

AN INTRODUCTION TO DIFFEOLGY

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ABSTRACT. This text presents the basics of Diffeology and the main domains: Homotopy, Fiber Bundles, Quotients, Singularities, Cartan-de Rham Calculus — which form the core of differential geometry — from the point of view of this theory. We show what makes diffeology special and relevant in regard to these traditional subjects.

INTRODUCTION

Since its creation, in the early '80s, Diffeology has become an alternative, or a natural extension, to current Differential Geometry. With its developments in Higher Homotopy Theory, Fiber Bundles, Modeling Spaces, Cartan-de Rham Calculus, Moment Map and Symplectic¹ program... Diffeology covers now a large spectrum of traditional fields and deploys them from singular quotients to infinite dimensional spaces — and mixing the two — treating mathematical objects that are or not strictly speaking manifolds, and other constructions, on an equal footing in a common framework. We shall see some of its achievements through a series of examples, chosen because they are not covered by the geometry of manifolds, either because they involve infinite dimensional spaces or singular quotients or both of them.

The relative success of Diffeology comes from the conjunction of two strong properties of the theory:

- (1) First of all, the category $\{\text{Diffeology}\}$ is stable under all set-theoretic constructions: sums, products, subsets and quotients. It is then a complete and co-complete category. It is also Cartesian closed, the space of smooth maps in diffeology has itself a natural *functional diffeology*.

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¹About what does mean “to be symplectic” in diffeology see [PIZ10, PIZ16, PIZ17].

- (2) Second of all, quotient spaces, that are trivial under almost all other generalizations of differential calculus², get naturally a meaningful diffeology. That is in particular the case of *irrational tori*, quotients of the real line by strict subgroups. They own, as we shall see, a non trivial diffeology, capturing faithfully the intrication of the subgroup into its ambient space. This crucial property will be the *raison d'être* of many new constructions, or wide generalizations of classical constructions, that cannot exist in almost all the other extensions of Differential Geometry².

The treatment of singularities, maybe more than the inclusion of infinite dimensional spaces, reveals how Diffeology changes the way we understand smoothness, and discriminates this theory among its various competitors, see for example the use of dimension in diffeology [PIZ07] which discriminates between the various quotients $\mathbf{R}^n/\mathbf{O}(n)$.

The diagram (Fig. 1) tries to show the inclusivity of Diffeology, with respect to differential constructions, in comparison with the classical theory.

CONNECTING A FEW DOTS — The story began in the early '80s, when Jean-Marie Souriau introduced his *diffeologies* in a paper titled “Groupes Différentiels” [Sou80]. It was defined as a formal but light structure on groups³. And it was designed for dealing easily with infinite dimensional groups of diffeomorphisms, in particular the group of symplectomorphisms or quantomorphisms. He named the groups equipped with such a structure *groupes différentiels*⁴, as announced in the title of his paper⁵. His definition was made of five axioms that we can decompose today into a first group of three which gave later the notion of diffeology on arbitrary sets, and the last two ones for the compatibility with the internal group multiplication. But it took three years from 1980 to 1983 to separate the first general three axioms from the specific last two ones, and extract the general structure of *espace différentiel* from the definition of *groupe différentiel*. That was the ongoing work of Paul Donato on the covering of *homogeneous spaces* of

²I am referring here to the various generalizations of C^∞ differential geometry *à la* Sikorski or Frölicher [KriMic97]... I am not considering the various algebraic generalizations that do not play on the same level of intuition and generality, and do not concern exactly the same sets/objects, lattices instead of quotients for example etc.

³Compared to functional analysis heavy structures.

⁴That translates in English as “Differential” or “Differentiable Groups”.

⁵Actually, diffeologies are built on the model of K.-T. Chen *Differentiable Spaces* [Che77], for which the structure is defined over convex Euclidean subsets instead of open Euclidean domains. That makes diffeology more suitable to extend differential geometry than Chen's differentiable spaces, which focuses more on homology and cohomology theories.

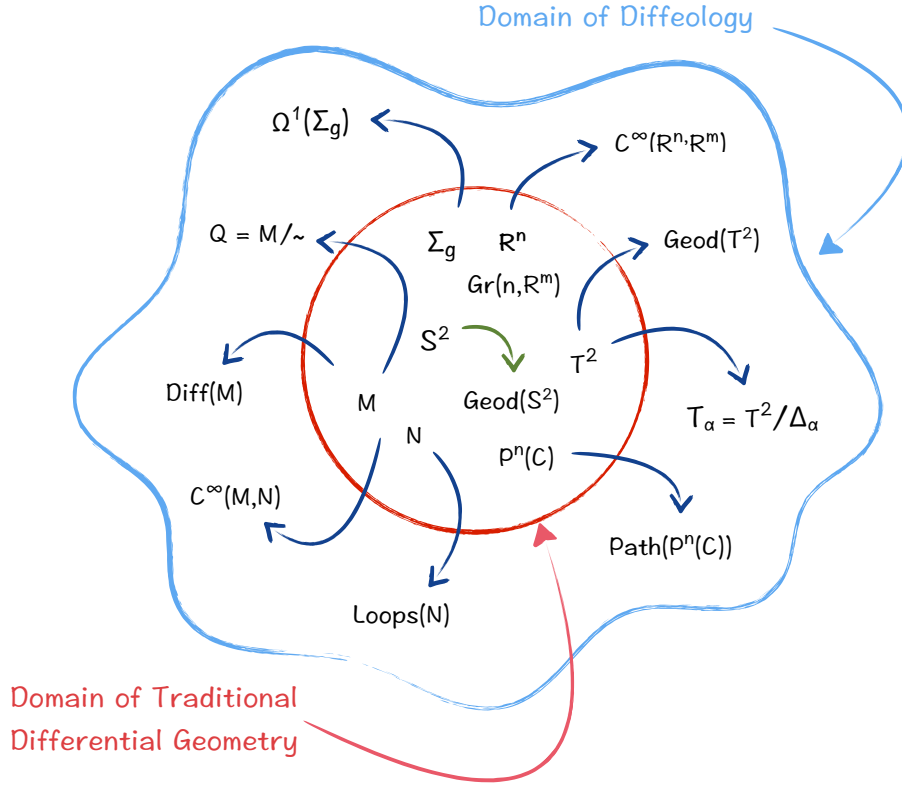


Figure 1 — The Scope of Diffeology.

differential groups, for one part, and mostly our joint work on the *irrational torus*, which made urgent and unavoidable a formal separation between Groups and Spaces in the domain of Souriau’s *differential structures*, as that gave a new spin at the theory. Actually, the first occurrence of the wording “espace différentiel” (for *differential space*), — between quotes — appears in the paper “Exemple de groupes différentiels : flots irrationnels sur le tore”, in July 1983 [DonIgl83, p.112 l.18]. The expression was used informally, for the purposes of the case, without giving a precise definition. The formal definition was published a couple of months later in “Groupes différentiels et physique mathématique”, in October 1983 [Sou83]. It took then a couple of years to bring the theory of *espaces différentiel* to a new level: with Souriau on generating quantum structures [Sou84], with Donato doctoral dissertation on covering of homogeneous spaces, defended in 1984 [Don84], and with my doctoral dissertation, defended in 1985, where was introduced higher homotopy theory and diffeological fiber bundles⁶, [Igl85].

⁶That is in this thesis that the first occurrence of *diffeological space* appears, in replacement of *differential space* which would have been confused with other structures.

What follows now is an attempt to introduce the main constructions and results in Diffeology, past and recent, through several meaningful examples. More details on the theory can be found in the textbook *Diffeology* [PIZ13] and related papers.

THANKS — I am grateful to the organizers of the Conference “New Spaces in Mathematics and Physics”, Gabriel Catren and Mathieu Anel, who gave me the opportunity to present the today state of Diffeology.

THE UNEXPECTED EXAMPLE: THE IRRATIONAL TORUS

Let us begin with the *irrational torus* T_α . At this time, in the early '80s, physicists were interested in quantizing 1-dimensional systems with a quasi periodic potential. That is, a function u from \mathbf{R} to \mathbf{R} which is the pullback of a smooth function U on a torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ along a line of slope α , with $\alpha \in \mathbf{R} - \mathbf{Q}$. If you prefer, $u(x) = U(e^{2i\pi x}, e^{2i\pi\alpha x})$, where $U \in C^\infty(T^2, \mathbf{R})$. That problem has drawn the attention of physicists and some mathematicians to the question about the statute of the quotient space

$$T_\alpha = T^2/\Delta_\alpha,$$

where $\Delta_\alpha \subset T^2$ is a 1-parameter subgroup, projection from \mathbf{R}^2 of the line $y = \alpha x$, that is, $\Delta_\alpha = \{(e^{2i\pi x}, e^{2i\pi\alpha x})\}_{x \in \mathbf{R}}$.

As a topological space T_α is trivial, because α is irrational and Δ_α “fills” the torus, that is, its closure is T^2 . And a trivial topological space is of no help. The various differentiable approaches⁷ lead also to dead ends, the only smooth maps from T_α to \mathbf{R} are constant maps, because the composition with the projection $\pi: T^2 \rightarrow T_\alpha$ should be smooth. For these reasons, the irrational torus was regarded by everyone as an extremely *singular* space.

But when we think a little bit, the irrational torus T_α is a group, moreover an Abelian group, and there is nothing more regular and homogeneous than a group. And that was the point that made us⁸, eager to explore T_α through the approach of *Diffeologies*. But, diffeologies were invented to study infinite dimensional groups, like groups of symplectomorphisms, and it was not clear that they could be of any help for the study of such “singular” spaces like T_α . Except that, because a diffeology on a set is defined by its *smooth parametrizations*, and because smooth parametrizations on T_α are just (locally) the composites of smooth parametrizations into T^2 by the projection $\pi: T^2 \rightarrow T_\alpha$, it was already clear that the diffeology of T_α was not the *trivial diffeology* made of all parametrizations, neither the *discrete diffeology* made only of locally constant parametrizations. Indeed, considering two smooth parametrizations⁹ P and P' in T^2 and there is a great chance that $\pi \circ P$ and $\pi \circ P'$ are different. And because P and P' are not any parametrizations but smooth ones, that makes T_α neither trivial, nor discrete. That was already a big

⁷Sikorski, Frölicher, ringed spaces etc.

