# A NON-TRIVIAL $(\mathbf{R},+)$ PRINCIPAL BUNDLE OVER A CONTRACTIBLE DIFFEOLOGICAL SPACE

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Abstract. We present an explicit example of an $(\mathbf{R},+)$ non-trivial principal bundle $\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha$ within the framework of diffeology, where both the fiber and the base space $\mathrm{D}_\alpha$ are smoothly contractible. This stands in stark contrast to classical bundle theory over manifolds, where fiber bundles over a contractible base are always trivial, as well as principal fiber bundles with contractible fiber. This example demonstrates why homotopy is not a sufficient tool for classifying fiber bundles in diffeology.

## Introduction

Classical fiber bundle theory, as developed by Steenrod and others, establishes fundamental results regarding the classification and triviality of bundles over (topological spaces and) smooth manifolds. A cornerstone result states that a fiber bundle over a contractible base space is necessarily trivial, and a fiber bundle with a contractible fiber admit a (continuous or a) smooth section, and if it is a principal bundle it is trivial.[1]

Diffeology defines smooth structures on arbitrary sets via collections of 'plots' (smooth maps from Euclidean domains), extending the category of manifolds, where they become a full subcategory.[2] This framework naturally accommodates quotient spaces, including those arising from dense group actions or actions with fixed points, which often lack a manifold structure but possess a meaningful diffeology. Within diffeology, the notion of a principal G-bundle $\pi : \mathrm{Y} \to \mathrm{X}$ can be defined rigorously, requiring the action map $\mathrm{F} : \mathrm{Y} \times \mathrm{G} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ to be an induction in $\mathrm{Y} \times \mathrm{Y}$ [PIZ13, §8.11].

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[1] See e.g., [Die70].

[2] The axiomatic of diffeology introduced by J.-M. Souriau in the 1980s has been extended to a comprehensive theory over the past few decades, covering the mains branches of differential geometry, see [PIZ13].

It has been observed that certain diffeological invariants, trivial for manifolds, reveal rich structures in more general diffeological spaces. Notably, the classifying group $\mathsf{Fl}(\mathrm{X}, \mathbf{R})$ for $(\mathbf{R}, +)$ principal bundles is non-trivial for spaces like the irrational torus $\mathrm{T}_\alpha$, reflecting arithmetic properties of $\alpha$ [PIZ86, PIZ25].

This note presents a specific example, implicit in [PIZ85], that directly confronts the classical triviality theorem. We construct an $(\mathbf{R}, +)$ principal bundle $\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha$ where the base $\mathrm{D}_\alpha$ is a quotient of the closed unit disk D by an irrational rotation action fixing the origin. We demonstrate that $\mathrm{D}_\alpha$ is smoothly contractible. We carefully verify that $\pi$ satisfies the diffeological definition of a principal bundle. Finally, we prove that $\pi$ is non-trivial, both by restricting it to a known non-trivial bundle over the boundary $\mathrm{T}_\alpha \subset \mathrm{D}_\alpha$, and by identifying its non-zero class in $\mathsf{Fl}(\mathrm{D}_\alpha, \mathbf{R})$. This example highlights that contractibility does not guarantee bundle triviality in diffeology, demonstrating the framework's ability to capture finer smooth obstructions, and the need of classification tools for diffeological fiber bundles adapted to the category, as for example constructions coming from Čech cohomology [PIZ24].

## Construction of the Bundle

Let $\mathrm{D} = \{z \in \mathbf{C} \mid |z| \leq 1\}$ be the closed unit disk equipped with its subset diffeology inherited from $\mathbf{C}$. Let $\alpha \in \mathbf{R} - \mathbf{Q}$ be a fixed irrational number.

Consider the action of the group $\mathbf{Z}$ on D by rotations:

$$n \cdot z = ze^{i2\pi n\alpha} \quad \text{for } n \in \mathbf{Z}, z \in \mathrm{D}.$$

This action is smooth. It fixes the origin $z = 0$. Let $\mathrm{D}_\alpha = \mathrm{D}/\mathbf{Z}$ be the quotient space equipped with the quotient diffeology. The projection map $\pi_\mathrm{D} : \mathrm{D} \to \mathrm{D}_\alpha$ is a subduction.

