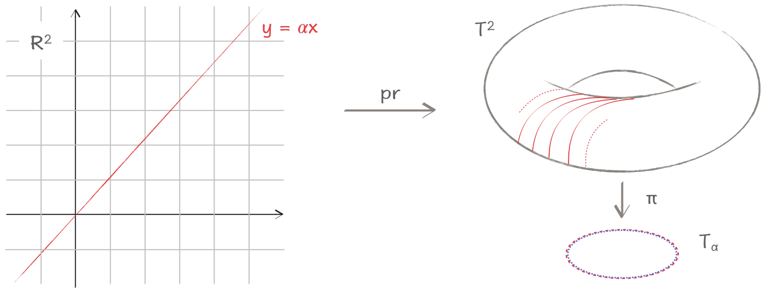


FIGURE 2. The Construction of T_α.

Proof. We now proceed to prove the theorems stated in the article.

(1) First, we verify that the projection $\mathrm{pr} : (x,t) \mapsto [x,t]$ from $\mathbf{R}\times\mathbf{R}$ to $\mathrm{Y} = \mathbf{R}\times_\tau \mathbf{R}$ is a covering. Let $\mathrm{P} : r \mapsto y_r$ be a plot of Y and consider

$$\mathrm{P}^*(\mathbf{R}\times\mathbf{R}) = \{(r,(x,t)) \in \mathrm{dom}(\mathrm{P}) \times \mathbf{R}\times\mathbf{R} \mid [x,t] = \mathrm{P}(r)\}.$$

Since $(x,t)\mapsto[x,t]$ is a subduction the plot P has a local lift $r \mapsto (x_r, t_r)$. Then, $\Psi : (r,k) \mapsto (r, (x_r + k, t_r + \tau(k)(x_r)))$ is a smooth bijection from $\mathrm{dom}(\mathrm{P})\times\mathrm{K}$ to $\mathrm{P}^*(\mathbf{R}\times\mathbf{R})$. Its inverse is given by $\Psi^{-1}(r,(x,t)) = (r, x - x_r)$ which is clearly smooth. Thus, $\mathrm{pr} : (x,t)\mapsto[x,t]$ is a diffeological fibration with fiber K, thus a covering, and since $\mathbf{R}\times\mathbf{R}$ is simply connected, it is the universal covering [PIZ13, §8.26].

(2) Next, let us prove that the projection $\pi : \mathrm{Y} = \mathbf{R}\times_\tau\mathbf{R} \to \mathrm{T} = \mathbf{R}/\mathrm{K}$ is a $(\mathbf{R},+)$ principal fibration, with its action of $\mathbf{R} : s_\mathrm{Y}[x,t] = [x, t+s]$. Let us note first that this action is well defined: if $[x,t] = [x',t']$, then $x' = x + k$ and $t' = t + \tau(k)(x)$, thus $t' + s = t + s + \tau(k)(x)$ and $[x', t'+s] = [x, t+s]$.

Next, for the ones familiar with diffeology, note that this construction is not exactly an associated bundle to the covering $\mathrm{pr} : \mathbf{R} \to \mathrm{T}$ as described in [PIZ13, §8.16], so we shall prove directly that it is indeed an $(\mathbf{R},+)$ principal bundle, by applying the criterion established in [PIZ13, §8.11]. Let us then check that the following action map is an induction,

$$\mathrm{F} : \begin{array}{ccc} \mathrm{Y}\times\mathbf{R} & \to & \mathrm{Y}\times\mathrm{Y} \\ (y,t) & \mapsto & (y, t_\mathrm{Y}(y)) \end{array}$$

(a) First of all F is injective: $\mathrm{F}(y,t) = \mathrm{F}(y',t')$ means $y = y'$ and $t_\mathrm{Y}(y) = t'_\mathrm{Y}(y)$. Let $y = [x,s]$, then $t_\mathrm{Y}(y) = t'_\mathrm{Y}(y)$ implies $[x, t+s] = [x, t'+s]$, i.e., there exist $k \in \mathrm{K}$ such that $x = x + k$ and $t' + s = t + s + \tau(k,t)$. But $x = x + k$ implies $k = 0$, and then, since $\tau(0,x) = 0$, $t + s = t' + s$ implies $t = t'$.

(b) Now,

$$\mathrm{F}(\mathrm{Y}\times\mathbf{R}) = \{(y,y') \in \mathrm{Y}\times\mathrm{T} \mid \pi(y) = \pi(y')\};$$

Consider a plot in $\mathrm{F}(\mathrm{Y} \times \mathbf{R})$, that is, a plot $r \mapsto (y_r, y'_r)$ in $\mathrm{Y} \times \mathrm{Y}$ such that $\pi(y_r) = \pi(y'_r)$, for all r. Since the map $\mathbf{R} \times \mathbf{R} \to \mathrm{Y} = \mathbf{R} \times_\tau \mathrm{Y}$ is a subduction, there are, locally everywhere, two smooth parametrizations $r \mapsto (x_r, t_r)$ and $r \mapsto (x'_r, t'_r)$ in $\mathbf{R} \times \mathbf{R}$ such that: $y_r = [x_r, t_r]$ and $y'_r = [x'_r, t'_r]$, with $\pi(y_r) = \pi(y'_r) = [x_r]$. Thus, there exists a smooth map $r \mapsto k_r$ such that $x'_r = x_r + k_r$, but since $\mathrm{K} \subset \mathbf{R}$ is discrete, locally everywhere: $k_r = k$ and $x'_r = x_r + k$. Hence,

$$\begin{aligned} y'_r = \big[x_r + k, t'_r\big] = \big[x_r, t'_r - \tau(k, x_r)\big] &= \big[x_r, t_r + (t'_r - t_r - \tau(k, x_r))\big] \\ &= [x_r, t_r + s_r] \quad \text{with} \quad s_r = t'_r - t_r - \tau(k, x_r). \end{aligned}$$

Then, $y'_r = [x_r, t_r + s_r] = (s_r)_{\mathrm{Y}}(y_r) = \mathrm{F}(y_r, s_r)$, and $r \mapsto s_r$ is a smooth parametrization in $\mathbf{R}$. Therefore, F is an induction and that achieves the proof that $\pi : \mathrm{Y} = \mathbf{R} \times_\tau \mathbf{R} \to \mathrm{T} = \mathbf{R}/\mathrm{K}$ is a $(\mathbf{R}, +)$ principal fiber bundle.

(3) Now, let us prove that H is a homomorphism. Let $\pi : \mathrm{Y} = \mathbf{R} \times_\tau \mathbf{R} \to \mathrm{T} = \mathbf{R}/\mathrm{K}$ and $\pi' : \mathrm{Y}' = \mathbf{R} \times_{\tau'} \mathbf{R} \to \mathrm{T} = \mathbf{R}/\mathrm{K}$. The proof that $\mathrm{H}(\tau + \tau') = \mathrm{H}(\tau) + \mathrm{H}(\tau')$ will be done in two steps. Let us introduce first the following notations:

- $\mathrm{Y} = \mathrm{Y}_\tau = \mathbf{R} \times_\tau \mathbf{R} := (\mathbf{R} \times \mathbf{R})/\mathrm{K}$ with the action of K defined by τ.
- $\mathrm{Y}' = \mathrm{Y}_{\tau'} = \mathbf{R} \times_{\tau'} \mathbf{R} := (\mathbf{R} \times \mathbf{R})/\mathrm{K}$ with the action of K defined by τ'.
- $\mathrm{Y}_{\tau,\tau'} = \mathbf{R} \times_{\tau,\tau'} \mathbf{R}^2 := (\mathbf{R} \times \mathbf{R}^2)/\mathrm{K}$ with the action of K on $\mathbf{R} \times \mathbf{R}^2$ associated with the pair (τ, τ'), defined, for all $k \in \mathrm{K}$ and $(t, t') \in \mathbf{R}^2$, by:
$$k_{\mathbf{R} \times \mathbf{R}^2}(x, t, t') = (x + k, t + \tau(k, x), t' + \tau'(k, x)).$$
- $\mathrm{Y}_{\tau+\tau'} = \mathbf{R} \times_{\tau+\tau'} \mathbf{R} := (\mathbf{R} \times \mathbf{R})/\mathrm{K}$ with the action of K defined by $\tau + \tau'$.