⁸Paul Donato and myself

⁹A *parametrization* in a set X is just any map P defined on some open subset U of some numerical space \mathbf{R}^n into X .

difference with the traditional topological or differential approaches we mentioned above that makes T_α coarse.

But how to measure this non-triviality? To what ends did this new approach lead? That was the true question. We gave some answers in the paper *Exemple de groupes différentiels : flots irrationnels sur le tore* [DonIgl83]. Thanks to what Paul Donato had already developed at this time on the covering of *homogeneous differential spaces*, and which made the core of his doctoral dissertation [Don84], we could compute the fundamental group of T_α and its universal covering \tilde{T}_α . We found that

$$\pi_1(T_\alpha) = \mathbf{Z} \times \mathbf{Z} \quad \text{and} \quad \tilde{T}_\alpha = \mathbf{R},$$

with $\pi_1(T_\alpha)$ included in \mathbf{R} as $\mathbf{Z} + \alpha\mathbf{Z}$. That was a first insightful result showing the capability of diffeology concerning these spaces regarded ordinarily as (highly) singular. Needless to say that since then they became completely admissible.

In the second half of the seventies, the quantum mechanics of quasi-periodic potentials hit the field of theoretical physics.¹⁰ And the question of the structure of the space of leaves of the linear foliation of the 2-torus sparked a new level of interest. In particular, French theoretical physicists used the techniques of *Non-commutative Geometry* developed by Alain Connes. So, the comparison between the two theories became a natural question. The notion of fundamental group or universal covering was missing at that time in noncommutative geometry¹¹, then it was not with these invariants that we could compare the two approaches. That came eventually from the following result:

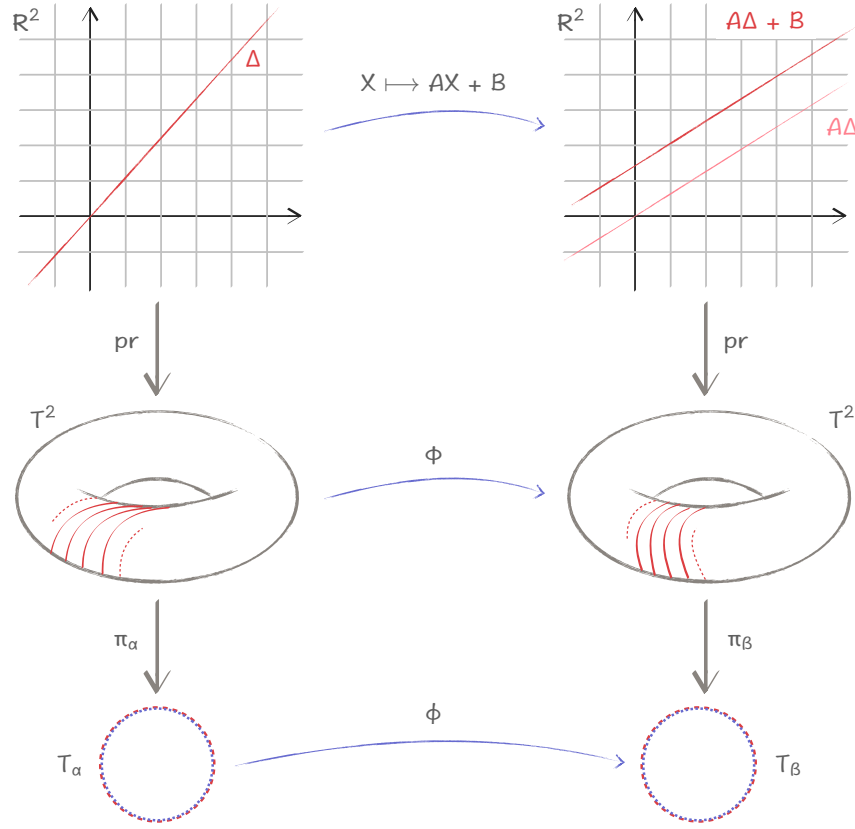
Theorem 1 (Donato-Iglesias, 1983). *Two irrational tori T_α and T_β are diffeomorphic if and only if α and β are conjugate modulo $\text{GL}(2, \mathbf{Z})$. That is, if there exists a matrix*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbf{Z}) \quad \text{such that} \quad \beta = \frac{a\alpha + b}{c\alpha + d}.$$

That result had its correspondence in Connes' theory, due to Marc Rieffel [Rie81], the \mathbf{C}^* -algebras associated with α and β are Morita-equivalent if and only if α and β are conjugate modulo $\text{GL}(2, \mathbf{Z})$. At this moment it was clear that diffeology was a serious alternative to noncommutative geometry. The advantage of diffeology was to stay close to the special intuition and concepts developed by geometers along centuries, if not millenia.

¹⁰The most famous paper on the question was certainly from Dinaburg and Sinai, on "The one-dimensional Schrödinger equation with a quasiperiodic potential" [DS75].

¹¹And are still missing today, except for a few attempts to fill the holes [IZL16].

Figure 2 — A diffeomorphism from T_α to T_β .

Actually, because of irrationality of α and β , we showed that a diffeomorphism $\phi: T_\alpha \rightarrow T_\beta$ could be fully lifted at the level of the covering \mathbf{R}^2 of T^2 into an affine diffeomorphism $\Phi(Z) = AZ + B$, where A preserves the lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$, that is, $A \in GL(2, \mathbf{Z})$, and $B \in \mathbf{R}^2$. The fact that these natural isomorphisms are preserved *a minima* in diffeology was of course an encouragement to continue the exploration of this example, and to push the test of diffeology even further.

And that is not all what could be said, and has been said, on irrational tori. Indeed, the category $\{\text{Diffeology}\}$ has many nice properties, in particular, as we shall see in the following, it is Cartesian closed. That means in particular that the set of smooth maps between diffeological spaces has a natural diffeology. We call it the *functional diffeology*. Thus any set of smooth maps between diffeological spaces has a fundamental group, in particular the group $\text{Diff}(T_\alpha)$ of diffeomorphisms of T_α . And its computation reserves us another surprise:

Theorem 2 (Donato-Iglesias, 1983). *The connected component of $\text{Diff}(T_\alpha)$ is T_α , acting by multiplication on itself. Its group of connected components is*

$$\pi_0(\text{Diff}(T_\alpha)) = \begin{cases} \{\pm 1\} \times \mathbf{Z} & \text{if } \alpha \text{ is quadratic,} \\ \{\pm 1\} & \text{otherwise.} \end{cases}$$

We recall that a number is quadratic if it is solution of a quadratic polynomial with integer coefficients. That result was indeed discriminating, since it did not appear in any other theory pretending to extend ordinary differential geometry. That clearly showed that diffeology, even for such a twisted example, was subtle enough to distinguish between numbers, quadratic or not. This property was not without reminding us about the periodicity of the continued fraction of real numbers.

This computation has been generalized on the $\pi_0(\text{Diff}(T_H))$ [IglLac90], where $H \subset \mathbf{R}^n$ is a totally irrational hyperplane and $T_H = \mathbf{T}^n/H$.

Theorem 3 (Iglesias-Lachaud, 1990). *Let $H \subset \mathbf{R}^n$ be a totally irrational hyperplane, that is, $H \cap \mathbf{Z}^n = \{0\}$. Let $T_H = \mathbf{T}^n/\text{pr}(H)$, where pr is the projection from \mathbf{R}^n to \mathbf{T}^n . Let $w = (1, w_2, \dots, w_n)$ be the normalized 1-form such that $H = \ker(w)$. The coefficients w_i are independent on \mathbf{Q} . Let $E_w = \mathbf{Q} + w_2\mathbf{Q} + \dots + w_n\mathbf{Q} \subset \mathbf{R}$ be the \mathbf{Q} -vector subspace of \mathbf{R} generated by the w_i . The subset*

$$K_w = \{\lambda \in \mathbf{R} \mid \lambda E_w \subset E_w\}$$

is an algebraic number field, a finite extension of \mathbf{Q} . Then, $\pi_0(\text{Diff}(T_H))$ is the group of the units of an order of K_w . Thanks to the Dirichlet theorem,

$$\pi_0(\text{Diff}(T_H)) \simeq \{\pm 1\} \times \mathbf{Z}^{r+s-1},$$

where r and s are the number of real and complex places of K_w .

Now, since we have seen what diffeology is capable of, we may have got your attention and it's time to give some details on what exactly a *diffeology* on an arbitrary set is. And then we shall see other applications of diffeology and examples, some of them famous.

WHAT IS A DIFFEOLOGY?

It is maybe time to give a precise meaning to what we claimed on the irrational torus. As a preamble, let us say that, contrarily to many constructions in differential geometry, diffeologies are defined just on sets, dry sets, without any pre-existing structure, neither topology nor anything else. That is important

enough to be underlined and that is what also makes the difference with the other approaches. A diffeology on a set X consists in declaring what parametrizations are smooth. Let us first introduce formally a fundamental word of this theory.

1. **PARAMETRIZATION** — We call *parametrization* in a set X , any map $P: U \rightarrow X$ such that U is some open subset of an Euclidean space. If we want to be specific we say that P is a n -parametrization when U is an open subset of \mathbf{R}^n . The set of all parametrizations in X is denoted by $\text{Params}(X)$. ►

Note that, there is no condition of injectivity on P , neither any topology precondition on X *a priori*. And also: we shall say *Euclidean domain* for “open subset of an Euclidean space”. Now,

2. **DEFINITION OF A DIFFEOLOGY** — A *diffeology* on X is any subset \mathcal{D} of $\text{Params}(X)$ that satisfies the following axioms.