Now, define the $\mathbf{Z}$-action on the product space $\mathrm{D} \times \mathbf{R}$:

$$n \cdot (z, t) = (ze^{i2\pi n\alpha}, t + n|z|^2) \quad \text{for } n \in \mathbf{Z}, (z, t) \in \mathrm{D} \times \mathbf{R}.$$

This action is not free, $\{0\} \times \mathbf{R}$ is the subspace of fixed points, but smooth. Let $\mathrm{W}_\alpha = (\mathrm{D} \times \mathbf{R})/\mathbf{Z}$ be the quotient space with the quotient diffeology. Let $\varpi : \mathrm{D} \times \mathbf{R} \to \mathrm{W}_\alpha$ be the projection map, $\varpi(z, t) = [z, t]$.

The projection $\mathrm{pr}_1 : \mathrm{D} \times \mathbf{R} \to \mathrm{D}$ is equivariant with respect to the $\mathbf{Z}$-actions, $(\mathrm{pr}_1(n \cdot (z, t)) = ze^{i2\pi n\alpha} = n \cdot \mathrm{pr}_1(z, t))$, and therefore descends to a smooth map between the quotients:

$$\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha \quad \text{defined by} \quad \pi([z, t]) = [z].$$

This structure, formed by the quotient space $\mathrm{W}_\alpha = (\mathrm{D}\times\mathbf{R})/\mathbf{Z}$ and its projection $\pi([z,t]) = [z]$ onto $\mathrm{D}_\alpha$, gives us what we shall call **the $\alpha$-warped bundle**. The relationships defining these spaces and maps are illustrated in the diagrams:

$$\begin{array}{ccc} \mathrm{D}\times\mathbf{R} & \xrightarrow{\ \varpi\ } & \mathrm{W}_\alpha \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathrm{D} & \xrightarrow[\ \pi_\alpha\ ]{} & \mathrm{D}_\alpha \end{array} \qquad\qquad \begin{array}{ccc} (z,t) & \xrightarrow{\ \varpi\ } & [z,t] \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ z & \xrightarrow[\ \pi_\alpha\ ]{} & [z] \end{array}$$

Finally, define an action of the additive group $(\mathbf{R},+)$ on $\mathrm{W}_\alpha$:

$$s\cdot[z,t] = [z,t+s] \quad \text{for } s\in\mathbf{R}, [z,t]\in\mathrm{W}_\alpha.$$

This action is well-defined. (If $[z,t]=[z',t']$, then $z' = n\cdot z$ and $t' = t+n|z|^2$. Then $s\cdot[z',t'] = [n\cdot z, t+n|z|^2+s]$. Also $s\cdot[z,t] = [z,t+s]$. We need $[n\cdot z, t+n|z|^2+s] = [z,t+s]$. This holds if $t+n|z|^2+s = (t+s)+n|n^{-1}\cdot z|^2 = t+s+n|z|^2$, which is true). The action is smooth since it lifts to the smooth action $(s,(z,t))\mapsto(z,t+s)$ on $\mathrm{D}\times\mathbf{R}$.

**Proposition (Principal Bundle).** The smooth projection $\pi:\mathrm{W}_\alpha\to\mathrm{D}_\alpha$ is a non trivial $(\mathbf{R},+)$ principal bundle, despite its contractible fiber $\mathbf{R}$ and base $\mathrm{D}_\alpha$.

*Proof.* Let us start by proving that $\pi:\mathrm{W}_\alpha\to\mathrm{D}_\alpha$ is a $(\mathbf{R},+)$ principal bundle:

According to the definition in [PIZ13, §8.11], $\pi:\mathrm{W}_\alpha\to\mathrm{D}_\alpha$ is an $(\mathbf{R},+)$ principal bundle if the action map

$$\mathrm{F}:\mathrm{W}_\alpha\times\mathbf{R}\to\mathrm{W}_\alpha\times\mathrm{W}_\alpha, \quad \text{defined by } \mathrm{F}(y,s) = (y, s\cdot y),$$

is an induction. An induction is an injective map which is a diffeomorphism onto its image equipped with the subset diffeology. The image is

$$\mathrm{W}_\alpha\times_{\mathrm{D}_\alpha}\mathrm{W}_\alpha = \mathrm{im}(\mathrm{F}) = \{(y_1,y_2)\in\mathrm{W}_\alpha\times\mathrm{W}_\alpha \mid \pi(y_1)=\pi(y_2)\}.$$

We emphasize that verifying this 'induction' property, particularly the smoothness of $\mathrm{F}^{-1}$, is often the most demanding part of establishing a principal bundle structure in diffeology, especially for non-trivial examples arising from quotients.