We write $[x, t]_\tau$, $[x, t']_{\tau'}$ and $[x, t, t']_{\tau,\tau'}$ the elements of Y_τ, $\mathrm{Y}_{\tau'}$ and $\mathrm{Y}_{\tau,\tau'}$. The first step consists to prove that

$$\mathrm{Y} \otimes \mathrm{Y}' \simeq \mathrm{Y}_{\tau,\tau'}.$$

Indeed, consider the diagrams

$$\begin{array}{ccc} (x, t, t') & \stackrel{j}{\longmapsto} & ((x, t), (x, t')) \\ {\scriptstyle \pi_{\tau,\tau'}} \downarrow & & \downarrow {\scriptstyle \pi_\tau \otimes \pi_{\tau'}} \\ [x, t, t']_{\tau,\tau'} & \underset{\underline{j}}{\longmapsto} & ([x, t]_\tau, [x, t']_{\tau'}) \end{array} \qquad \begin{array}{ccc} \mathbf{R} \times \mathbf{R}^2 & \stackrel{j}{\hookrightarrow} & (\mathbf{R} \times \mathbf{R}) \times (\mathbf{R} \times \mathbf{R}) \\ {\scriptstyle \pi_{\tau,\tau'}} \downarrow & & \downarrow {\scriptstyle \pi_\tau \otimes \pi_{\tau'}} \\ \mathrm{Y}_{\tau,\tau'} & \underset{\underline{j}}{\longrightarrow} & \mathrm{Y}_\tau \otimes \mathrm{Y}_{\tau'} \end{array}$$

The map j is an induction, that is a diffeomorphism onto its image equipped with the subset diffeology. Then, it descends to the quotient in $\underline{j} : [x, t, t']_{\tau,\tau'} \mapsto$

$([x,t]_\tau, [x,t]_{\tau'})$, where it is a smooth bijection onto its image. Since j is an induction, the inverse $\underline{j}^{-1}$ is smooth, and then $\underline{j}$ a strict map [PIZ13, §1.54], i.e. a diffeomorphism onto its image.

The second step consists to show that the quotient $(\mathrm{Y}\otimes\mathrm{Y}')/\overline{\mathbf{R}}$, which is equivalent to $\mathrm{Y}_{\tau,\tau'}/\overline{\mathbf{R}}$, with $\overline{\mathbf{R}}$ acts according to $s_{\tau,\tau'}\,[x,t,t']_{\tau,\tau'} = [x,t+s,t'-s]_{\tau,\tau'}$ is realized by the projection

$$\mathrm{pr}_{\tau,\tau'} : \mathrm{Y}_{\tau,\tau'} \to \mathrm{Y}_{\tau+\tau'} \quad \text{with} \quad \mathrm{pr}_{\tau,\tau'} : \left[x,t,t'\right]_{\tau,\tau'} \mapsto \left[x,t+t'\right]_{\tau+\tau'}.$$

That achieves to prove that $\mathrm{H}(\tau) + \mathrm{H}(\tau') = \mathrm{H}(\tau+\tau')$.

(4) We now prove that

$$\ker(\mathrm{H}) = \mathbf{B}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})).$$

A cocycle τ defines a trivial principal bundle if and only if there is a global smooth section $\boldsymbol{\sigma} : \mathrm{T} \to \mathrm{Y}$. Then, $\mathrm{pr}\circ\boldsymbol{\sigma} : \mathbf{R} \to \mathrm{Y}$ has a smooth lift $x \mapsto (x, \sigma(x))$ in $\mathbf{R}\times\mathbf{R}$ such that $\boldsymbol{\sigma}([x]) = [x,\sigma(x)]$. This can be regarded as a consequence of the monodromy theorem [PIZ13, §8.25], since the projection from $\mathbf{R}\times\mathbf{R}$ to its quotient Y is a covering, actually the universal covering. Now, for all $k \in \mathrm{K}$, on the one hand $\boldsymbol{\sigma}([x]) = \boldsymbol{\sigma}([x+k])$ gives $[x,\sigma(x)] = [x+k, \sigma(x+k)]$, and on the other hand $[x,\sigma(x)] = [x+k, \sigma(x)+\tau(k,x)]$. Thus, $\sigma(x+k) = \sigma(x) + \tau(k,x)$, that is, $\tau(k,x) = \sigma(x+k) - \sigma(x)$, meaning $\tau = \Delta\sigma$, and therefore $\ker(\mathrm{H}) = \mathbf{B}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R}))$. In consequence, two maps τ and τ' satisfying (3) define equivalent $(\mathbf{R},+)$ principal bundles if and only if there exists a map $\sigma \in \mathcal{C}^\infty(\mathbf{R})$ such that: $\tau'(k,x) = \tau(k,x) + \sigma(x+k) - \sigma(x)$. Therefore, $\mathrm{H} : \mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) \to \mathbf{Fl}(\mathrm{T}, \mathbf{R})$ descends to the quotient $\mathrm{H} : \mathbf{H}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) \to \mathbf{Fl}(\mathrm{T}, \mathbf{R})$.

(5) Finally, let us prove that $\mathrm{H} : \mathbf{Z}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) \to \mathbf{Fl}(\mathrm{T}, \mathbf{R})$ is surjective. Let $\pi : \mathrm{Y} \to \mathrm{T}$ be an $(\mathbf{R},+)$ principal bundle. Consider the pullback of $\pi : \mathrm{Y} \to \mathrm{X}$ by the projection $\mathrm{pr} : \mathbf{R} \to \mathrm{T}$, that is, the first projection $\mathrm{pr}_1 : \mathrm{pr}^*(\mathrm{Y}) \to \mathbf{R}$, with

$$\mathrm{pr}^*(\mathrm{Y}) = \{(x,y) \in \mathbf{R}\times\mathrm{Y} \mid \mathrm{pr}(x) = \pi(y)\}.$$

Since it is a principal fiber bundle over a manifold with a contractible fiber, it has a smooth section and then it is trivial [Die70]. Let $\Phi : \mathbf{R}\times\mathbf{R} \to \mathrm{pr}^*(\mathrm{Y})$ such an isomorphism from the trivial bundle to $\mathrm{pr}^*(\mathrm{Y})$. Since $\mathrm{pr}_1 \circ \Phi = \mathrm{pr}_1$, the isomorphism Φ writes

$$\Phi(x,t) = (x, \phi(x,t)),$$

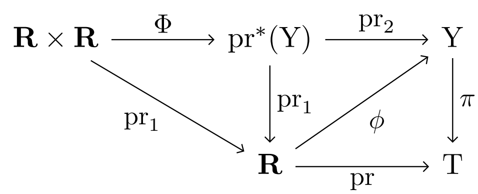
where $\phi : \mathbf{R}\times\mathbf{R} \to \mathrm{Y}$ is smooth. As an isomorphism of principal fiber bundle, it satisfies $\Phi(s_{\mathbf{R}\times\mathbf{R}}(x,t)) = s_{\mathrm{pr}^*(\mathrm{Y})}(\Phi(x,t))$, that is, $\Phi(x,s+t) = (x, \phi(x,s+t))$, and then $s_{\mathrm{pr}^*(\mathrm{Y})}(x,\phi(x,t)) = (x, s_\mathrm{Y}(\phi(x,t)))$. Thus,

$$\phi(x,s+t) = s_\mathrm{Y}(\phi(x,t)), \quad \text{and then} \quad \phi(x,t) = t_Y(\phi(x,0)).$$

Let us rename

$$\phi(x) := \phi(x, 0).$$

The map $\phi : \mathbf{R} \to \mathrm{Y}$ is a smooth lift of pr, that is, $\pi \circ \phi = \mathrm{pr}$, which is summarized by the following diagram.



The map $\Phi : (x, t) \mapsto (x, t_{\mathrm{Y}}(\phi(x)))$ being an isomorphism from $\mathbf{R} \times \mathbf{R}$ to $\mathrm{pr}^*(\mathrm{Y})$, and since π and pr are subductions, the second projection $\mathrm{pr}_2 : \mathrm{pr}^*(\mathrm{Y}) \to \mathrm{Y}$ is a subduction.

Thus, Y is equivalent to the quotient of $\mathbf{R} \times \mathbf{R}$ by the relation $(x, t) \sim (x', t')$ if and only if $\mathrm{pr}_2 \circ \Phi(x, t) = \mathrm{pr}_2 \circ \Phi(x', t')$, that is, $t_{\mathrm{Y}}(\phi(x)) = t'_{\mathrm{Y}}(\phi(x'))$. This implies:

(1) There exist $k \in \mathrm{K}$ such that $x' = x + k$.
(2) There exists a map $\tau : \mathrm{K} \times \mathbf{R} \to \mathbf{R}$ such that

$$\phi(x + k) = \tau(k, x)_{\mathrm{Y}}(\phi(x)).$$

The map τ satifies the *cocycle condition*:

$$\tau(k + k', x) = \tau(k, x + k') + \tau(k', x).$$

Therefore, $(x', t') \sim (x, t)$ if and only if

$$x' = x + k \quad \text{and} \quad t' = t + \tau(k, x).$$

This cocycle $\tau \in \mathcal{C}^\infty(\mathrm{K}, \mathbf{R})$ lifts the action on K on $\mathbf{R}$ to $\mathbf{R} \times \mathbf{R}$ according to Eq. (E). And since $\mathrm{pr}_2 \circ \Phi$ is a subduction, Y is isomorphic to the quotient of $\mathbf{R} \times \mathbf{R}$ by this action:

$$\mathrm{Y} \simeq \mathbf{R} \times_\tau \mathbf{R} = (\mathbf{R} \times \mathbf{R})/\mathrm{K}.$$

This achieves the proof that the induced homomorphism $\mathrm{H} : \mathbf{H}^1(\mathrm{K}, \mathcal{C}^\infty(\mathbf{R})) \to \mathsf{FI}(\mathrm{T}, \mathbf{R})$ is surjective. As shown in Part (4), it is also injective, and hence an isomorphism. ▶

3.3. Irrational Tori and Small Divisors: the Group $\mathsf{FI}(\mathrm{T}_\alpha, \mathbf{R})$.

We shall now explore the geometry delivered by the group $\mathsf{FI}(\mathrm{T}_\alpha, \mathbf{R})$. This will illustrate the sensibility of the geometry (i.e., diffeology) of the quotient $\mathrm{T}_{\mathrm{K}} = \mathbf{R}/\mathrm{K}$, captured by the group $\mathsf{FI}(\mathrm{T}_{\mathrm{K}}, \mathbf{R})$, for the key example of $\mathrm{K} = \mathbf{Z} + \alpha\mathbf{Z}$, where $\alpha \in \mathbf{R} - \mathbf{Q}$ and $\mathbf{R}/\mathrm{K} = \mathrm{T}_\alpha$, leading to distinct behaviors depending on the arithmetic nature of α.