- (1) **COVERING** : \mathcal{D} contains the constant parametrizations.
- (2) **LOCALITY** : Let P be a parametrization in X . If for all $r \in \text{dom}(P)$ there is an open neighbourhood V of r such that $P \upharpoonright V \in \mathcal{D}$, then $P \in \mathcal{D}$.
- (3) **SMOOTH COMPATIBILITY** : For all $P \in \mathcal{D}$, for all $F \in C^\infty(V, \text{dom}(P))$, where V is a Euclidean domain, $P \circ F \in \mathcal{D}$.

A space equipped with a diffeology is called a *diffeological space*. The elements of the diffeology of a diffeological space are called the *plots* of (or in) the space¹². ►

The first and foremost examples of diffeological spaces are the Euclidean domains, equipped with their *smooth diffeology*, that is, the ordinary smooth parametrizations. Pick for example the smooth \mathbf{R}^2 , we have a diffeology on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ by lifting locally the parametrizations in \mathbf{R}^2 . That is, a plot of T^2 will be a parametrization $P: r \mapsto (z_r, z'_r)$ such that, for every point in the domain of P , there exist two smooth parametrizations θ and θ' , in \mathbf{R} , defined in the neighbourhood of this point, with $(z_r, z'_r) = (e^{2i\pi\theta(r)}, e^{2i\pi\theta'(r)})$. That is the usual diffeology which makes T^2 the manifold we know. But that procedure can be extended naturally to T_α . Indeed, a parametrization $P: r \mapsto \tau_r$ in T_α is a plot if there exists locally, in the neighbourhood of every point in the domain of P , a parametrization $\zeta: r \mapsto (z_r, z'_r)$, such that $\tau_r = \pi(\zeta(r))$. That construction is summarized by

¹²There is a discussion about diffeology as a sheaf theory in [Igl87, Annex]. But we do not develop this formal point of view in general, because the purpose of diffeology is to minimise the technical tools in favour of a direct, more geometrical, intuition.

the sequence of arrows:

$$\mathbf{R}^2 \xrightarrow{\text{pr}} \mathbf{T}^2 \xrightarrow{\pi} \mathbf{T}_\alpha$$

That is exactly the diffeology we consider when we talk about the irrational torus. But this construction of diffeologies by *pushforward* is actually one of the fundamental constructions of the theory. However to go there we need previously to precise an important property of diffeologies.

3. COMPARING DIFFEOLOGIES — Inclusion defines a partial order in diffeology. If \mathcal{D} and \mathcal{D}' are two diffeologies on a set X , one says that \mathcal{D} is finer than \mathcal{D}' if $\mathcal{D} \subset \mathcal{D}'$, or \mathcal{D}' is coarser than \mathcal{D} . Moreover, diffeologies are stable by intersection, that gives the following property:

PROPOSITION. *This partial order, called fineness, makes the set of diffeologies on a set X , a lattice. That is, every set of diffeologies has an infimum and a supremum. As usual, the infimum of a family is obtained by intersection of the elements of the family. And the supremum is obtained by intersecting the diffeologies coarser than any element of the family.*

The infimum of every diffeology, the finest diffeology, is called the *discrete diffeology*. It consists of locally constant parametrizations. The supremum of every diffeology, the coarsest diffeology, is called the *coarse* or *trivial diffeology*, it is made of all the parametrizations. As we shall see these bounds will be useful to define diffeologies defined by properties (boolean functions). ►

Now, we can write the construction by pushforward:

4. PUSHING FORWARD DIFFEOLOGIES — Let $f : X \rightarrow X'$ be a map, let X be a diffeological space, with diffeology \mathcal{D} . Then, there exists a finest diffeology on X' such that f is smooth. It is called the *pushforward* of the diffeology of X . We denote it by $f_*(\mathcal{D})$. If f is surjective, its plots are the parametrizations P in X' that can be written $\text{Sup}_i f \circ P_i$, where the P_i are plots of X such that the $f \circ P_i$ are compatible, that is, coincide on the intersection of their domains, and Sup denotes the smallest common extension of the family $\{f \circ P_i\}_{i \in \mathcal{I}}$.

In particular, the diffeology of \mathbf{T}^2 is the pushforward of the smooth diffeology of \mathbf{R}^2 by pr , and the diffeology on \mathbf{T}_α is the pushforward of the diffeology of \mathbf{T}^2 by π . Or, what is equivalent, the pushforward of the smooth diffeology of \mathbf{R}^2 by the projection $\pi \circ \text{pr}$.

NOTE 1. Let $\pi : X \rightarrow X'$ be a map between diffeological spaces. We say that π is a *subduction* if it is surjective and if the pushforward of the diffeology of X coincides with the diffeology of X' . In particular $\text{pr} : \mathbf{R}^2 \rightarrow \mathbf{T}^2$ and $\pi : \mathbf{T}^2 \rightarrow \mathbf{T}_\alpha$

are two subductions. Subductions made a subcategory, since the composite of two subductions is again a subduction. ►

NOTE 2. Let X be a diffeological space and \sim be an equivalence relation on X . Let $Q = X/\sim$ be the quotient set¹³, that is,

$$Q = \{\text{class}(x) \mid x \in X\} \quad \text{and} \quad \text{class}(x) = \{x' \mid x' \sim x\}.$$

The pushforward on Q of the diffeology of X by the projection class is called the *quotient diffeology*. Equipped with the quotient diffeology, Q is called the *quotient space* of X by \sim . This is the first important property of the category $\{\text{Diffeology}\}$, it is closed by quotient.

Then, after having equipped the irrational tori with a diffeology (actually the quotient diffeology), we would compare different irrational tori with respect to diffeomorphisms. For that we need a precise definition.

5. SMOOTH MAPS — Let X and X' be two diffeological spaces. A map $f : X \rightarrow X'$ is *smooth* if for any plot P in X , $f \circ P$ is a plot in X' . The set of smooth maps from X to X' is denoted, as usual, by $C^\infty(X, X')$.

NOTE 1. The composition of smooth maps is smooth. Diffeological spaces, together with smooth maps, make a category which we denote by $\{\text{Diffeology}\}$.

NOTE 2. The isomorphisms of the category $\{\text{Diffeology}\}$ are the bijective maps, smooth as well as their inverses. They are called *diffeomorphisms*. ►

6. WHAT ABOUT MANIFOLDS ? — The time has come now to make a comment on manifolds. Every manifold is naturally a diffeological space, its plots are the smooth parametrizations in the usual sense. That makes the category $\{\text{Manifolds}\}$ a full and faithful subcategory of $\{\text{Diffeology}\}$. But we should insist on that diffeology must not be understood as a generalization of the theory of manifolds. It happens that, between many other things, diffeology extends the theory of manifolds, but its true nature is to extend the differential calculus on domains in Euclidean spaces, and it is the way it should be regarded. ►

Let us continue to explore our example of the irrational torus T_α . We still have to describe its fundamental group and its universal covering. Actually, the way it was treated in the founding paper [DonIgl83] used special definitions adapted only to groups and homogeneous spaces. Because at this time diffeology was only about groups. It was clear at this moment that considering diffeology only on groups was insufficient, and that we missed a real independent theory of fiber bundles

¹³I regard always a quotient set as a subset of the set of all subsets of a set.

and homotopy in diffeology. That was the content of the doctoral dissertation “*Fibrations Diffeologiques et Homotopie*” [Igl85].

Let us begin with the covering thing. A covering is a special kind of fiber bundle with a discrete fiber. But all these terms must be understood in the sense of diffeology, especially the word discrete. Let me give an example:

7. PROPOSITION — The rational numbers \mathbf{Q} are discrete in \mathbf{R} . ►

This is completely natural for everyone, except that this is false as far as topology is concerned. But we are talking diffeology: if we equip \mathbf{Q} with the diffeology *induced* by \mathbf{R} , it is not difficult to prove¹⁴ that \mathbf{Q} is discrete, that is, its diffeology is discrete. And that is the meaning we want to give to be a *discrete subset* of a diffeological space. Well, we still have to elaborate a little bit about *induced diffeology*.

8. PULLING BACK DIFFEOLOGIES — Let $f : X \rightarrow X'$ be a map, let X' be a diffeological space, with diffeology \mathcal{D}' . Then, there exists a coarsest diffeology on X such that f is smooth. It is called the *pullback* of the diffeology of X' . We denote it by $f^*(\mathcal{D}')$. Its plots are the parametrizations P in X such that $f \circ P$ is a plot of X' .

In particular that gives to any subset $A \subset X$, where X is a diffeological space, a *subset diffeology*, that is, $j^*(\mathcal{D})$, where $j : A \rightarrow X$ is the inclusion and \mathcal{D} is the diffeology of X . A subset equipped with the subset diffeology is called a *diffeological subspace*. Now, it is clear what does mean a *discrete subset* of a diffeological space, it is a subset such that its induced diffeology is discrete.

NOTE. Let $j : X \rightarrow X'$ be a map between diffeological spaces. We say that j is an *induction* if j is injective and if the pullback of the diffeology of X' coincides with the diffeology of X . For example, in the case of the irrational torus, the injection $t \mapsto (e^{2i\pi t}, e^{2i\pi \alpha t})$ from \mathbf{R} to T^2 is an induction. That means precisely that if $r \mapsto (z_r, z'_r)$ is smooth in T^2 but takes its values in Δ_α , then there exists a smooth parametrization $r \mapsto t_r$ in \mathbf{R} such that $z_r = e^{2i\pi t_r}$ and $z'_r = e^{2i\pi \alpha t_r}$. ►

Then, with this construction, the category $\{\text{Diffeology}\}$ which was closed by quotient is also closed by inclusion.

Now, continuing with our example, there are two reasons we need a good concept of fiber bundle in diffeology.

¹⁴A nice application of the intermediate value theorem.