(1) *Injectivity of* F. — Suppose $\mathrm{F}([z,t],s) = \mathrm{F}([z',t'],s')$. This means (a) $[z,t]=[z',t']$ and (b) $[z,t+s] = [z',t'+s']$. From (a), $z' = ze^{i2\pi n\alpha}$ and $t' = t+n|z|^2$ for some $n\in\mathbf{Z}$. Substituting into (b) yields $[z,t+s]=[z,t+s']$. This equality requires existence of $m\in\mathbf{Z}$ such that $z = ze^{i2\pi m\alpha}$ and $t+s = (t+s')+m|z|^2$. Since $\alpha$ is irrational, $z = ze^{i2\pi m\alpha}$ implies $z=0$ or $m=0$. If $z\neq 0$, then $m=0$, so $t+s=t+s'$, implying $s=s'$. If $z=0$, then

$t + s = (t + s') + m|0|^2 = t + s'$, implying $s = s'$. In all cases, $s = s'$. Thus F is injective.

(2) *Smoothness of* $\mathrm{F}^{-1}$ — We verify that the map $\mathrm{F}^{-1} : \mathrm{W}_\alpha \times_{\mathrm{D}_\alpha} \mathrm{W}_\alpha \to \mathrm{W}_\alpha \times \mathbf{R}$ is smooth according to the definition in diffeology. Let $P : \mathrm{U} \to \mathrm{W}_\alpha \times_{\mathrm{D}_\alpha} \mathrm{W}_\alpha$ be an arbitrary plot, $P(r) = (y_{1,r}, y_{2,r})$, and $r \mapsto (y_{1,r}, s_r)$ be the composite $\mathrm{F}^{-1} \circ \mathrm{P}$. It is a plot in $\mathrm{W}_\alpha \times \mathbf{R}$ if and only if $y_1$ is a plot in $\mathrm{W}_\alpha$, what it is by hypothesis, and $r \mapsto s_r$ is a plot in $\mathbf{R}$, where $s_r$ is the unique scalar such that $y_{2,r} = s_r \cdot y_{1,r}$.

We will first prove this for 1-dimensional plots (smooth curves or paths) using local lifts and interval decomposition, and then use Boman's theorem [Bom67] to extend to arbitrary plots.

(2.1) *Smoothness for 1-Dimensional Plots.* — Consider an arbitrary 1-plot $\gamma : \mathrm{I} \to \mathrm{W}_\alpha \times_{\mathrm{D}_\alpha} \mathrm{W}_\alpha$, where $\mathrm{I} = ]a, b[ \subseteq \mathbf{R}$ is an open interval. Let $\gamma(t) = (y_{1,t}, y_{2,t})$. We need to show that the map $\mathrm{Q}_\gamma = \mathrm{F}^{-1} \circ \gamma : \mathrm{I} \to \mathrm{W}_\alpha \times \mathbf{R}$ is a plot. Let $\mathrm{Q}_\gamma(t) = (y_{1,t}, s(t))$, where, we recall, $s(t)$ is uniquely defined by $s(t) \cdot y_{1,t} = y_{2,t}$. We already know the first component $t \mapsto y_{1,t}$ is a plot. We must show that the second component $s : \mathrm{I} \to \mathbf{R}$ is smooth ($\mathcal{C}^\infty$).

Fix an arbitrary $t_0 \in \mathrm{I}$. Since $\varpi : \mathrm{D} \times \mathbf{R} \to \mathrm{W}_\alpha$ is a subduction, we can find an open interval $\mathrm{V} \subseteq \mathrm{I}$ containing $t_0$ and smooth lifts $\tilde\gamma_1, \tilde\gamma_2 : V \to \mathrm{D} \times \mathbf{R}$ such that:

- $\tilde\gamma_1(t) = (w_1(t), \tau_1(t))$ with $\varpi(\tilde\gamma_1(t)) = y_{1,t}$.
- $\tilde\gamma_2(t) = (w_2(t), \tau_2(t))$ with $\varpi(\tilde\gamma_2(t)) = y_{2,t}$.