Consider the representation of the irrational torus $\mathrm{T}_\alpha = \mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z})$ as $(\mathbf{R}/\mathbf{Z})/\alpha\mathbf{Z} = \mathrm{T}/\alpha\mathbf{Z}$, where $\mathrm{T} = \mathbf{R}/\mathbf{Z}$, with $n(x) = x+n$, $x \in \mathbf{R}$ and $n \in \mathbf{Z}$, and then $\alpha\mathbf{Z}$ acts on T by $\alpha m\,[x] = [x+\alpha m]$, where $[x] \in \mathrm{T}$ denotes the class of x modulo $\mathbf{Z}$. Of course, $\mathrm{T} \simeq \mathrm{S}^1 \subset \mathbf{C}$ by the identification $[x] \mapsto z$, with $z = e^{i2\pi x}$, and we will use indifferently $[x]$ or z depending on context.

Let's start by describing the cocycle τ in this particular case.

(A) The Cocycle $\boldsymbol{\tau}$.

Let us denote by $\mathrm{pr} : \mathrm{T} \to \mathrm{T}_\alpha$ the projection, and let $\pi : \mathrm{Y} \to \mathrm{T}_\alpha$ be an $(\mathbf{R},+)$ principal fiber bundle. The pullback $\mathrm{pr}_1 : \mathrm{pr}^*(\mathrm{Y}) \to \mathrm{T}$ is an $(\mathbf{R},+)$ principal fiber bundle over a manifold $\mathrm{T} \simeq \mathrm{S}^1$, thus, it is trivial: $\mathrm{pr}^*(\mathrm{Y}) \simeq \mathrm{T}\times\mathbf{R}$. And the previous reconstruction of Y, for K acting on $\mathbf{R}\times\mathbf{R}$, can be mimicked in this situation with $\mathbf{Z}$ acting on $\mathrm{T}\times\mathbf{R}$ through a cocycle $\tau : \mathbf{Z} \to \mathcal{C}^\infty(\mathrm{T},\mathbf{R})$ satisfying

$$\tau(m+m')([x]) = \tau(m)\big([x+m'\alpha]\big) + \tau(m')([x]).$$

Proposition. *Every cocycle τ is uniquely defined by a function $f \in \mathcal{C}^\infty(\mathrm{T},\mathbf{R})$, and every function $f \in \mathcal{C}^\infty(\mathrm{T},\mathbf{R})$ uniquely defines a cocycle τ via the formula, for all positive integers m:*

$$(\clubsuit)\quad \tau(m)([x]) = \sum_{k=0}^{m-1} f([x+k\alpha]), \quad \textit{with} \quad f = \tau(1),\ \tau(0) = 0,$$

$$\textit{and} \quad \tau(-m)([x]) = -\sum_{k=0}^{m-1} f([x+(k-m)\alpha]),$$

Thus, the cohomology group $\mathsf{FI}(\mathrm{T}_\alpha,\mathbf{R})$ is a quotient of $\mathcal{C}^\infty(\mathrm{T},\mathbf{R})$, which we will now determine.

(B) The Action of Z on $\mathrm{S}^1 \times \mathbf{R}$.

Thanks to the explicit expression of the cocycle τ in equation ($\clubsuit$), the action of $\mathbf{Z}$ on $\mathrm{S}^1\times\mathbf{R}$ writes, with $m > 0$:

$$\underline{m}_{\mathrm{S}^1\times\mathbf{R}}(z,t) = \left(ze^{i2\pi m\alpha}, t + \sum_{k=0}^{m-1} f\left(ze^{i2\pi m\alpha}\right)\right)$$

$$\underline{-m}_{\mathrm{S}^1\times\mathbf{R}}(z,t) = \left(ze^{-i2\pi m\alpha}, t - \sum_{k=0}^{m-1} f\left(ze^{i2\pi(k-m)\alpha}\right)\right),$$

where we use the complex notation $z = e^{i2\pi x}$ for $[x]$. Let us introduce now the following diffeomorphism $h \in \mathrm{Diff}(\mathrm{S}^1\times\mathbf{R})$

$$h : z \mapsto \left(ze^{i2\pi\alpha}, t + f(z)\right) \quad \text{and} \quad h^{-1}(z,t) = \left(ze^{-i2\pi\alpha}, t - f\left(ze^{-i2\pi\alpha}\right)\right).$$

For all $m >$, let us denote by h^m the composite m-times of h, by h^{-m}, the composite m-times of h^{-1}, and $h^0 = \mathbf{1}_{\mathrm{S}^1\times\mathbf{R}}$. Then, the action of $\mathbf{Z}$ on $\mathrm{S}^1 \times \mathbf{R}$ writes, for all $m \in \mathbf{Z}$

$$\underline{m}_{\mathrm{S}^1\times\mathbf{R}}(z,t) = h^m(z,t).$$

Thus, the space $\mathrm{Y} = \mathrm{S}^1 \times_\tau \mathbf{R}$ is the quotient of $\mathrm{S}^1 \times \mathbf{R}$ by the $\mathbf{Z}$-iterations of the diffeomorphism h, what we denote by:

$$\mathrm{Y} : (\mathrm{S}^1 \times \mathbf{R})/\langle h\rangle.$$

(C) The Cohomological Equation.

A function $f \in \mathcal{C}^\infty(\mathrm{T},\mathbf{R})$ defines a coboundary if and only if there exists $g \in \mathcal{C}^\infty(\mathrm{T},\mathbf{R})$ satisfying

$$f([x]) = g([x+\alpha]) - g([x]).$$

Let $F, G \in \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R})$ denote the 1-periodic functions representing f and g, respectively. The above equation is then equivalent to

$$\mathrm{F}(x) = \mathrm{G}(x+\alpha) - \mathrm{G}(x).$$

This equation in G, the *cohomological equation*, is central to the study of dynamical systems, particularly circle rotations, with deep historical roots in problems involving small divisors (see, e.g., Arnold [Arn65, Arn80], Moser [Mos66], Herman [Her79], Katok and Hasselblatt [KH95], and Ghy's survey [Ghy07]).

(D) The Decomposition of $\mathsf{FI}(\mathrm{T}_\alpha,\mathbf{R})$.

Let $\mathcal{C}^\infty_{\mathrm{per},0}(\mathbf{R}) \subset \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R})$ be the vector subspace of 1-periodic smooth real functions with mean value 0:

$$\mathcal{C}^\infty_{\mathrm{per},0}(\mathbf{R}) = \left\{ \mathrm{F} \in \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}) \;\middle|\; \int_0^1 \mathrm{F}(x)\,dx = 0 \right\}.$$

Consider the linear map

$$\Delta_\alpha : \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}) \to \mathcal{C}^\infty_{\mathrm{per},0}(\mathbf{R}), \quad \text{with} \quad \Delta_\alpha(\mathrm{G}) = [x \mapsto \mathrm{G}(x+\alpha) - \mathrm{G}(x)],$$

Recalling that $\mathsf{FI}(\mathrm{T}_\alpha,\mathbf{R})$ is the quotient of $\mathcal{C}^\infty(\mathrm{T},\mathbf{R})$ (or equivalently $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R})$) by the image of the related map $\mathrm{G} \mapsto \Delta_\alpha(\mathrm{G})$, we can analyze this quotient by decomposing $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R})$ into its mean value and zero-mean components. Let:

$$\mathrm{F} \mapsto (c,\delta) \quad \text{with} \quad \begin{cases} c = \displaystyle\int_0^1 \mathrm{F}(x)\,dx & \in \mathbf{R}, \\ \delta = \mathrm{F} - \displaystyle\int_0^1 \mathrm{F}(x)\,dx & \in \mathcal{C}^\infty_{\mathrm{per},0}(\mathbf{R}). \end{cases}$$