- (a) Coverings of diffeological spaces should be defined as fiber bundles with discrete fiber. (We shall see then that we are even able to build a universal covering, unique up to isomorphism, for every diffeological space.)
- (b) If we look closely at the projection $\pi: T^2 \rightarrow T_\alpha$, we observe that this looks like a fiber bundle with fiber $\Delta_\alpha \simeq \mathbf{R}$. Thus, if the long homotopy sequence could apply to diffeological fiber bundles, we should get immediately $\pi_1(T_\alpha) = \pi_1(T^2) = \mathbf{Z}^2$, since the fiber \mathbf{R} is contractible.

FIBER BUNDLES

Of course, the classical definition of locally trivial bundle is helpless here since T_α has a trivial topology. The situation is more subtle, we are looking for a definition¹⁵ that satisfies the following two conditions:

- (1) The quotient of a diffeological group by any subgroup is a diffeological fibration, whatever the subgroup is.
- (2) For diffeological fibrations, the long exact homotopy sequence applies.

And since we refer to diffeological groups, we have to clarify their definition.

9. **DIFFEOLOGICAL GROUP** — A diffeological group is a group G which is also a diffeological space such that the multiplication $m: (g, g') \mapsto g g'$, and the inversion $\text{inv}: g \mapsto g^{-1}$, are smooth. ►

That needs a comment on the product of diffeological spaces, since we refer to the multiplication in a diffeological group G which is defined on the product $G \times G$.

10. **PRODUCT OF DIFFEOLOGICAL SPACES** — Let $\{X_i\}_{i \in \mathcal{I}}$ be any family of diffeological spaces. There exists on the product $X = \prod_{i \in \mathcal{I}} X_i$ a coarsest diffeology such that every projection $\pi_i: X \rightarrow X_i$ is smooth. It is called the *product diffeology*. A plot in X is just a parametrization $r \mapsto (x_{i,r})_{i \in \mathcal{I}}$ such that each $r \mapsto x_{i,r}$ is a plot of X_i .

NOTE. The category $\{\text{Diffeology}\}$ is then closed by products. ►

There are two equivalent definitions of diffeological bundles, this one is the pedestrian version.

11. **DIFFEOLOGICAL FIBER BUNDLES** — Let $\pi: Y \rightarrow X$ be a map with X and Y two diffeological spaces. We say that π is a fibration, with fiber F , if, for every

¹⁵The definition of fiber bundles in diffeology — their two equivalent versions — has been introduced in “Fibrés Difféologiques et Homotopie” [Igl85, Thesis].

plot $P: U \rightarrow X$, the pullback

$$\text{pr}_1: P^*(Y) \rightarrow U \quad \text{with} \quad P^*(Y) = \{(r, y) \in U \times Y \mid P(r) = \pi(y)\},$$

is locally trivial with fiber F . That is, every point in U has an open neighbourhood V such that there exists a diffeomorphism $\varphi: V \times F \rightarrow \text{pr}_1^{-1}(V) \subset P^*(Y)$, with $\text{pr}_1 \circ \varphi = \text{pr}_1$.

$$\begin{array}{ccccc} V \times F & \xrightarrow{\varphi} & P^*(Y) \upharpoonright V & \xrightarrow{\text{pr}_2} & Y \\ & \searrow \text{pr}_1 & \downarrow \text{pr}_1 & & \downarrow \pi \\ & & V & \xrightarrow{P \upharpoonright V} & X \end{array}$$

This definition extends the usual definition of smooth fiber bundles in differential geometry. But it contains more:

PROPOSITION. *The projection $\pi: G \rightarrow G/H$, where G is a diffeological group and $H \subset G$ is any subgroup, is a diffeological fibration.*

We have now the formal framework where $\pi: T^2 \rightarrow T_\alpha$ is a legitimate fiber bundle. This definition of fiber bundle satisfies also the long sequence of homotopy, we shall come back on the subject later.

Since we have a definition of fiber bundles, we inherit naturally the notion of a diffeological covering:

DEFINITION OF COVERINGS. *A covering of a diffeological space X is a diffeological fibration $\pi: \hat{X} \rightarrow X$ with a discrete fiber.*

NOTE. The projection $\pi: \mathbf{R} \rightarrow \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$ is a connected, simply connected covering, since \mathbf{R} is a diffeological group and $\mathbf{Z} + \alpha\mathbf{Z}$ a subgroup. The fact that $\mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$ is diffeomorphic to T_α is an exercise left to the reader. ►

There is an alternative to the definition of fiber bundles involving a groupoid of diffeomorphisms. This alternate definition is maybe less intuitive, but more internal to diffeology. It is based on the existence of a natural diffeology on the set of smooth maps between diffeological spaces.

12. THE FUNCTIONAL DIFFEOLOGY — Let X and X' be two diffeological spaces. There exists on $C^\infty(X, X')$ a coarsest diffeology such that the *evaluation map*

$$\text{ev}: C^\infty(X, X') \times X \rightarrow X' \quad \text{defined by} \quad \text{ev}(f, x) = f(x),$$

is smooth. Thus diffeology is called the *functional diffeology*.

NOTE 1. A parametrization $r \mapsto f_r$ is a plot for the functional diffeology if the map $(r, x) \mapsto f_r(x)$ is smooth.

NOTE 2. There exists a natural diffeomorphism between $C^\infty(X, C^\infty(X', X''))$ and $C^\infty(X \times X', X')$. That makes the category $\{\text{Diffeology}\}$ *Cartesian closed*. Which is a pretty nice property. ►

13. FIBER BUNDLES: THE GROUPOID APPROACH — Let $\pi: Y \rightarrow X$ be a map with X and Y two diffeological spaces. Consider the groupoid \mathbf{K} whose objects are the points of X and the arrows from x to x' are the diffeomorphisms from Y_x to $Y_{x'}$, where the preimages $Y_x = \pi^{-1}(x)$ are equipped with the subset diffeology.

There is a *functional diffeology* on \mathbf{K} that makes it a *diffeological groupoid*. The set $\text{Obj}(\mathbf{K}) = X$ is obviously equipped with its own diffeology. Next, a parametrization $r \mapsto \varphi_r$ in $\text{Mor}(\mathbf{K})$, defined on a domain U , will be a plot if

- (1) $r \mapsto (\text{src}(\varphi_r), \text{trg}(\varphi_r))$ is a plot of $X \times X$.
- (2) The maps $\text{ev}: U_{\text{src}} \rightarrow Y$ and $\overline{\text{ev}}: U_{\text{trg}} \rightarrow Y$, defined by $\text{ev}(r, y) = \varphi_r(y)$ and $\overline{\text{ev}}(r, y) = \varphi_r^{-1}(y)$, on $U_{\text{src}} = \{(r, y) \in U \times Y \mid y \in \text{def}(\varphi_r)\}$ and $U_{\text{trg}} = \{(r, y) \in U \times Y \mid y \in \text{def}(\varphi_r^{-1})\}$, are smooth, where these two sets are equipped with the subset diffeology of the product $U \times Y$.

We have then the following theorem [Igl85] :

THEOREM. *The map π is a fibration if and only if the characteristic map*

$$\chi: \text{Mor}(\mathbf{K}) \rightarrow X \times X, \quad \text{defined by} \quad \chi(f) = (\text{src}(f), \text{trg}(f)),$$

is a subduction.

Thanks to this approach we can construct, for every diffeological fiber bundle, a principal fiber bundle — by splitting the groupoid — with which the fiber bundle is associated. It is possible then to refine this construction and define fiber bundles with structures (linear etc.). ►

The next step about fiber bundles will be the establishment of the long homotopy sequence, but that requires a preliminary preparation, beginning by the definition of the homotopy groups.

HOMOTOPY THEORY

The basis of homotopy begins by understanding what does mean to be *homotopic*, that is, being in the same “place” or “component”.

14. HOMOTOPY AND CONNEXITY — Let X be a diffeological space, we denote by $\text{Paths}(X)$ the space of (smooth) paths in X , that is, $C^\infty(\mathbf{R}, X)$. The ends of a path γ are denoted by

$$\hat{0}(\gamma) = \gamma(0), \quad \hat{1}(\gamma) = \gamma(1) \quad \text{and} \quad \text{ends}(\gamma) = (\gamma(0), \gamma(1)).$$

We say that two points x and x' are *connected* or *homotopic* if there exists a path γ such that $x = \hat{0}(\gamma)$ and $x' = \hat{1}(\gamma)$.

To be connected defines an equivalence relation whose equivalence classes are called *connected components*, or simply *components*. The set of components is denoted by $\pi_0(X)$.

PROPOSITION. *The space X is the sum of its connected components, that is,*

$$X = \coprod_{X_i \in \pi_0(X)} X_i.$$

Moreover, the partition in connected components is the finest partition of X that makes X the sum of its parts. ►

It is time to give the precise definition of the sum of diffeological spaces that founds the previous proposition.

15. SUM OF DIFFEOLOGICAL SPACES — Let $\{X_i\}_{i \in \mathcal{I}}$ be a family of diffeological spaces. There exists a finest diffeology on the sum $X = \coprod_{X_i \in \mathcal{I}} X_i$ such that each injection $j_i = x \mapsto (i, x)$, from X_i to X , is smooth. We call it the *sum diffeology*. The plots of X are the parametrization $r \mapsto (i_r, x_r)$ such that $r \mapsto i_r$ is locally constant. ►

With that definition we close, by the way, one of the most interesting aspects of the category $\{\text{Diffeology}\}$. This category is stable by all the set theoretic constructions: sum, product, part, quotient. It is a complete and co-complete category, every direct or inverse limit of diffeological spaces have their natural diffeology. Moreover, the category is Cartesian closed.

That is by itself very interesting, as a generalization of the smooth category of Euclidean domains. But there are a few other generalizations that have these same properties (Frölicher spaces for example). What makes really the difference is that, these nice properties apply to a category that includes non trivially extremely singular spaces, as we have seen with the irrational tori. And that is what makes $\{\text{Diffeology}\}$ so unique. But let us now come back to the Homotopy Theory.