The maps $w_1, \tau_1, w_2, \tau_2$ are smooth functions $\mathrm{V} \to \mathrm{D}$ or $\mathrm{V} \to \mathbf{R}$. The condition $s(t) \cdot y_{1,t} = y_{2,t}$ implies $\varpi(w_1(t), \tau_1(t) + s(t)) = \varpi(w_2(t), \tau_2(t))$. This holds iff there exists an integer $m_t \in \mathbf{Z}$ such that:

$$w_2(t) = w_1(t) e^{i2\pi m_t \alpha}, \quad \text{and} \quad \tau_2(t) = \tau_1(t) + s(t) + m_t |w_1(t)|^2.$$

Consider the open set

$$\mathrm{V}_* = \{t \in \mathrm{V} \mid w_1(t) \neq 0\}.$$

On $\mathrm{V}_*$, $m_t$ is determined by $e^{i2\pi m_t \alpha} = w_2(t)/w_1(t)$. Since $w_1, w_2$ are smooth and $\alpha$ is irrational, the integer-valued function $t \mapsto m_t$ must be locally constant on $\mathrm{V}_*$. The set $\mathrm{V}_*$ is an open subset of the interval V, so it is a countable disjoint union of open intervals, $\mathrm{V}_* = \bigcup_n \mathrm{I}_n$, where $\mathrm{I}_n = ]a_n, b_n[$. On each interval $\mathrm{I}_n$, $m_t$ must be constant, say $m_t = m_n$ for $t \in \mathrm{I}_n$. For all $n$, $(w_1 \restriction [b_n, a_{n+1}])(t) = 0$. So,

$$\begin{aligned} \text{on } \; ]a_n, b_n[ : &\quad \tau_2(t) = \tau_1(t) + m_n |w_1(t)|^2 + s(t), \\ \text{and on } \; [b_n, a_{n+1}[ : &\quad \tau_2(t) = \tau_1(t) + s(t) = \tau_1(t) + \underbrace{m_n |w_1(t)|^2}_{=0} + s(t). \end{aligned}$$

Thus,

$$\text{restricted to } ]a_n, a_{n+1}[ : \quad \tau_2(t) = \tau_1(t) + m_n|w_1(t)|^2 + s(t),$$
$$\text{and then, on } ]a_n, a_{n+1}[ : \quad s(t) = \tau_2(t) - \tau_1(t) - m_n|w_1(t)|^2.$$

Thus $s \restriction ]a_n, a_{n+1}[ : t \mapsto \tau_2(t) - \tau_1(t) - m_n|w_1(t)|^2$. Since $\tau_1, \tau_2, w_1$ are smooth on V and $m_n$ is a constant, the restriction $s \restriction ]a_n, a_{n+1}[$ is smooth.

Similarly,
$$s \restriction ]b_n, b_{n+1}[ : t \mapsto \tau_2(t) - \tau_1(t) - m_{n+1}|w_1(t)|^2.$$
Hence, the restriction $s \restriction ]b_n, b_{n+1}[$ is smooth.

Now, the collection of open intervals $\{]a_n, a_{n+1}[\}_n \cup \{]b_n, b_{n+1}[\}_n$ covers V (modulo potential adjustments at the endpoints of V itself, which can be handled by shrinking V). On each interval in this cover, $s \restriction ]a_n, a_{n+1}[$ or $s \restriction ]b_n, b_{n+1}[$ is smooth. By the locality principle for smoothness, since the restriction of $s$ on every element of this open cover of V is smooth, $s$ is smooth on V.[3]

Since $t_0 \in \mathrm{I}$ was arbitrary, the function $s : \mathrm{I} \to \mathbf{R}$ is smooth. This proves that for any 1-plot $\gamma : \mathrm{I} \to \mathrm{W}_\alpha \times_{\mathrm{D}_\alpha} \mathrm{W}_\alpha$, the second component $s = \mathrm{pr}_2 \circ \mathrm{F}^{-1} \circ \gamma$ is smooth.