The mean value c is clearly not affected by the coboundary condition $\Delta_\alpha\mathrm{G}$. The obstructions for the zero-mean part δ lie precisely in the cokernel of

Δ_α restricted to functions mapping to $\mathcal{C}^\infty_{\mathrm{per},0}(\mathbf{R})$. This leads directly to the isomorphism:

$$\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R}) \simeq \mathbf{R} \times \operatorname{coker}(\Delta_\alpha).$$

This decomposition separates each bundle class into two components: the mean value c, which we identify as the *drift parameter* (determining the average vertical shift in the bundle construction and related to the flow speed $1/c$), and a zero-mean part $\delta = f - c$, the *fluctuation component*, whose class lies in $\operatorname{coker}(\Delta_\alpha)$. The naturalness of this decomposition is particularly obvious in the expression of the generating diffeomorphism of the $\mathbf{Z}$ action on $\mathrm{S}^1 \times \mathbf{R}$, namely:

$$h(z,t) = \big(ze^{i2\pi\alpha}, t + c + \delta(z)\big),$$

Thanks to this decomposition into a significant product, we now analyze the structure of the bundle $\pi : \mathrm{Y} = (\mathrm{S}^1 \times \mathbf{R})/\langle h \rangle \to \mathrm{T}_\alpha$ in $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$, based on the value of the drift parameter c.

(D.1) Non-zero drift: $\mathbf{c \neq 0}$.

In this case, the total space Y is a manifold, always diffeomorphic to T^2, only the nature of the flow changes depending on the fluctuation δ. The drift c governs the manifold nature of the total space.

(D.1.a) Case $c \neq 0$ and $\delta = 0$.

This case describes the subspace $\mathbf{R} \times \{0\} \subset \mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$. The cocycle τ is not trivial and equivalent to the morphism

$$\tau(m) : z \mapsto mc \ \text{ and } \ f(z) = \tau(1)(z) = c = \text{const}.$$

The corresponding diffeomorphism generating the quotient is

$$h : (z,t) \mapsto \big(ze^{i2\pi\alpha}, t + c\big)\,.$$

The structure of Y and the $(\mathbf{R}, +)$ dynamics are given by:

Proposition. *The total space* $\mathrm{Y} = (\mathrm{S}^1 \times \mathbf{R})/\langle h \rangle$ *is diffeomorphic to the torus* T^2*. The principal bundle* $\pi : \mathrm{Y} = \mathrm{S}^1 \times_\tau \mathbf{R} \to \mathrm{T}_\alpha$ *is isomorphic to the standard fibration* $\pi_\alpha : \mathrm{T}^2 \to \mathrm{T}_\alpha$*. The* $(\mathbf{R}, +)$ *action corresponds, under this isomorphism, to the standard linear flow on* T^2 *run at speed* $1/c$*.*

This action of $(\mathbf{R}, +)$ on T^2 is also known as the linear flow running at speed $1/c$. It is denoted in the dynamic systems literature by $\mathrm{L}_{1/c}$.

This case, where the class δ in $\operatorname{coker}(\Delta_\alpha)$ is zero, is particularly significant because it encompasses *all* non-trivial bundles when α is Diophantine. Indeed, in this case, the cohomological equation above is known to be solvable for any 1-periodic functions with zero mean. This solvability implies that $\operatorname{coker}(\Delta_\alpha) = 0$ (cf. [KH95, Thm. 12.2.2]). Consequently, for Diophantine

α, every class in $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$ is represented by a constant function. In other words:

Theorem. *For α Diophantine, the group $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$ is isomorphic to $\mathbf{R}$. The total space of any non-trivial $(\mathbf{R}, +)$-principal bundle over T_α (corresponding to $c \neq 0$) is diffeomorphic to T^2, and the fibration π is equivalent to the standard linear flow on T^2 run at speed $1/c$.*

(D.1.b) Case $c \neq 0$ and $\delta \neq 0$.

Since $\delta \neq 0$ this case concerns uniquely numbers α that are not Diophantine.[9] They are irrational numbers that are very well approximated by rationals. Precisely, α is *non-Diophantine* if it is irrational and for every integer $k \geq 1$, there exists infinitely many pairs of integers (m, n) with $n \geq 1$, such that:[10]

$$0 < |n\alpha - m| < \frac{1}{n^k}.$$

For non-Diophantine α the structure of $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$ is made explicit by this theorem:

Theorem. *For non-Diophantine numbers α, unlike the Diophantine case where $\mathrm{coker}(\Delta_\alpha)$ reduces to $\{0\}$, the cokernel is infinite-dimensional:*

$$\dim(\mathrm{coker}(\Delta_\alpha)) = \infty.$$

Consequently

$$\dim(\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})) = 1 + \infty.$$

The infinite dimensionality of $\mathrm{coker}(\Delta_\alpha)$ for non-Diophantine α is a well-known consequence in the specialized field studying small divisor problems. On this general question, see for example the discussions in [Her79, Arn83, KH95, Yoc95, For02]. But for the sake of self-consistency we shall give a proof below.

The geometry of these cases is given by the following theorem:

Theorem. *For a cocycle τ corresponding to a pair (c, δ) with $c \neq 0$ and $\delta \neq 0$, the space $\mathrm{Y} = \mathrm{S}^1 \times_\tau \mathbf{R}$ is still diffeomorphic to the 2-torus T^2. The transmutation of the action $s\,[z, t] = [z, t + s]$ on Y to T^2 is a $(\mathbf{R}, +)$ principal fibration which is not conjugate to the standard linear flow of slope α, but*

[9] There is a convention sometimes found in dynamical systems literature, referring to any non-Diophantine irrational number (equivalently, any irrational with infinite Liouville-Roth irrationality measure) as a Liouville number. This should be distinguished from the stricter definition, often used in number theory requiring $|\alpha - p/q| < 1/q^n$ for all $n \geq 1$, identifies a subset of these numbers. For our purposes, the relevant class is precisely the set of non-Diophantine numbers.

[10] Note that stating existence for infinitely many pairs (m, n) is equivalent to stating existence of at least one pair (m, n) with $n \geq 2$ for each k, because if one exists for k, one must exist for $k + 1$ etc., leading to infinitely many distinct pairs overall as k increases.

which shares the same (diffeological) space of orbits (the base space of the fiber bundle) T_α.

We shall see in details in the proof, but the fact that Y is a manifold is a consequence that, when $c \neq 0$, the diffeomorphism h generating the quotient acts freely and properly discontinuously. Then, because we know that $\mathbf{R}^2$ is its universal covering and $\mathbf{Z}^2$ its fundamental group, Y can only be diffeomorphic to T^2.

(D.2) Zero drift: $\mathbf{c} = \mathbf{0}$.

In this case, the total space Y is no longer a manifold but the product $\mathrm{T}_\alpha \times \mathbf{R}$ as a set. It is equipped with the usual product diffeology in the trivial case when the fluctuation vanishes, and with an "exotic" diffeology in case of a non-zero fluctuation.

(D.2.a) Case $c = 0$ and $\delta = 0$.

This case corresponds to the unique class of the trivial bundle, whose cocycle τ is cohomologous to the zero-cocycle $\tau(n, z) = 0$. The space $\mathrm{Y} = \mathrm{S}^1 \times_{\tau=0} \mathbf{R} = (\mathrm{S}^1/\mathbf{Z}) \times \mathbf{R} = \mathrm{T}_\alpha \times \mathbf{R}$ is not a manifold. The projection π is the projection on the first factor and the action of $\mathbf{R}$ is the translation by any number on the second factor.

(D.2.b) Case $c = 0$ and $\delta \neq 0$.

This case requires α to be non-Diophantine, as δ represents a non-zero class in $\operatorname{coker}(\Delta_\alpha)$. The defining function $f = \delta$ has zero mean ($\int \delta = 0$). This condition guarantees the existence of solutions $g : \mathrm{S}^1 \to \mathbf{R}$ to the cohomological equation $\delta = g \circ \mathrm{R}_\alpha - g$. Although a continuous solution is known to exist under this condition (cf. [KH95]), the crucial point for our diffeological analysis is that since $[\delta] \neq 0$ in $\operatorname{coker}(\Delta_\alpha)$, no such solution g can be smooth.

The existence of a solution g allows us to construct a "principal projection isomorphism", namely, $\psi : \mathrm{Y} \to \mathrm{T}_\alpha \times \mathbf{R}$ given by $\psi([z, t]) = ([z], t - g(z))$, intertwining the action of $(\mathbf{R}, +)$ and projecting on $\mathbf{1}_{\mathrm{T}_\alpha}$. We can use this bijection to push the diffeology $\mathcal{D}_\mathrm{Y}$ forward onto the product set $\mathrm{T}_\alpha \times \mathbf{R}$, resulting in a diffeological space $(\mathrm{T}_\alpha \times \mathbf{R}, \mathcal{D}'_\mathrm{prod})$ which is diffeologically isomorphic to $(\mathrm{Y}, \mathcal{D}_\mathrm{Y})$ and carries the projection pr_1 as its bundle structure.