16. THE FUNDAMENTAL GROUP AND COVERINGS — Let X be a connected diffeological space, that is, $\pi_0(X) = \{X\}$. Let $x \in X$ be some point. We denote

by $\text{Loops}(X, x) \subset \text{Paths}(X)$ the subspaces of *loops* in X , based at x . That is, the subspace of paths ℓ such that $\ell(0) = \ell(1) = x$. As a diffeological space, with its functional diffeology, this space has a set of components. We define the *fundamental group*¹⁶ of X , at the point x as

$$\pi_1(X, x) = \pi_0(\text{Loops}(X, x), \hat{x}),$$

where \hat{x} is the constant loop $t \mapsto x$. The group multiplication on $\pi_1(X, x)$ is defined as usual, by concatenation:

$$\tau \cdot \tau' = \text{class} \left[t \mapsto \begin{cases} \ell(2t) & \text{if } t \leq 1/2 \\ \ell'(2t - 1) & \text{if } t > 1/2 \end{cases} \right],$$

where $\tau = \text{class}(\ell)$ and $\tau' = \text{class}(\ell')$. The inverse is given by $\tau^{-1} = \text{class}[t \mapsto \ell(1 - t)]$. Actually, we work with stationary paths for the concatenation to be smooth. A stationary path is a path that is constant on a small interval around 0 and also around 1. We have now a few main results.

PROPOSITION. *The groups $\pi_1(X, x)$ are conjugate to each other when x runs over X . We denote $\pi_1(X)$ their type.*

We say that X is *simply connected* if its fundamental group is trivial, $\pi_1(X) = \{0\}$. Now we have the following results.

THEOREM [UNIVERSAL COVERING]. *Every connected diffeological space X has a unique — up to isomorphism — simply connected covering $\pi: \tilde{X} \rightarrow X$. It is a principal fiber bundle with group $\pi_1(X)$. It is called the universal covering: every other connected covering is a quotient of \tilde{X} by a subgroup of $\pi_1(X)$.*

Actually the universal covering is the *half* of the *Poincaré Groupoid*, quotient of the space $\text{Paths}(X)$ by fixed-end homotopy relation [PIZ13, §5.15]. This is the second meaningful construction of groupoid in the development of diffeology.

MONODROMY THEOREM. *Let $f: Y \rightarrow X$ be a smooth map, where Y is a simply connected diffeological space. With the notations above, there exists a unique lifting $\tilde{f}: Y \rightarrow \tilde{X}$ once we fix $\tilde{f}(y) = \tilde{x}$, with $x = f(y)$ and $\tilde{x} \in \pi^{-1}(x)$.*

These are the more relevant constructions and results, familiar to the differential geometer, concerning the fundamental group on diffeological spaces. ►

17. TWO EXAMPLES OF COVERINGS — Let us now come back to our irrational torus $T_\alpha = T^2/\Delta_\alpha = [\mathbf{R}^2/\mathbf{Z}^2]/\Delta_\alpha$, that is, $T_\alpha = \mathbf{R}^2/[\mathbf{Z}^2 \times \{(x, \alpha x)\}_{x \in \mathbf{R}}] = [\mathbf{R}^2/\{(x, \alpha x)\}_{x \in \mathbf{R}}]/\mathbf{Z}^2$. The quotient $\mathbf{R}^2/\{(x, \alpha x)\}_{x \in \mathbf{R}}$ can be realized by \mathbf{R} with the projection $(x, y) \mapsto y - \alpha x$. Then, the action of \mathbf{Z}^2 on \mathbf{R}^2 induces the

¹⁶Formally speaking, the homotopy groups are objects of the category {Pointed Sets}. In particular $\pi_0(X, x) = (\pi_0(X), x)$.

action on \mathbf{Z}^2 on \mathbf{R} by $(n, m)(x, y) = (x + n, y + m) \mapsto y + m - \alpha(x + n) = y - \alpha x + (m - \alpha n)$. That is, $(n, m): t \mapsto t + m - \alpha n$. Therefore, $T_\alpha \simeq \mathbf{R}/\mathbf{Z} + \alpha\mathbf{Z}$. Since \mathbf{R} is simply connected, thanks to the theorem on existence and unicity of the universal covering, $\tilde{T}_\alpha = \mathbf{R}$ and the $\pi_1(T_\alpha)$ injects in \mathbf{R} as $\mathbf{Z} + \alpha\mathbf{Z}$. Of course there is no need all this gear of sophisticated tools to get this result, as we have seen in [DonIgl83]. But this shows how the pedestrian computation integrates seamlessly the general theory.

Another example¹⁷ of this theory, this one in infinite dimension: the universal covering of $\text{Diff}(S^1)$. The group is equipped with its functional diffeology. Let f be a diffeomorphism of $S^1 = \mathbf{R}/\mathbf{Z}$, assume that f fixes 1. Let $\pi: t \mapsto e^{2i\pi t}$, from \mathbf{R} to S^1 , be the universal covering. The composite $f \circ \pi$ is a plot. Thanks to the monodromy theorem, since \mathbf{R} is simply connected, $f \circ \pi$ has a unique smooth lifting $\tilde{f}: \mathbf{R} \rightarrow \mathbf{R}$, such that $\tilde{f}(0) = 0$.

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\tilde{f}} & \mathbf{R} \\ \pi \downarrow & & \downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Then, $\pi \circ \tilde{f} = f \circ \pi$ implies that $\tilde{f}(t + 1) = \tilde{f}(t) + k$, $k \in \mathbf{Z}$, but f cannot be injective unless $k = \pm 1$. Next, f being a diffeomorphism implies that \tilde{f} is a diffeomorphism of \mathbf{R} , that is a strictly increasing or decreasing function. Assume that \tilde{f} is increasing, then $\tilde{f}(t + 1) = \tilde{f}(t) + 1$. Thus, the *positive diffeomorphisms* of S^1 are the quotient of the increasing diffeomorphisms of \mathbf{R} satisfying that condition. Now, let $\tilde{f}_s(t) = f(t) + s(t - \tilde{f}(t))$, with $s \in [0, 1]$. We still have $\tilde{f}_s(t + 1) = \tilde{f}_s(t) + 1$ and $\tilde{f}'_s(t) = s + (1 - s)\tilde{f}'(t)$, which still is positive. Thus, since $\tilde{f}_0(t) = \tilde{f}(t)$ and $\tilde{f}_1(t) = t$, the group

$$\widetilde{\text{Diff}}_+(S^1) = \{\tilde{f} \in \text{Diff}_+(\mathbf{R}) \mid \tilde{f}(t + 1) = \tilde{f}(t) + 1\}$$

is contractible, hence simply connected. It is the universal covering of $\text{Diff}_+(S^1)$, the group of the positive diffeomorphisms of S^1 . The monodromy theorem indicates that there is only \mathbf{Z} different liftings \tilde{f} of a given diffeomorphism f . Hence, $\pi_1(\text{Diff}_+(S^1)) = \mathbf{Z}$. \blacktriangleright

¹⁷It was first elaborated by Paul Donato in his dissertation [Don84]. We just reinterpret it with our tools.

18. HIGHER HOMOTOPY GROUPS — Let X be a diffeological space and $x \in X$. Since $\text{Loops}(X, x)$ is a diffeological space that contains $\hat{x}: t \mapsto x$, there is no obstruction to define the *higher homotopy groups* by recursion:

$$\pi_n(X, x) = \pi_{n-1}(\text{Loops}(X, x), \hat{x}), \quad n \geq 1.$$

One can also define the recursion of diffeological spaces:

$$\begin{aligned} X_0 &= X, \quad x_0 = x \in X_0; \quad X_1 = \text{Loops}(X_0, x_0), \quad x_1 = [t \mapsto x_0] \in X_1; \dots \\ \dots; \quad X_n &= \text{Loops}(X_{n-1}, x_{n-1}), \quad x_n = [t \mapsto x_{n-1}] \in X_n; \dots \end{aligned}$$

Thus, $\pi_n(X, x) = \pi_{n-1}(\text{Loops}(X, x), \hat{x})$, that is, $\pi_n(X, x) = \pi_{n-1}(X_1, x_1) = \dots = \pi_1(X_{n-1}, x_{n-1}) = \pi_0(X_n, x_n)$. Since $\pi_n(X, x)$ is the fundamental group of a diffeological space, it is a group. And that is the formal definition of the *n-th homotopy group*¹⁸ of X at the point x .

We can feel in particular here, the benefits of considering all these spaces: X , $\text{Paths}(X)$, $\text{Loops}(X)$ etc., on an equal footing. Being all diffeological spaces, the recursion does not need any supplementary construction than the ones already defined. ►

19. THE HOMOTOPY SEQUENCE OF A FIBER BUNDLE — One of the most important property of diffeological fiber bundles is their long homotopy sequence. Let $\pi: Y \rightarrow X$ be a fiber bundle with fiber F . Then, there is a long exact sequence of group homomorphisms [Igl85],

$$\begin{aligned} \dots \rightarrow \pi_n(F) \rightarrow \pi_n(Y) \rightarrow \pi_n(X) \rightarrow \pi_{n-1}(F) \rightarrow \dots \\ \dots \rightarrow \pi_0(F) \rightarrow \pi_0(Y) \rightarrow \pi_0(X) \rightarrow 0. \end{aligned}$$

As usual in these cases, if the fiber is homotopy trivial then the base space has the homotopy of the total space. And that is what happens for the irrational torus $\pi_k(T_\alpha) = \pi_k(T^2)$, $k \in \mathbf{N}$.