(2.2) *Extension to Arbitrary Plots using Boman's Theorem.* — Now return to the arbitrary plot $\mathrm{P} : \mathrm{U} \to \mathrm{W}_\alpha \times_{\mathrm{D}_\alpha} \mathrm{W}_\alpha$ from an open set $\mathrm{U} \subseteq \mathbf{R}^k$. We need to show that $\mathrm{S}_\mathrm{P} = \mathrm{pr}_2 \circ \mathrm{F}^{-1} \circ \mathrm{P} : \mathrm{U} \to \mathbf{R}$ is smooth ($\mathcal{C}^\infty$).

By Boman's theorem [Bom67], $\mathrm{S}_\mathrm{P}$ is $\mathcal{C}^\infty$ if and only if for every smooth curve $\gamma : \mathrm{I} \to \mathrm{U}$ (where $\mathrm{I} \subseteq \mathbf{R}$ is an open interval), the composition $\mathrm{S}_\mathrm{P} \circ \gamma : \mathrm{I} \to \mathbf{R}$ is $\mathcal{C}^\infty$.

Consider the composition:
$$\mathrm{S}_\mathrm{P} \circ \gamma = (\mathrm{pr}_2 \circ \mathrm{F}^{-1} \circ P) \circ \gamma = \mathrm{pr}_2 \circ \mathrm{F}^{-1} \circ (P \circ \gamma).$$
Let $\mathrm{P}_\gamma = \mathrm{P} \circ \gamma : \mathrm{I} \to \mathrm{W}_\alpha \times_{\mathrm{D}_\alpha} \mathrm{W}_\alpha$. Since P is a plot and $\gamma$ is smooth, $\mathrm{P}_\gamma$ is a 1-dimensional plot.

From Step 2.1, we know that for any 1-plot $\mathrm{P}_\gamma$, the map $\mathrm{pr}_2 \circ \mathrm{F}^{-1} \circ P_\gamma : \mathrm{I} \to \mathbf{R}$ is smooth ($\mathcal{C}^\infty$). Therefore, $\mathrm{S}_\mathrm{P} \circ \gamma$ is $\mathcal{C}^\infty$ for all smooth curves $\gamma : \mathrm{I} \to \mathrm{U}$.

By Boman's theorem, this implies that the function $\mathrm{S}_\mathrm{P} : \mathrm{U} \to \mathbf{R}$ is $\mathcal{C}^\infty$. Condition (2) is satisfied.

(2.3) *Conclusion:* Since both conditions (1) and (2) are satisfied for any plot $\mathrm{P} : \mathrm{U} \to \mathrm{W}_\alpha \times_{\mathrm{D}_\alpha} \mathrm{W}_\alpha$, the map $\mathrm{Q} = \mathrm{F}^{-1} \circ \mathrm{P} : \mathrm{U} \to \mathrm{W}_\alpha \times \mathbf{R}$ is always a plot of $\mathrm{W}_\alpha \times \mathbf{R}$. Therefore, the map $\mathrm{F}^{-1}$ is smooth.

Therefore, $\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha$ is an $(\mathbf{R},+)$ principal bundle.

[3] Indeed, this is exactly what has been chosen as the second axiom of diffeology, called *axiom of locality*.

Next, let us prove its non-triviality.

(3) *Non-Triviality via Restriction.* — Let $\mathrm{S}^1 = \{z \in \mathbf{C} \mid |z| = 1\} \subset \mathrm{D}$ be the boundary unit circle. The $\mathbf{Z}$-action $n \cdot z = ze^{i2\pi n\alpha}$ restricts to $\mathrm{S}^1$. Let $\mathrm{T}_\alpha = \mathrm{S}^1/\mathbf{Z}$ be the irrational torus, which sits inside $\mathrm{D}_\alpha$ as the image of the boundary. Let $j : \mathrm{T}_\alpha \hookrightarrow \mathrm{D}_\alpha$ be the inclusion map.