This bundle $(\mathrm{T}_\alpha \times \mathbf{R}, \mathcal{D}'_\mathrm{prod}) \to \mathrm{T}_\alpha$ represents the class $[\delta] \in \mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$. Its non-triviality (since $[\delta] \neq 0$) manifests directly in the fact that the canonical zero section $s_0 : \mathrm{T}_\alpha \to \mathrm{T}_\alpha \times \mathbf{R}$, $s_0([z]) = ([z], 0)$, is *not smooth* with respect to the pushforward diffeology $\mathcal{D}'_\mathrm{prod}$. As shown in the proof, the smoothness of s_0 is equivalent to the smoothness of g, which fails in this case.

Thus, although set-theoretically equivalent to the trivial bundle in the category of principal projections, the bundle corresponding to $(c = 0, \delta \neq 0)$ is smoothly non-trivial. Its diffeological structure, whether viewed on Y or pushed forward to $\mathrm{T}_\alpha \times \mathbf{R}$, reflects the non-smooth nature of the solution g to the cohomological equation.

Proof I. **Everything except infinite dimension claim.** We will only prove what cannot be immediately deduced from the text. In particular, parts (A), (B) and (C) require no more proof than is explicitly written.

For the part (D), we shall inspect every statement closely.

(D.1.a) Case $c \neq 0, \delta = 0$. Here f is cohomologous to the constant c. The action of $\mathbf{Z}$ on $\mathrm{S}^1 \times \mathbf{R}$ is given by $m : (z, t) \mapsto (z^{i2\pi m\alpha}, t + mc)$. The following projection

$$\mathrm{pr} : \mathrm{S}^1 \times \mathbf{R} \to \mathrm{T}^2 \quad \text{with} \quad \mathrm{pr} : \begin{pmatrix} z \\ t \end{pmatrix} \mapsto \begin{pmatrix} z_1 = e^{i2\pi t/c} \\ z_2 = \bar{z} e^{i2\pi t\alpha/c} \end{pmatrix}$$

realizes the quotient $(\mathrm{S}^1 \times \mathbf{R})/\mathbf{Z}$. The action $s : (z, t) \mapsto (z, t+s)$ is transmuted to

$$s_{\mathrm{T}^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i2\pi(t+s)/c} \\ \bar{z} e^{i2\pi(t+s)\alpha/c} \end{pmatrix} = \begin{pmatrix} z_1 e^{i2\pi s/c} \\ z_2 e^{i2\pi s\alpha/c} \end{pmatrix}.$$

This is the linear flow on T^2 of slope α run at speed $1/c$.

For the Theorem in (D.1.a): If α is Diophantine, the equation $\varphi = \Delta_\alpha g$ has a smooth solution g for any smooth φ with zero mean [KH95, Thm. 12.2.2]. Thus $\mathrm{coker}(\Delta_\alpha) = \{0\}$. The decomposition $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R}) \simeq \mathrm{R} \times \{0\} \cong \mathrm{R}$ follows. All non-trivial bundles $(c \neq 0)$ fall into this case, have total space T^2, and correspond to scaled linear flows.

(D.1.b) Case $c \neq 0, \delta \neq 0$. This requires α to be non-Diophantine, since $\delta \neq 0$ in the cokernel. The generator is $h(z, t) = (ze^{i2\pi\alpha}, t + f(z))$, where $f = c + \delta$.

The action is free as α is irrational. Let us prove that the action is properly discontinuous. Let

$$S_n(z) = \sum_{k=0}^{n-1} f(ze^{i2\pi k\alpha}) = nc + D_n(z), \quad \text{where} \quad D_n(z) = \sum_{k=0}^{n-1} \delta(ze^{i2\pi k\alpha}).$$

It is a standard result from the ergodic theory of circle rotations that for a continuous function $\delta : \mathrm{S}^1 \to \mathbf{R}$, the equation $\delta = g \circ T_\alpha - g$ admits a continuous solution $g : \mathrm{S}^1 \to \mathbf{R}$ if and only if $\int_{\mathrm{S}^1} \delta = 0$. Since δ is smooth with $\int \delta = 0$, there exists a continuous function $g : \mathrm{S}^1 \to \mathbf{R}$ such that $\delta = g \circ T_\alpha - g$. As S^1 is compact, g is bounded, say $|g(z)| \leq M$. Then $|D_n(z)| = |g(z_n) - g(z)| \leq 2M$ for all n and all $z \in \mathrm{S}^1$. The fluctuation term

$D_n(z)$ is uniformly bounded. The height of $h^n(z,t)$ is $t+S_n(z) = t+nc+D_n(z)$. Since $c \neq 0$ and $D_n(z)$ is uniformly bounded, $|t + S_n(z)| \to \infty$ as $|n| \to \infty$, uniformly for (z,t) in any compact set K. For any compact K, $h^n(K) \cap K = \varnothing$ for large enough $|n|$. The action is properly discontinuous. Therefore, the quotient $\mathrm{Y} = (\mathrm{S}^1 \times \mathbf{R})/\langle h \rangle$ is a smooth Hausdorff manifold. We have seen above that its universal cover is $\mathbf{R}^2$ and its fundamental group is $\mathbf{Z}^2$, whatever the quotient is a manifold or not, since manifolds constitute a full and faithful subcategory of diffeological spaces. Therefore, as in case (D.1.a), Y is diffeomorphic to T^2. This proves the first part of the Theorem/Proposition in (D.1.b). The second part, that the resulting flow $s \cdot [z,t] = [z, t+s]$ on $\mathrm{Y} \cong \mathrm{T}^2$ is not smoothly conjugate to the standard linear flow when $\delta \neq 0$, is a deeper result from dynamical systems which we accept here (see e.g., Herman [Her79]).

The result concerning the dimension, namely $\dim(\mathbf{Fl}(\mathrm{T}_\alpha, \mathbf{R})) = 1 + \infty$, is proved separately below (see Proof II).

(D.2.a) Case $c = 0, \delta = 0$. The isomorphism $\mathrm{H} : \mathbf{H}^1(\mathbf{Z}, \mathcal{C}^\infty(\mathrm{S}^1, \mathbf{R})) \to \mathbf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$ sends $(0,0)$ to the trivial bundle $\mathrm{pr}_1 : \mathrm{T}_\alpha \times \mathbf{R} \to \mathrm{T}_\alpha$.

(D.2.b) Case $c = 0, \delta \neq 0$. Again, this requires α to be non-Diophantine. The generating function is $f = \delta$, with $\int \delta = 0$ but $[\delta] \neq 0$ in $\mathrm{coker}(\Delta_\alpha)$.

Since $\int \delta = 0$, the equation $\delta(x) = g(x+\alpha) - g(x)$ admits a unique (up to a constant) solution $g : \mathrm{T} \to \mathbf{R}$. However, because $[\delta] \neq 0$ in the smooth cokernel $\mathrm{coker}(\Delta_\alpha)$, this solution g cannot be smooth.

Using this function g, define the map $\psi : \mathrm{Y} \to \mathrm{T}_\alpha \times \mathbf{R}$ by $\psi([z,t]) = ([z], t - g(z))$. This map is a bijection, and it intertwines the $\mathbf{R}$-actions and projections, making it a principal projection isomorphism.

Let $\mathcal{D}_\mathrm{Y}$ be the quotient diffeology on $\mathrm{Y} = (\mathrm{S}^1 \times \mathbf{R})/\langle h \rangle$. Let $\mathcal{D}'_{\mathrm{prod}}$ be the pushforward diffeology on the set $\mathrm{T}_\alpha \times \mathbf{R}$ induced by ψ. By construction, $\psi : (\mathrm{Y}, \mathcal{D}_\mathrm{Y}) \to (\mathrm{T}_\alpha \times \mathbf{R}, \mathcal{D}'_{\mathrm{prod}})$ is a diffeological isomorphism.

The bundle $(\mathrm{T}_\alpha \times \mathbf{R}, \mathcal{D}'_{\mathrm{prod}}) \to \mathrm{T}_\alpha$ represents the class $[\delta]$. To check if this bundle is smoothly trivial, we examine its canonical zero section $s_0 : \mathrm{T}_\alpha \to \mathrm{T}_\alpha \times \mathbf{R}$, defined by $s_0([z]) = ([z], 0)$.

The section s_0 is smooth (as a map into the pushforward diffeology $\mathcal{D}'_{\mathrm{prod}}$) if and only if the composition $\varphi \circ s_0 : \mathrm{T}_\alpha \to \mathrm{Y}$ is smooth (a plot in $\mathcal{D}_\mathrm{Y}$), where $\varphi = \psi^{-1}$. We have $\varphi([z], t) = [z, g(z) + t]$, so $(\varphi \circ s_0)([z]) = \varphi([z], 0) = [z, g(z)]$. Let $s = \varphi \circ s_0$ be this section.