Let us consider another example, in infinite dimension this time. Let S^∞ be the *infinite dimensional sphere* in the Hilbert space $\mathcal{H} = \ell^2(\mathbf{C})$. We equip first \mathcal{H} with the fine diffeology of vector space [PIZ13, §3.7]. Then, we prove that $S^\infty \subset \mathcal{H}$, equipped with the subset diffeology, is contractible [PIZ13, §4.10]. Then we consider the *infinite projective space* $\mathbf{CP}^\infty = S^\infty/S^1$, where S^1 acts on the ℓ^2 sequences by multiplication. The projection $\pi: S^\infty \rightarrow \mathbf{CP}^\infty$ is then a diffeological principal fibration with fiber S^1 . The homotopy exact sequence gives then $\pi_2(\mathbf{CP}^\infty) = \mathbf{Z}$ and $\pi_k(\mathbf{CP}^\infty) = 0$ if $k \neq 2$, what is what we expected.

¹⁸It is abelian for $n = 2$.

That would prove if necessary that we can work on singular constructions or infinite dimensional spaces using the same tools and the same intuition than when we deal with ordinary differential geometry. ►

20. CONNECTIONS ON FIBER BUNDLES AND HOMOTOPY INVARIANCE — Let $\pi: Y \rightarrow X$ be a principal fiber bundle with group G . That fabricates a new principal fibration $\pi_*: \text{Paths}(Y) \rightarrow \text{Paths}(X)$ with structure group $\text{Paths}(G)$. Roughly speaking, a *connection* on π is a reduction of this *paths fiber bundle* to the subgroup $G \subset \text{Paths}(G)$, consisting of constant paths [PIZ13, §8.32]. We require for this reduction to satisfy a few axioms: locality (sheaf condition on the interval of \mathbf{R}), compatibility with concatenation etc. The main point is that once we have a path γ in X and a point y over $x = \gamma(t)$, then there exists a unique *lift* $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(t) = y$, it is called the *horizontal lift*. Moreover, if $y' = g_Y(y)$, the lift γ' of γ passing at y' at the time t is the shifted $\gamma' = g_Y \circ \gamma$. That property is exactly what we call a reduction of π_* to G .

An important consequence of the existence of a connection on a principal fiber bundle is the homotopy invariance of pullbacks.

PROPOSITION. *Let $\pi: Y \rightarrow X$ be a principal fiber bundle with group G , equipped with a connection. Let $t \mapsto f_t$ be a smooth path in $C^\infty(X', X)$, where X' is any diffeological space. Then, the pullbacks $\text{pr}_0: f_0^*(Y) \rightarrow X'$ and $\text{pr}_1: f_1^*(Y) \rightarrow X'$ are equivalent.*

COROLLARY. *Any diffeological fiber bundle equipped with a connection over a contractible space is trivial.*

We know that this is always true in ordinary differential geometry, because every principal bundle over a manifold can be equipped with a connection. ►

21. THE GROUP OF FLOWS OF A SPACE — Connections are usually defined in ordinary differential geometry by a differential form with values in some Lie algebra. As we have seen above, that is not the way chosen in diffeology. There are a few good reasons for that. First of all, for such important property like the homotopy invariance of pullbacks, a broad definition of connection is enough. Moreover, we have no indisputable concept of Lie algebra in diffeology¹⁹, and choosing one definition rather than another would link a universal concept, like *parallel transport*, to an arbitrary choice.

And there is also a new diffeological construction where the difference between general-connections versus form-valued-connections is meaningful enough to

¹⁹Even for the moment map in symplectic diffeology we do not need the definition of a Lie algebra, as we shall see later on.

justify *a posteriori* our choice. That is the computation of the *group of flows* over the irrational torus [PIZ13, §8.39]. Let us begin by a definition:

DEFINITION. *We shall call flow over a diffeological space X , any $(\mathbf{R}, +)$ -principal bundle over X .*

There is an additive operation on the set $\text{Flows}(X)$ of (equivalence classes of) flows. Let $a = \text{class}(\pi: Y \rightarrow X)$ and $a' = \text{class}(\pi': Y' \rightarrow X)$ be two classes of flows. Consider the pullback $\pi^*(Y') = \{(y, y') \in Y \times Y' \mid \pi(y) = \pi'(y')\}$. It is a $(\mathbf{R}^2, +)$ principal bundle over X by $(y, y') \mapsto \pi(y) = \pi'(y')$. Let Y'' be the quotient of $\pi^*(Y')$ by the antidiagonal action of \mathbf{R} , that is, $\underline{t}(y, y') = (t_Y(y), -t_{Y'}(y'))$. And let $\pi'': Y'' \rightarrow X$ be the projection $\pi''(\text{class}(y, y')) = \pi(y) = \pi'(y')$. We define then $a + a' = a''$ with $a'' = \text{class}(\pi'': Y'' \rightarrow X)$.

The set $\text{Flows}(X)$, equipped with this addition, is an abelian group. The neutral element is the class of the trivial bundle, the inverse of a flow is the same bundle but with the inverse action of \mathbf{R} . Note that this group is a kind of Picard Group on a diffeological space but with \mathbf{R} instead of S^1 as structure group. And if this group doesn't appear in ordinary differential geometry, it is because every principal bundle with fiber \mathbf{R} over a manifold is trivial. But that is not the case in diffeology, and we know one such non trivial bundle, the irrational torus $\pi: T^2 \rightarrow T_\alpha$.

Let $\pi: Y \rightarrow T_\alpha$ be a flow. Consider the pullback $\text{pr}_1: \text{pr}^*(Y) \rightarrow \mathbf{R}$, where $\text{pr}: \mathbf{R} \rightarrow T_\alpha$ is the universal covering. It is an \mathbf{R} -principal fiber bundle over \mathbf{R} , then it is trivial. Let $\varphi: \mathbf{R} \times \mathbf{R} \rightarrow \text{pr}^*(Y)$ be an isomorphism. Thus $Y \simeq \mathbf{R} \times \mathbf{R} / \text{pr}_2 \circ \varphi$. But $\text{pr}_2 \circ \varphi$ is any lifting on the second factor of $\mathbf{R} \times \mathbf{R}$, of the action of $\mathbf{Z} \oplus \alpha\mathbf{Z}$ on the first one.

$$\begin{array}{ccccc}
 \mathbf{R} \times \mathbf{R} & \xrightarrow{\varphi} & \text{pr}^*(Y) & \xrightarrow{\text{pr}_2} & Y \\
 & \searrow \text{pr}_1 & \downarrow \text{pr}_1 & & \downarrow \pi \\
 & & \mathbf{R} & \xrightarrow{\text{pr}} & T_\alpha
 \end{array}$$

A lifting of a subgroup $\Gamma \subset \mathbf{R}$ on the second factor of $\mathbf{R} \times \mathbf{R}$, where Γ acts by translation on the first factor is given, for all $k \in \Gamma$ and $(x, \tau) \in \mathbf{R} \times \mathbf{R}$, by

$$k: (x, t) \mapsto (x + k, t + \tau(k)(x)),$$

where $\tau: \Gamma \rightarrow C^\infty(\mathbf{R})$ is a cocycle satisfying

$$\tau(k + k')(x) = \tau(k)(x + k') + \tau(k')(x).$$

Two cocycles τ and τ' define the same flow if they differ from a coboundary $\delta\sigma$:

$$\tau'(k)(x) = \tau(k)(x) + \sigma(x+k) - \sigma(x), \quad \text{with } \sigma \in C^\infty(\mathbf{R}).$$

In other words,

$$\text{Flows}(\mathbf{R}/\Gamma) = H^1(\Gamma, C^\infty(\mathbf{R})).$$

Applied to $\Gamma = \mathbf{Z} + \alpha\mathbf{Z}$, that gives $\text{Flows}(T_\alpha)$ equivalent to the group of real 1-periodic functions f , after some normalization, modulo the relation:

$$f \sim f' \quad \text{if} \quad f'(x) = f(x) + g(x + \alpha) - g(x).$$

This relation is known as the *small divisors Arnold's cohomology relation*. The solution depends on the arithmetic of α : if α is a diophantine or a Liouville number, $\text{Flows}(T_\alpha)$ is 1-dimensional or ∞ -dimensional²⁰.

Moreover, every flow $\pi: Y \rightarrow T = \mathbf{R}/\Gamma$ defined by a cocycle τ , can be naturally equipped with the connection associated with the covering $\text{pr}: \mathbf{R} \rightarrow T$ [PIZ13, §8.36]. However, not all these bundles support a connection form [PIZ13, §8.37], only those whose cocycle τ defining π is equivalent to a homomorphism from Γ to \mathbf{R} , see [PIZ13, Exercise 139]. In other words, if the cocycle τ is not cohomologous to a homomorphism, then there is no connection which can be defined by a connection form. In particular, for T_α , the only flow equipped with connections defined by a connection form is the Kronecker flow (with arbitrary speeds). ►

MODELING DIFFEOLOGY

Now we have seen a few constructions in diffeology and applications to unusual situations: singular quotients, infinite dimension spaces. It will be interesting to revisit some constructions of differential geometry, and see what diffeology can do with them.

22. MANIFOLDS — Every manifold owns a natural diffeology, for which the plots are the smooths parametrizations. There is a definition internal to the category diffeology²¹:

DEFINITION. *An n -manifold is a diffeological space locally diffeomorphic to \mathbf{R}^n at each point.*

With this definition, as diffeological spaces, $\{\text{Manifolds}\}$ form a full subcategory of the category $\{\text{Diffeology}\}$. ►

²⁰As $H^1(\Gamma, C^\infty(\mathbf{R}))$, the group $\text{Flows}(\mathbf{R}/\Gamma)$ is obviously a real vector space.

²¹We could use too the concept of generating families, see [PIZ13, §1.66].

Of course, this definition needs to precise the use of the wording *locally diffeomorphic*.

23. LOCAL SMOOTH MAPS, D-TOPOLOGY &C. — Very soon after the initial works in diffeology, it was clear that we needed to enrich the theory with local considerations, which were missing until then. To respect the spirit of diffeology we defined directly the concept of local smoothness [Igl85], as follows.