Consider the pullback bundle $j^*(\mathrm{W}_\alpha) \to \mathrm{T}_\alpha$. The total space $j^*(\mathrm{W}_\alpha)$ consists of pairs $([z]_{\mathrm{T}_\alpha}, y)$ where $y \in \mathrm{W}_\alpha$ and $j([z]_{\mathrm{T}_\alpha}) = \pi(y)$. This space is naturally diffeomorphic to the quotient $(\mathrm{S}^1 \times \mathbf{R})/\mathbf{Z}$, where the $\mathbf{Z}$-action is the restriction of the action defining $\mathrm{W}_\alpha$ to $|z| = 1$:

$$n \cdot (z,t) = (ze^{i2\pi n\alpha}, t + n|1|^2) = (ze^{i2\pi n\alpha}, t+n) \quad \text{for } |z| = 1.$$

Let $\mathrm{W}_\alpha|_{\mathrm{T}_\alpha} = j^*(\mathrm{W}_\alpha)$. The restricted projection $\pi|_{\mathrm{T}_\alpha} : \mathrm{W}_\alpha|_{\mathrm{T}_\alpha} \to \mathrm{T}_\alpha$ is the $(\mathbf{R}, +)$ principal bundle over $\mathrm{T}_\alpha$ associated with the cocycle $\tau(n,z) = n$ (constant function $f(z) = 1$ for $n = 1$).

This restricted bundle is precisely the standard example of a non-trivial $(\mathbf{R}, +)$ principal bundle over the irrational torus, representing a non-zero class in $\mathsf{Fl}(\mathrm{T}_\alpha, \mathbf{R})$, that is, the irrational winding on the 2-torus [PIZ25]. If the original bundle $\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha$ were trivial, then its pullback $j^*(\mathrm{W}_\alpha) \to \mathrm{T}_\alpha$ via the inclusion $j$ would also be trivial. Since the pullback bundle is non-trivial, we conclude that the original bundle $\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha$ must be non-trivial.

(4) *Contractibility of the base* — Let us end by checking that the base $\mathrm{D}_\alpha = \mathrm{D}/\mathbf{Z}_\alpha$ is indeed smoothly contractible: The base space $\mathrm{D}_\alpha = \mathrm{D}/\mathbf{Z}$ is the quotient of the contractible closed disk D. The standard contraction $H : [0,1] \times \mathrm{D} \to \mathrm{D}$, $H(s,z) = sz$, is $\mathbf{Z}$-equivariant in the second variable ($H(s, n \cdot z) = s(n \cdot z) = n \cdot (sz) = n \cdot H(s,z)$) and thus descends to a smooth map $\bar{H} : [0,1] \times \mathrm{D}_\alpha \to \mathrm{D}_\alpha$, $\bar{H}(s,[z]) = [sz]$. This map provides a smooth homotopy from the identity map of $\mathrm{D}_\alpha$ (at $s = 1$) to the constant map onto the cone point $[0]$ (at $s = 0$). Thus, $\mathrm{D}_\alpha$ is smoothly contractible.

We have indeed an example of a principal bundle in diffeology over a contractible space, with a contractible fiber which, despite all this, is non-trivial. ☐

Note that, we could have started with the whole $\mathbf{C}$ instead of the closed disc D and avoided considering $\mathrm{S}^1$ as the border. This choice was made on purpose to show that we can use the freedom of structure in diffeology, and work with a manifold with boundary [PIZ13, §4.16] or not, without betraying the logic of diffeology.

## Class in $\mathsf{Fl}(\mathrm{D}_\alpha, \mathbf{R})$

The non-triviality can also be seen by identifying the class represented by $\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha$ in the classifying group $\mathsf{Fl}(\mathrm{D}_\alpha, \mathbf{R})$ [PIZ25]. Recall that $\mathsf{Fl}(\mathrm{D}_\alpha, \mathbf{R}) \cong \mathcal{C}^\infty(\mathrm{D}, \mathbf{R}) / \operatorname{im}(\Delta_\alpha)$, where $\Delta_\alpha(\sigma)(z) = \sigma(ze^{i2\pi\alpha}) - \sigma(z)$.