The map $s : \mathrm{T}_\alpha \to \mathrm{Y}$ is smooth if and only if its lift $\tilde{s} : \mathrm{S}^1 \to \mathrm{S}^1 \times \mathbf{R}$, given by $\tilde{s}(z) = (z, g(z))$, is smooth when composed with plots into S^1. This requires the map $z \mapsto g(z)$ to be smooth.

Since g is not smooth in this case, the section $s = \varphi \circ s_0$ is not smooth, and consequently the zero section s_0 is not smooth with respect to the pushforward diffeology $\mathcal{D}'_{\text{prod}}$.

Therefore, the bundle represented by $(c = 0, [\delta] \neq 0)$ is smoothly non-trivial, confirming $[\delta] \neq 0$ in $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$. The non-triviality is captured by the non-standard smooth structure $\mathcal{D}'_{\text{prod}}$ placed on the product set $\mathrm{T}_\alpha \times \mathbf{R}$. ▶

Proof II. **The infinite dimension claim.** We shall prove using Fourier transforms, that, for α non-Diophantine, the cokernel of $\Delta_\alpha : \mathcal{C}^\infty(\mathrm{T}, \mathbf{R}) \to \mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$, with $\Delta_\alpha(g) = g \circ \mathrm{R}_\alpha - g$, where R_α is the rotation of angle α (that is, the multiplication by $e^{i2\pi\alpha}$), is infinite-dimensional when $\alpha \in \mathbf{R} - \mathbf{Q}$ is not Diophantine. This will justify the notation $\dim(\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})) = 1 + \infty$. The strategy is to construct an infinite sequence of smooth functions $\{f_m\}$ and demonstrate that they cannot lie in the image of Δ_α; specifically, we show that if we assumed $f_m = \Delta_\alpha g_m$ for some smooth g_m, the Fourier coefficients of this hypothetical g_m would necessarily violate the rapid decay condition required for smoothness (i.e., membership in the Schwartz space $\mathcal{S}_\mathbf{R}(\mathbf{Z})$), leading to a contradiction.

(a) Framework.

Let $\mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$ be the space of smooth, 1-periodic, real-valued functions on $\mathbf{R}$. We identify $\mathrm{T} = \mathbf{R}/\mathbf{Z}$. Any function $g \in \mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$ has a Fourier series expansion

$$g(x) = \sum_{k \in \mathbf{Z}} \hat{g}(k) e^{i2\pi kx}$$

where the Fourier coefficients are given by

$$\hat{g}(k) = \int_0^1 g(x) e^{-i2\pi kx} dx.$$

The condition that $g(x)$ is real-valued for all x is equivalent to the symmetry condition on its Fourier coefficients:

$$\hat{g}(k)^* = \hat{g}(-k) \quad \forall k \in \mathbf{Z},$$

where the asterisk denotes the complex conjugate.

The smoothness of g is equivalent to the rapid decay of its Fourier coefficients, meaning the sequence $\hat{g} = (\hat{g}(k))_{k \in \mathrm{Z}}$ belongs to the Schwartz space $\mathcal{S}(\mathbf{Z})$. This means for every $\mathrm{N} \geq 0$, there exists $\mathrm{C_N} > 0$ such that

$$|\hat{g}(k)| \leq \frac{\mathrm{C_N}}{(1 + |k|)^{\mathrm{N}}}, \ \forall k \in \mathbf{Z}.$$

Let

$$\mathcal{S}_\mathbf{R}(\mathbf{Z}) = \{\hat{g} \in \mathcal{S}(\mathrm{Z}) : \hat{g}(k)^* = \hat{g}(-k)\}.$$

The Fourier transform $\mathcal{F} : g \mapsto \hat{g}$ establishes a (topological) vector space isomorphism (over R) between $\mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$ and $\mathcal{S}_\mathbf{R}(\mathbf{Z})$. The operator $\Delta_\alpha g(x) = g(x+\alpha) - g(x)$ clearly maps $\mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$ to itself, since if g is smooth and real-valued, so is $\Delta_\alpha g$.

(b) Action of Δ_α in Fourier Space.

Let $h = \Delta_\alpha g$. Its Fourier coefficients are

$$\hat{h}(k) = \lambda_k \hat{g}(k) \quad \text{with} \quad \lambda_k = e^{i2\pi k\alpha} - 1, \quad \text{and then} \quad \widehat{\Delta_\alpha g}(k) = \lambda_k \hat{g}(k).$$

The operator Δ_α corresponds via Fourier transform to the multiplication operator

$$\mathrm{M}_\lambda : \hat{g} \mapsto (\lambda_k \hat{g}(k))_{k\in\mathbf{Z}}.$$

We verify that M_λ maps $\mathcal{S}_\mathbf{R}(\mathbf{Z})$ to itself. If $\hat{g} \in \mathcal{S}_\mathbf{R}(\mathbf{Z})$, let $\hat{h} = M_\lambda \hat{g}$. Then $\hat{h} \in \mathcal{S}(\mathrm{Z})$ because $\mathcal{S}(\mathrm{Z})$ is an algebra under pointwise multiplication and (λ_k) is a bounded sequence (thus a multiplier). We check the reality condition:

$$\hat{h}(k)^* = \big(\lambda_k \hat{g}(k)\big)^* = \lambda_k^* \hat{g}(k)^*, \quad \text{and} \quad \hat{h}(-k) = \lambda_{-k}\hat{g}(-k).$$

Since α is real, $\lambda_k^* = (e^{i2\pi k\alpha} - 1)^* = e^{-i2\pi k\alpha} - 1 = \lambda_{-k}$. Since $\hat{g} \in \mathcal{S}_\mathbf{R}(\mathbf{Z})$, we have $\hat{g}(k)^* = \hat{g}(-k)$. Therefore, $\hat{h}(k)^* = \lambda_{-k}\hat{g}(-k) = \hat{h}(-k)$. So, $\hat{h} \in \mathcal{S}_\mathbf{R}(\mathbf{Z})$. The image $\mathrm{im}(\Delta_\alpha)$ corresponds via $\mathcal{F}$ to $\mathrm{im}(\mathrm{M}_\lambda|_{\mathcal{S}_\mathbf{R}(\mathbf{Z})}) \subset \mathcal{S}_\mathbf{R}(\mathbf{Z})$. The cokernel is $\mathrm{coker}(\Delta_\alpha) = \mathcal{C}^\infty(\mathrm{T}, \mathbf{R}) / \mathrm{im}(\Delta_\alpha)$, which is isomorphic (as a real vector space) to:

$$\mathrm{coker}(\Delta_\alpha) \simeq \mathcal{S}_\mathbf{R}(\mathbf{Z}) / \mathrm{im}(\mathrm{M}_\lambda|_{\mathcal{S}_\mathbf{R}(\mathbf{Z})}).$$

(c) The Non-Diophantine Condition.

An irrational number α is *not* Diophantine if for every $\nu \geq 2$ and every $\mathrm{C} > 0$, there exists $k \neq 0$ such that

$$\|k\alpha\| < \frac{\mathrm{C}}{|k|^{\nu-1}}.$$

where $\|x\|$ denotes the distance to the nearest integer $\|x\| = \min_{p\in\mathrm{Z}} |x - p|$. The magnitude of λ_k is related to $\|k\alpha\|$:

$$|\lambda_k| = |e^{i2\pi k\alpha} - 1| = |e^{i\pi k\alpha}(e^{i\pi k\alpha} - e^{-i\pi k\alpha})| = |1 \cdot 2i\sin(\pi k\alpha)| = 2|\sin(\pi k\alpha)|.$$

Since $|\sin(\pi x)| = \sin(\pi\|x\|)$ and $\sin(x) \approx x$ for small x, we have $|\lambda_k| \approx 2\pi\|k\alpha\|$ when $\|k\alpha\|$ is small. This implies that $|\lambda_k|$ can decay faster than any polynomial rate for certain sequences of k's tending to infinity. Specifically, we can construct an infinite sequence $\mathrm{K}^+ = \{k_p\}_{p\geq 1}$ of distinct positive integers such that $k_p \to \infty$ as $p \to \infty$ and satisfying:

$$|\lambda_{k_p}| < \frac{1}{k_p^{2p}} \quad \text{for all } p \geq 1.$$

◀ **Justification**. Since α is not Diophantine, for any integer $\mathrm{N} \geq 1$, the set $\mathrm{S_N} = \{k \in \mathbf{Z} - \{0\} : |\lambda_k| < 1/|k|^{\mathrm{N}}\}$ is infinite. We construct K^+ inductively. Choose $k_1 \in \mathrm{S}_2$ with $k_1 > 0$. Assume $k_1, \ldots, k_{p-1}$ have been chosen. Since S_{2p} is infinite, we can choose $k_p \in \mathrm{S}_{2p}$ such that $k_p > k_{p-1}$. If this k_p is negative, we can use $|k_p|$ instead, since $|\lambda_{-k}| = |\lambda_k|$ and $|-k|^{\mathrm{N}} = |k|^{\mathrm{N}}$, so $\mathrm{S_N}$ is symmetric. Thus we can always choose $k_p > k_{p-1} > 0$. This yields the desired sequence $\mathrm{K}^+ = \{k_p\}_{p\geq 1}$ of distinct positive integers with $k_p \to \infty$ and $|\lambda_{k_p}| < 1/k_p^{2p}$. ▶

Let $\mathrm{K} = \mathrm{K}^+ \cup (-\mathrm{K}^+)$. This set K is infinite, symmetric ($k \in \mathrm{K} \iff -k \in \mathrm{K}$), and $0 \notin \mathrm{K}$. For any $k \in \mathrm{K}$, if $|k| = k_p$, we have $|\lambda_k| = |\lambda_{k_p}| < 1/|k|^{2p}$.