DEFINITION. *Let X and X' be two diffeological spaces. Let f be a map from a subset $A \subset X$ into X' . We say that f is local smooth if for each plot P in X , $f \circ P$ is a plot of X' .*

Note that $f \circ P$ is defined on $P^{-1}(A)$, and a first condition for $f \circ P$ to be a plot of X' is that $P^{-1}(A)$ is open. That leads to the second definition.

DEFINITION. *We say that a subset $A \subset X$ is D-open if $P^{-1}(A)$ is open for all plots P in X . The D-open subsets in X define a topology on X called the D-Topology.*

We have then the following proposition linking these two definitions:

PROPOSITION. *A map f defined on a subset $A \subset X$ to X' is local smooth if and only if: A is D-open and $f : A \rightarrow X'$ is smooth when A is equipped with the subset diffeology.*

To avoid misunderstanding and signify that f is local smooth — not just smooth for the subset diffeology — we note $f : X \supset A \rightarrow X'$.

Next, since we have local smooth maps, we have *local diffeomorphisms* too.

DEFINITION. *We say that $f : X \supset A \rightarrow X'$ is a local diffeomorphism if f is injective, local smooth, and if its inverse $f^{-1} : X' \supset f(A) \rightarrow X$ is local smooth. We say that f is a local diffeomorphism at $x \in X$ if there is a superset A of x such that $f \upharpoonright A : X \supset A \rightarrow X'$ is a local diffeomorphism.*

In particular, these definitions give a precise meaning to the sentence “*the space X is locally diffeomorphic to X' at each/some point*”. ►

24. ORBIFOLD AS DIFFEOLOGIES. — The word *orbifold* was coined by Thurston [Thu78] in 1978 as a replacement for *V-manifold*, a structure invented by Ichiro Satake in 1956 [Sat56].

These new objects have been introduced to describe the smooth structure of spaces that look like manifolds, except around a few points, where they look like quotients of Euclidean domains by finite linear groups. Satake captured

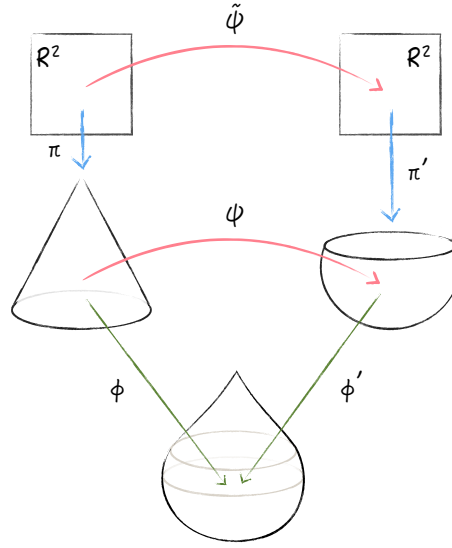


Figure 3 — The Teardrop as Satake's Orbifold.

the smooth structure around the singularities by a family of compatible *local uniformizing systems* defining the orbifold²².

The Figure 3 gives an idea about what would be an uniformizing system for the “Teardrop” with one conic singularity.

The main problem with Satake's definition is that it does not lead to a satisfactory notion of smooth maps between orbifolds, and therefore prevents the conception of a category of orbifolds. Indeed, in [Sat57, page 469], Satake writes this footnote:

“The notion of C^∞ -map thus defined is inconvenient in the point that a composite of two C^∞ -maps defined in a different choice of defining families is not always a C^∞ map.”

And, for a mathematician, that is very annoying.

Considering orbifolds as diffeologies solved the problem. Indeed, in [IKZ10] we defined a *diffeological orbifold* by a modeling process in the same spirit than for smooth manifolds

DEFINITION. *An orbifold is a diffeological space which is locally diffeomorphic, at each point, to some quotient space \mathbf{R}^n / Γ , for some finite subgroup Γ of the linear group $\mathrm{GL}(n, \mathbf{R})$, depending possibly on the point.*

²²We will not discuss this construction here. The original description by Satake is found in [Sat56], and a discussion on this definition in [IKZ10].

The Figure 4 gives an idea about what is a diffeological orbifold: the Teardrop as some diffeology on the sphere S^2 .

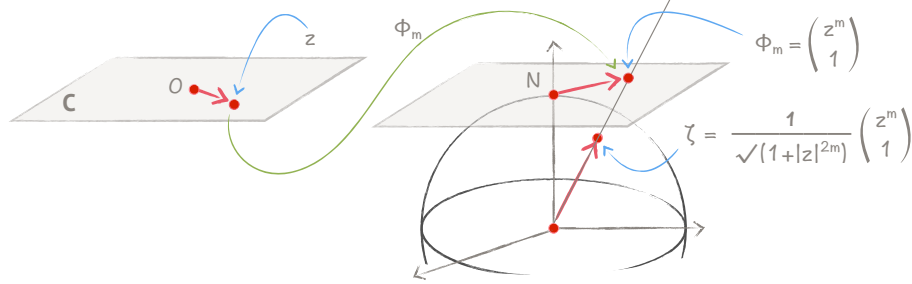


Figure 4 — The Teardrop as Diffeology.

Once this definition given, we could prove *op. cit.* that, according to this definition, every Satake defining family of local uniformizing system was associated with a diffeology of orbifold, and conversely that, every diffeological orbifold was associated with a Satake defining family of local uniformizing system. And we proved that these constructions are inverse from each other, modulo equivalence.

Thus, the diffeology framework fulfilled Satake's program by embedding orbifolds into diffeological spaces, and providing them naturally with good workable smooth mappings.

The difficulty met by Satake is subtle and can be explained as follows: he tried to define smooth maps between orbifolds as maps which have equivariant liftings on the level of Euclidean domains, before quotienting. But the embedding of orbifolds into {Diffeology} shows that, if that is indeed satisfied for local diffeomorphisms, see [IKZ10, Lemma 20, 21, 22], it is not necessarily the case for ordinary smooth maps as shows this counter-example 25 *op. cit.*

Consider the cone orbifolds $\mathcal{Q}_n = \mathbb{R}^2 / \mathbb{Z}_n$, and let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$f(x, y) = \begin{cases} 0 & \text{if } r > 1 \text{ or } r = 0 \\ e^{-1/r} \rho_n(r)(r, 0) & \text{if } \frac{1}{n+1} < r \leq \frac{1}{n} \text{ and } n \text{ is even} \\ e^{-1/r} \rho_n(r)(x, y) & \text{if } \frac{1}{n+1} < r \leq \frac{1}{n} \text{ and } n \text{ is odd,} \end{cases}$$

where $r = \sqrt{x^2 + y^2}$ and ρ_n is a smooth non-zero real function which is zero outside the interval $[1/(n+1), 1/n]$. Then, for all integers m dividing n , f projects onto a smooth map $\varphi: \mathcal{Q}_m \rightarrow \mathcal{Q}_n$ which cannot be lifted locally in an equivariant smooth map, on an neighborhood of 0.

Once again, diffeology structured the problem in a way that almost solves it. ►

25. NONCOMMUTATIVE GEOMETRY & DIFFEOLOGY: THE CASE OF ORBIFOLDS. — The question of a relation between Diffeology and Noncommutative Geometry appeared immediately with the study of the irrational torus [DonIgl83]. The condition of diffeomorphy between two irrational tori T_α and T_β , that is, α and β conjugate modulo $GL(2, \mathbf{Z})$, coincided clearly in noncommutative geometry, with the Morita-equivalent of the \mathbf{C}^* -algebras associated with the foliations [Rie81]. That suggested a structural relationship between Diffeology and Noncommutative Geometry deserving to be explored, but that question had been left aside since then.

We recently reopened the case of this relationship, considering orbifolds. And we exhibited a simple construction associating naturally, with every diffeological orbifold a \mathbf{C}^* -algebra, such that two diffeomorphic orbifolds give two Morita equivalent algebra [IZL16].

Before continuing, we need to recall a general definition [PIZ13, §1.66 & 1.76].

DEFINITION. *Let X be a diffeological space. We define the nebula of a set \mathcal{F} of plots in X as the diffeological sum*

$$\mathcal{N} = \coprod_{P \in \mathcal{F}} \text{dom}(P) = \{(P, r) \mid P \in \mathcal{F}, r \in \text{dom}(P)\},$$

where each domain is equipped with its standard smooth diffeology. Then we define the evaluation map

$$\text{ev}: \mathcal{N} \rightarrow X \quad \text{by} \quad \text{ev}: (P, r) \mapsto P(r).$$

We say that \mathcal{F} is a generating family of X if ev is a subduction.

Now, let \mathcal{Q} be an orbifold. A local diffeomorphism F from a quotient \mathbf{R}^n / Γ to \mathcal{Q} will be called a *chart*. A set \mathcal{A} of charts whose images cover \mathcal{Q} will be called an *atlas*. With every atlas \mathcal{A} is associated a special generating family \mathcal{F} by considering the strict lifting of the charts $F \in \mathcal{A}$ to the corresponding \mathbf{R}^n . Precisely, let $\pi_\Gamma = \mathbf{R}^n \rightarrow \mathbf{R}^n / \Gamma$, $F: \mathbf{R}^n / \Gamma \supset \text{dom}(F) \rightarrow \mathcal{Q}$, then $\mathcal{F} = \{F \circ \pi_\Gamma\}_{F \in \mathcal{A}}$. We call \mathcal{F} the *strict generating family* associated with \mathcal{A} , and we denote by \mathcal{N} its nebula.