The bundle $\mathrm{W}_\alpha$ was constructed using the $\mathbf{Z}$-action corresponding to the cocycle $\tau(n, z) = n|z|^2$. This cocycle is represented by the function $f = \tau(1) \in \mathcal{C}^\infty(\mathrm{D}, \mathbf{R})$, namely $f(z) = |z|^2$. Thus, the class of the bundle is:

$$\operatorname{class}(\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha) = [f] \in \mathsf{Fl}(\mathrm{D}_\alpha, \mathbf{R}) \quad \text{where } f(z) = |z|^2.$$

A necessary condition for a function $f$ to be in the image $\operatorname{im}(\Delta_\alpha)$ is that its average over circles concentric with the origin must be zero. Let $M(f)(r) = \int_0^1 f(re^{i2\pi x})dx$. For $f(z) = |z|^2$, we compute:

$$M(f)(r) = \int_0^1 |re^{i2\pi x}|^2 dx = \int_0^1 r^2 dx = r^2.$$

Since $M(f)(r) = r^2$ is not identically zero for $r \in [0, 1]$, the function $f(z) = |z|^2$ cannot be in $\operatorname{im}(\Delta_\alpha)$. Therefore, the class $[f]$ is non-zero, and the bundle is non-trivial.

In the case where $\alpha$ is Diophantine, the map $M$ induces an isomorphism $\mathsf{Fl}(\mathrm{D}_\alpha, \mathbf{R}) \cong \mathcal{C}^\infty([0, 1], \mathbf{R})$. Under this isomorphism, our bundle represents the class corresponding to the non-zero function $F(r) = r^2$.

## Conclusion

We have constructed an explicit $(\mathbf{R}, +)$ principal bundle $\pi : \mathrm{W}_\alpha \to \mathrm{D}_\alpha$ over the quotient space $\mathrm{D}_\alpha = \mathrm{D}/\mathbf{Z}$. We verified that $\pi$ satisfies the rigorous diffeological definition of a principal bundle, requiring the action map F to be an induction [PIZ13, §8.11]. The main result is that this bundle is non-trivial, despite the fact that both the fiber $\mathbf{R}$ and the base space $\mathrm{D}_\alpha$ are smoothly contractible.

This "anomaly" stands in stark contrast to classical bundle theory. In classical algebraic topology and differential geometry, principal G-bundles over a base B are classified by homotopy classes of maps $[\mathrm{B}, \mathrm{BG}]$. When G (like $(\mathbf{R}, +)$) and B are contractible, the classifying space BG is weakly contractible, leading to the conclusion that such bundles over sufficiently nice bases are always trivial [Die70]. Classical proofs often rely on constructing global sections using homotopy extension or partitions of unity, tools generally unavailable for *smooth* maps in the category of diffeological spaces, especially over quotients like $\mathrm{D}_\alpha$ which lack local Euclidean structure.

Our example demonstrates that this classical principle fails in the broader category of diffeology, highlighting that homotopy theory alone is insufficient for classifying diffeological bundles. Even considering the diffeological homotopy theory [PIZ13, Chap. 5], the smooth contractibility of $\mathrm{D}_\alpha$ and $\mathbf{R}$ would likely predict triviality.[4] The non-triviality observed here is a purely *smooth* phenomenon, undetectable by standard homotopic methods. Indeed, the obstruction is not homotopic – the base space *is* contractible. It is a genuinely diffeological feature, tied to the specific smooth structure inherited by the quotient $\mathrm{D}_\alpha$ from the $\mathbf{Z}$-action on D. It arises from the impossibility of finding a *smooth* global section, which manifests as the non-solvability of the associated cohomological equation $f = \Delta_\alpha(\sigma)$ for $f(z) = |z|^2$ within the class of smooth functions $\sigma$.

This strongly suggests that a successful classification theory for principal bundles in diffeology, especially over non-manifold bases or with non-discrete groups, must employ tools sensitive to the fine smooth structure captured by plots. Cohomological approaches, such as diffeological Čech cohomology [PIZ24], appear more promising. Notably, the diffeological invariant $\mathsf{FI}(\mathrm{D}_\alpha, \mathbf{R})$, identified here with the non-zero class $[|z|^2]$ and corresponding to the Čech group $\mathsf{H}_\delta^{1,0}(\mathrm{D}_\alpha, \mathbf{R})$, directly captures this smooth obstruction and serves as evidence for the necessity of such specifically diffeological tools.

Ultimately, this example serves as a clear reminder that in diffeology, 'smoothly contractible' does not imply 'smoothly trivial' for principal bundles.

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[4]Actually all the homotopy group $\pi_k(\mathrm{W}_\alpha)$ are trivial, thanks to the long exact homotopy sequence of fiber bundles in diffeology [PIZ85, PIZ13], that reduces the homotopy of the total space to the homotopy of the base which is contractible, then trivial.

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