(d) Constructing Elements in the Cokernel.

We construct an infinite set of functions $\{f_m\}_{m\geq 1}$ in $\mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$ whose equivalence classes $[f_m]$ in $\operatorname{coker}(\Delta_\alpha)$ are linearly independent over $\mathbf{R}$.

(d.1) Partition the sequence. We routinely partition the infinite set K^+ into infinitely many disjoint infinite subsets $\mathrm{K}_m^+ = \{k_{p_j}^{(m)}\}_{j\geq 1}$ for $m = 1, 2, \ldots$. Let $\mathrm{K}_m = \mathrm{K}_m^+ \cup (-\mathrm{K}_m^+)$. The sets K_m ($m \geq 1$) are pairwise disjoint, symmetric, infinite, contain no zero, and their union is K.

(d.2) Define functions f_m via Fourier coefficients. For each $m \geq 1$, define the sequence $\hat{f}_m = (\hat{f}_m(k))_{k\in\mathbf{Z}}$ by:

$$\hat{f}_m(k) = \begin{cases} |\lambda_k|^{1/2} & \text{if } k \in \mathrm{K}_m \\ 0 & \text{if } k \notin \mathrm{K}_m \end{cases}$$

(d.3) Check $\hat{f}_m \in \mathcal{S}_{\mathrm{R}}(\mathrm{Z})$. The reality condition holds by the same argument as in the previous version (symmetry of K_m and $|\lambda_k| = |\lambda_{-k}|$).

For the rapid decay, consider $k \in \mathrm{K}_m$. Then $k \in K$, so $|k| = k_p$ for some $p \geq 1$ (specifically, $p = p_j^{(m)}$ if $k = \pm k_{p_j}^{(m)}$). We have $|\lambda_k| < 1/|k|^{2p}$. Thus,

$$|\hat{f}_m(k)| = |\lambda_k|^{1/2} < \left(\frac{1}{|k|^{2p}}\right)^{1/2} = \frac{1}{|k|^p}.$$

Since $k \in \mathrm{K}_m$ and $|k| \to \infty$, the corresponding index $p = p(k)$ also tends to infinity. For any fixed $N \geq 0$, we have $p > N$ for $|k|$ sufficiently large. Thus, for $|k|$ large enough and $k \in \mathrm{K}_m$, $|\hat{f}_m(k)| < 1/|k|^N$. Since $\hat{f}_m(k) = 0$ for $k \notin \mathrm{K}_m$, the sequence $\hat{f}_m$ decays faster than any fixed polynomial rate, hence $\hat{f}_m \in \mathcal{S}(\mathbf{Z})$.

Since $\hat{f}_m$ satisfies both conditions, $\hat{f}_m \in \mathcal{S}_{\mathbf{R}}(\mathbf{Z})$. Let $f_m = \mathrm{F}^{-1}(\hat{f}_m)$. Then $f_m \in \mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$.

(d.4) Show $f_m \notin \operatorname{im}(\Delta_\alpha)$. Assume, for contradiction, that $f_m = \Delta_\alpha g_m$ for some $g_m \in \mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$. This requires

$$\hat{g}_m(k) = \frac{\hat{f}_m(k)}{\lambda_k} \text{ (for } k \neq 0\text{) to be in } \mathcal{S}_\mathrm{R}(\mathrm{Z}).$$

Consider $k \in \mathrm{K}_m$. Then $k \neq 0$ and $|k| = k_p$ for some $p = p(k)$.

$$|\hat{g}_m(k)| = \left|\frac{\hat{f}_m(k)}{\lambda_k}\right| = \frac{|\lambda_k|^{1/2}}{|\lambda_k|} = \frac{1}{|\lambda_k|^{1/2}}.$$

Since $|\lambda_k| < 1/|k|^{2p}$ for $k \in \mathrm{K}_m$, we get

$$|\hat{g}_m(k)| > \frac{1}{(1/|k|^{2p})^{1/2}} = \frac{1}{1/|k|^p} = |k|^p.$$

As $|k| \to \infty$ for $k \in \mathrm{K}_m$, the corresponding index $p = p(k)$ also tends to infinity. The magnitude $|\hat{g}_m(k)|$ grows super-polynomially ($|k|^p$ grows faster than any fixed power $|k|^N$ as $p \to \infty$). This contradicts the requirement that $\hat{g}_m \in \mathcal{S}(\mathrm{Z})$. Therefore, $f_m \notin \operatorname{im}(\Delta_\alpha)$.

(d.5) Linear Independence. Suppose $\sum_{m=1}^{\mathrm{M}} c_m[f_m] = [0]$ in $\operatorname{coker}(\Delta_\alpha)$ for real coefficients c_m. Then $\mathrm{F} = \sum_{m=1}^{\mathrm{M}} c_m f_m \in \operatorname{im}(\Delta_\alpha)$. Let $\hat{\mathrm{G}} \in \mathcal{S}_\mathbf{R}(\mathbf{Z})$ be such that $\hat{\mathrm{F}}(k) = \lambda_k \hat{\mathrm{G}}(k)$ for all k. For $k \neq 0$, $\hat{\mathrm{G}}(k) = \hat{\mathrm{F}}(k)/\lambda_k$. Consider $k \in \mathrm{K}_{m_0}$ for some $1 \leq m_0 \leq \mathrm{M}$. Then $\hat{\mathrm{F}}(k) = c_{m_0}\hat{f}_{m_0}(k)$ (since the supports K_m are disjoint). Let $|k| = k_p$ for $p = p(k)$.

$$\hat{\mathrm{G}}(k) = \frac{c_{m_0}\hat{f}_{m_0}(k)}{\lambda_k} = \frac{c_{m_0}|\lambda_k|^{1/2}}{\lambda_k}.$$

$$|\hat{\mathrm{G}}(k)| = |c_{m_0}|\frac{|\lambda_k|^{1/2}}{|\lambda_k|} = |c_{m_0}|\frac{1}{|\lambda_k|^{1/2}}.$$

If $c_{m_0} \neq 0$, then

$$|\hat{\mathrm{G}}(k)| > |c_{m_0}||k|^p.$$

As $|k| \to \infty$ for $k \in K_{m_0}$, $p = p(k) \to \infty$, so $|\hat{\mathrm{G}}(k)|$ grows super-polynomially. This prevents $\hat{\mathrm{G}}$ from being in $\mathcal{S}(\mathbf{Z})$. Therefore, we must have $c_{m_0} = 0$. Since this holds for each $m_0 = 1, \ldots, \mathrm{M}$, the classes $[f_m]$ are linearly independent over $\mathbf{R}$.

Conclusion. We have constructed an infinite set of functions $\{f_m\}_{m \geq 1}$ in $\mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$ whose equivalence classes $[f_m]$ in the cokernel $\operatorname{coker}(\Delta_\alpha) = \mathcal{C}^\infty(\mathrm{T}, \mathbf{R})/\operatorname{im}(\Delta_\alpha)$ are linearly independent over R. Therefore, the dimension of the cokernel of Δ_α acting on $\mathcal{C}^\infty(\mathrm{T}, \mathbf{R})$ is infinite when α is irrational and not Diophantine. ▶

4. Conclusion

In this paper, we have explored how the diffeology of irrational tori $\mathrm{T}_\alpha = \mathrm{T}^2/\mathcal{S}_\alpha$ encodes subtle arithmetic properties of the irrational slope α. Building on previous work linking $\pi_0(\mathrm{Diff}(\mathrm{T}_\alpha))$ to the algebraic nature of α [PIZ85, IZL90], we introduced and studied the group $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ classifying $(\mathbf{R}, +)$ principal bundles (flows) over a diffeological space X. This invariant, trivial for manifolds, proves sensitive to the finer structure of diffeological spaces like T_α.

Our central result is the explicit computation $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R}) \simeq \mathbf{R} \times \mathrm{coker}(\Delta_\alpha)$, where $\mathrm{coker}(\Delta_\alpha)$ relates to the solvability of the cohomological equation for circle rotations by α. The cokernel of Δ_α vanishes, and $\dim \mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R}) = 1$ if and only if α is Diophantine, otherwise $\dim \mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R}) = 1 + \infty$. This establishes a direct link between the Diophantine approximation properties of α and the structure of the diffeological invariant $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$.

