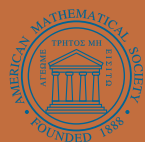


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Diffeology

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לא עָלֶיךָ הַמְּלָאכָה לְגַמּוֹר וְאִין אַתָּה בְּן־חֹרִין לְבַטֵּל מִמֶּנָּה

רבי טרפון, משנה אבות ב טז

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