Also, we clarify the diffeology of the total space Y: it is diffeomorphic to T^2 when the drift parameter $c \neq 0$. When $c = 0$, the bundle can be represented on the set $\mathrm{T}_\alpha \times \mathbf{R}$, becoming diffeologically trivial if and only if the fluctuation component $[\delta]$ vanishes in $\mathrm{coker}(\Delta_\alpha)$; otherwise, this representation endows $\mathrm{T}_\alpha \times \mathbf{R}$ with an exotic diffeology corresponding to a non-trivial bundle.

Fundamentally, these findings illustrate how, by defining smoothness intrinsically via plots, diffeology allows spaces lacking a manifold structure —such as the irrational torus— to be treated as genuine geometric objects, studied through their own inherent structures and invariants, a potential clearly realized in the arithmetic connections uncovered herein.

Note 1. This paper represents an expanded and detailed account of results originally outlined in the 1986 Comptes Rendus note [PIZ86]. Although that earlier work has received limited attention, the renewed interest in the interplay between Noncommutative Geometry and Diffeology, in particular, motivates this wider publication of those results, now presented with full proofs and detailed exposition.

Note 2. Unlike classical S^1-bundle theory for manifolds, classified by $\mathsf{H}^2(\mathrm{M}, \mathbf{Z})$ which detects topological twisting often resulting in different total spaces, for a diffeological space X, $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ captures finer *analytical obstructions*, as is the case for T_α. For non-zero drift $c \neq 0$ (corresponding to the $\mathbf{R}$ factor in $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R}) \simeq \mathbf{R} \times \mathrm{coker}(\Delta_\alpha)$), the total space is always T^2, but the invariant distinguishes between smoothly non-conjugate flows. This $\mathbf{R}$ factor corresponds precisely to the subgroup $\mathsf{Fl}^\bullet(\mathrm{T}_\alpha) \subset \mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$, whose elements are the bundles admitting a smooth connection 1-form (the Kronecker flows), see [PIZ88, PIZ24]. Generally, for any diffeological space X, the curvature of such connection 1-forms defines a characteristic class $c_1 \in \mathbf{H}^2_{\rm dR}(\mathrm{X})$ for bundles in $\mathbf{Fl}^\bullet(\mathrm{X})$. However, in the special case of $\mathrm{X} = \mathrm{T}_\alpha$, this class necessarily vanishes, since $\mathbf{H}^2_{\rm dR}(\mathrm{T}_\alpha) = 0$ (meaning that the fiber bundle is reducible to a covering, but we know that). The non-triviality for T_α comes from the second factor: the $\mathrm{coker}(\Delta_\alpha)$ part captures obstructions beyond those detected by connection forms, demonstrating how this diffeological invariant reflects subtle analytical properties.

Note 3. It is interesting to note that $\mathrm{T}_{\sqrt{p}}$ (where $p \in \mathbf{N}$ and $p \neq q^2$ for $q \in \mathbf{N}$) exhibits two key properties: (1) $\pi_0(\mathrm{Diff}(\mathrm{T}_{\sqrt{p}})) = \{\pm 1\} \times \mathbf{Z}$ (the $\mathbf{Z}$ factor) and (2) $\mathbf{Fl}(\mathrm{T}_{\sqrt{p}}, \mathbf{R}) = \mathbf{R}$, signifying the rigidity of the fibration $\pi : \mathrm{T}^2 \to \mathrm{T}_{\sqrt{p}}$. This rigidity arises from $\sqrt{p}$ being Diophantine (by Roth's theorem) and the constraints imposed by the torus' extensive internal $(\mathrm{Diff}(\mathrm{T}_{\sqrt{p}}))$ symmetries. This phenomenon of rigidity constrained by symmetries may appear in other places in diffeology and deserves a particular attention.

Note 4. It is noteworthy that the geometric complexity captured by $\mathbf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$ behaves, perhaps counter-intuitively, with respect to the arithmetic approximation properties of α. While the rational case forces geometric simplicity ($\mathrm{T}_\alpha = S^1$, $\mathbf{Fl} = \{0\}$), it is the non-Diophantine case, where α is exceptionally well-approximated by rationals, that yields the richest bundle structure ($\dim(\mathbf{Fl}) = \infty$). Conversely, the Diophantine case, where α resists rational approximation, leads to analytical regularity that restricts the geometry significantly ($\mathbf{Fl} \simeq \mathbf{R}$).

Note 5. Our study focused on the irrational torus $\mathrm{T}_\alpha = \mathrm{T}/\alpha\mathbf{Z}$, whose definition relies intrinsically on the linear rotation $\mathrm{R}_\alpha(z) = ze^{i2\pi\alpha}$. One could consider analogous diffeological spaces constructed from non-linear dynamics. Let $h \in \mathrm{Diff}(\mathrm{S}^1)$ be a smooth orientation-preserving diffeomorphism without periodic points, hence topologically conjugate to an irrational rotation R_α, where $\alpha = \rho(h)$ is its rotation number. If h is *not* smoothly conjugate to R_α (which requires α to be non-Diophantine and h to be suitably chosen, cf. Herman [Her79]), we can define the "exotic irrational torus" $\mathrm{T}_h = \mathrm{S}^1/\langle h\rangle$ equipped with the quotient diffeology.

It would be natural to extend the diffeological investigation presented here to these spaces T_h: (1) Specify the condition for two diffeomorphisms h and h' to give diffeomorphic quotients. (2) Compute the group of components $\pi_0(\mathrm{Diff}(\mathrm{T}_h))$, which would involve the structure of the normalizer of $\langle h\rangle$ within $\mathrm{Diff}(\mathrm{S}^1)$. (3) Compute the group $\mathbf{Fl}(\mathrm{T}_h, \mathbf{R})$. Following the same procedure as in Art. 3.2, this classification leads directly to the first cohomology group

$\mathsf{H}^1(\mathbf{Z}_h, \mathcal{C}^\infty(\mathrm{S}^1))$, where the $\mathbf{Z}$-action is generated by $g \mapsto g \circ h^{-1}$. This group is isomorphic to $\mathcal{C}^\infty(\mathrm{S}^1)/\operatorname{im}(\Delta_h)$, where $\Delta_h(g) = g \circ h - g$.

The central question again becomes the solvability of the cohomological equation $f = g \circ h - g$ for $g \in \mathcal{C}^\infty(\mathrm{S}^1)$. Since h is chosen precisely because it is *not* smoothly linearizable, this equation is known to have obstructions beyond the mean value. Specifically, the cokernel $\operatorname{coker}(\Delta_h)$ acting on zero-mean functions will be infinite-dimensional. Consequently, we expect $\mathsf{FI}(\mathrm{T}_h, \mathbf{R}) \cong \mathbf{R} \times \operatorname{coker}(\Delta_h)$ with $\dim(\operatorname{coker}(\Delta_h)) = \infty$.

This suggests that the invariant $\mathsf{FI}(\cdot, \mathbf{R})$ is sensitive not just to the topological dynamics (the rotation number α) but to the finer *smooth conjugacy class* of the generating diffeomorphism. Investigating T_h would thus further illustrate the connection between diffeological invariants and subtle analytical properties of dynamical systems.

Note 6. A compelling direction for future research arises from extending the analogy between linear flows on T^2 and geodesic flows on hyperbolic surfaces $\Sigma = \mathbf{D}/\Gamma$. The Kronecker foliation $\mathcal{F}_\alpha$ on T^2 consists of leaves parallel to a direction defined by α. Analogously, fix an endpoint $u \in \partial\mathbf{D}$ and consider the foliation $\mathcal{F}_u$ on Σ obtained by projecting all geodesics in the universal cover $\mathbf{D}$ that terminate at u. The leaf space $\mathrm{X}_u = \Sigma/\mathcal{F}_u$ equipped with the quotient diffeology serves as a hyperbolic analog of the irrational torus T_α. Investigating the structure of $\mathsf{FI}(\mathrm{X}_u, \mathbf{R})$ could potentially reveal how diffeological invariants capture information about the geometry of Σ and the endpoint u, mirroring the connection to Diophantine properties found for T_α.

Note 7. Furthermore, the non-triviality of $\mathsf{FI}(\mathrm{X}, \mathbf{R})$ appears deeply connected to the structure of the gauge groupoid $\mathbf{G}_\mathrm{X}$ associated with the quasifold X developed in [IZP21]. Specifically, obstructions to trivializing $(\mathbf{R}, +)$-bundles likely relate to analytical properties tied to the interplay between local symmetries and chart transitions encoded in $\mathbf{G}_\mathrm{X}$. Exploring this connection, potentially linking bundle classification to properties of the groupoid $\mathbf{C}^*$-algebra, presents a promising avenue for future research bridging the results herein with those of [IZP21].

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