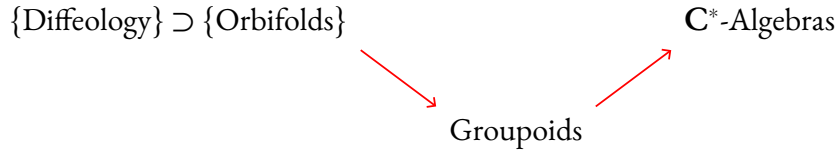
Next, we consider the groupoid \mathbf{G} whose objects are the points of the nebula \mathcal{N} , and the arrows, the germs of local diffeomorphisms ϕ of \mathcal{N} that project on the identity by ev , that is, $\text{ev} \circ \phi = \text{ev}$. We call \mathbf{G} the *structure groupoid* of \mathcal{Q} . Then, for a suitable but natural *functional diffeology* on \mathbf{G} , we have [IZL16]:

THEOREM 1. *The groupoid \mathbf{G} is Hausdorff and etale. The groupoids associated with different atlases are equivalent as categories.*

Hence, thanks to the étale property we can associate a \mathbf{C}^* -algebra \mathfrak{A} with \mathbf{G} by the process described by Renaud in [Ren80]. We have then

THEOREM2. *The groupoids associated with different atlases of an orbifold are equivalent in the sense on Muhly-Renault-Williams [MRW87]. Therefore, their \mathbf{C}^* -algebras are Morita-equivalent.*

This Morita-equivalence between the \mathbf{C}^* -algebras associated with different atlases of the orbifold is the condition required to make this construction meaningful and categorical. That construction is the first bridge between diffeology and noncommutative geometry, it gives an idea where and how diffeology and noncommutative geometry respond to each other, at least at the level of orbifolds²³.



As an example, let us consider the simple orbifold $\Delta_1 = \mathbf{R}/\{\pm 1\}$. The structure of the orbifold is represented by the pushforward of the standard diffeology from \mathbf{R} to $[0, \infty[$, by the square map $\text{sqr}: t \mapsto t^2$. The singleton $\mathcal{F} = \{\text{sqr}\}$ is a strict generating family, and the structure groupoid \mathbf{G} is the groupoid of the action of $\Gamma = \{\pm 1\}$, that is,

$$\text{Obj}(\mathbf{G}) = \mathbf{R} \quad \text{and} \quad \text{Mor}(\mathbf{G}) = \{(t, \varepsilon, \varepsilon t) \mid \varepsilon = \pm 1\} \simeq \mathbf{R} \times \{\pm 1\}.$$

A continuous function f on $\text{Mor}(\mathbf{G})$ to \mathbf{C} is a pair of functions $f = (a, b)$, where $a(t) = f(t, 1)$ and $b(t) = f(t, -1)$. With this convention, the algebra of the orbifold is then represented by a submodule of $M_2(\mathbf{C}) \otimes \mathcal{C}^0(\mathbf{R}, \mathbf{C})$,

$$f = (a, b) \mapsto M = \left[t \mapsto \begin{pmatrix} a(t) & b(-t) \\ b(t) & a(-t) \end{pmatrix} \right],$$

with $M^*(t) = [\tau M(t)]^*$. The right upscript τ represents the transposition, and the $*$ the complex conjugation element by element.

Note that we still trace the orbifold in the characteristic polynomial $P_M(\lambda)$, which is invariant by the action of $\{\pm 1\}$, and is then defined on the orbifold Δ_1 itself, $P_M(\lambda): t \mapsto \lambda^2 - \lambda \text{tr}(M(t)) + \det(M(t))$, where $\text{tr}(M(t)) = a(t) + a(-t)$ and $\det(M(t)) = a(t)a(-t) - b(t)b(-t)$ are obviously $\{\pm 1\}$ invariant. ►

²³There is a more general subcategory of $\{\text{Diffeology}\}$ for which such a construction leads to the same conclusion. It is the subject of a work in progress.

26. MANIFOLDS WITH BOUNDARY AND CORNERS. — One day, in 2007, I received an e-mail from a mathematical physicist, wondering how diffeology behaves around the corners... Here is an excerpt:

“I have just one worry about this theory. I found it very difficult to check that there’s a diffeology on the closed interval $[0, 1]$ such that a smooth function $f : [0, 1] \rightarrow \mathbf{R}$ is smooth in the usual sense, even at the endpoints. ... / ... This problem would become very easy to solve using Chen’s definition of smooth space, which allows for plots whose domain is any convex subset of \mathbf{R}^n .”

And our colleague to end his remark by this wish, that people interested in diffeology “are hoping you could either solve this problem in the context of diffeologies, or switch to Chen’s definition...”.

The good news is that it is not necessary to give up diffeology to be happy. There is indeed a diffeology on $[0, 1]$ such that “a smooth function $f : [0, 1] \rightarrow \mathbf{R}$ is smooth in the usual sense”. And that diffeology is simply the subset diffeology; precisely [PIZ13, §4.13]:

THEOREM. *Let $H_n \subset \mathbf{R}^n$ be the half-space defined by $x_1 \geq 0$, equipped with the subset diffeology. Let $f \in C^\infty(H_n, \mathbf{R})$, then there exists a smooth function F , defined on an open superset of H_n in \mathbf{R}^n , such that $f = F \upharpoonright H_n$.*

This proposition is a consequence of a famous Whitney theorem on extension of smooth even functions [Whi43]. By the way, it gives a solid basis to the vague concept of “smooth in the usual sense”, as to be smooth in the usual sense means then to be smooth as diffeological space.

We can investigate further and characterize the local diffeomorphisms of half-spaces [PIZ13, §4.14].

THEOREM. *A map $f : A \rightarrow H_n$, with $A \subset H_n$, is a local diffeomorphism for the subset diffeology if and only if: A is open in H_n , f is injective, $f(A \cap \partial H_n) \subset \partial H_n$, and for all $x \in A$ there exist an open ball $\mathcal{B} \subset \mathbf{R}^n$ centered at x and a local diffeomorphism $F : \mathcal{B} \rightarrow \mathbf{R}^n$ such that f and F coincide on $\mathcal{B} \cap H_n$.*

Thanks to this theorem, it is then easy to include the manifolds with boundary into the category {Diffeology}, in the same way we included the categories {Manifolds} and {Orbifolds}.

DEFINITION. *An n -dimensional manifold with boundary is a diffeological space X which is diffeomorphic, at each point, to the half-space H_n . We say that X is modeled on H_n .*

Thanks to the previous theorems, it is clear that this definition covers completely, and not more, the usual definition of manifold with boundary one can find for example in [GuiPol74] or [Lee06]. In other words, the ordinary category $\{\text{Manifolds with Boundary}\}$ is a natural full subcategory of $\{\text{Diffeology}\}$. The category $\{\text{Manifolds}\}$ being itself a full subcategory of $\{\text{Manifolds with Boundary}\}$.

Moreover, we have a similar result for the subset diffeology of corners, thanks to a Schwartz theorem²⁴ [Sch75].

THEOREM. *Let K^n be the positive n -corner in \mathbf{R}^n defined by $x_i \geq 0$, with $i = 1, \dots, n$, equipped with the subset diffeology. Let $f \in C^\infty(K^n, \mathbf{R})$, then there exists a smooth function F , defined on an open superset of K^n in \mathbf{R}^n , such that $f = F \upharpoonright K^n$.*

Actually, this property extends to any differential k -form on K^n : it is the restriction of a smooth k -form on \mathbf{R}^n [GIZ17-a].

As for manifolds with boundary, it is natural to define the n -dimensional *manifolds with corners* as diffeological spaces which are locally diffeomorphic to K^n at each point. We get then naturally the category $\{\text{Manifolds with Corners}\}$ as a new subcategory of $\{\text{Diffeology}\}$.

Back to the alternative Chen versus Souriau: the three axioms of diffeologies are indeed identical to the preceding three axioms of Chen's differentiable spaces [Che77], except for the domains of plots which are open instead of being convex. Chen's spaces had been introduced with homology and cohomology in mind, that is why he chose the convex subsets as domains for his plots. On another side, the choice of open subsets positions diffeology as a competitor to differential geometry itself. And now, the fact that smooth maps for half-spaces or corners, equipped with the subset diffeology, coincide with what was guessed heuristically to describe manifolds with boundary or corners, is another confirmation that there is no need to amend the theory in any way. For example, if we would we could define smooth simplices in diffeological spaces as smooth maps from the standard simplices, equipped with its subset diffeology. And that would cover the usual situation in differential geometry. ►

27. FRÖLICHER SPACES AS REFLEXIVES DIFFEOLOGICAL SPACES. — We recall that a Frölicher structure on a set X is defined by a pair of sets: $\mathcal{F} \subset \text{Maps}(X, \mathbf{R})$ and $\mathcal{C} \subset \text{Maps}(\mathbf{R}, X)$ that satisfies the double condition:

$$\begin{aligned}\mathcal{C} &= \{c \in \text{Maps}(\mathbf{R}, X) \mid \mathcal{F} \circ c \subset C^\infty(\mathbf{R}, \mathbf{R})\}, \\ \mathcal{F} &= \{f \in \text{Maps}(X, \mathbf{R}) \mid f \circ \mathcal{C} \subset C^\infty(\mathbf{R}, \mathbf{R})\}.\end{aligned}$$

²⁴In this case, it is a simple corollary of the Whitney theorem [Whi43, Remark p. 310].

A set X equipped with a Frölicher structure is called a Frölicher space [KriMic97]. Now, let X be a diffeological space, \mathcal{D} be its diffeology and $C^\infty(X, \mathbf{R})$ its set of real smooth maps.

DEFINITION. *We say that X is reflexive if \mathcal{D} coincides with the coarsest diffeology on X , for which the set of real smooth maps is exactly $C^\infty(X, \mathbf{R})$.*

Then, thanks to Boman's theorem [Bom67], one can show²⁵ that a Frölicher space, equipped with the coarsest diffeology such that the elements of \mathcal{F} are smooth, is reflexive. And one can check conversely that a reflexive diffeological space satisfies the Frölicher conditions above, see [PIZ13, Exercises 79, 80]. In other words,

PROPOSITION. *The category of Frölicher spaces coincides with the subcategory of reflexive diffeological spaces.* ►

CARTAN-DE RHAM CALCULUS

With Fiber bundles and homotopy, differential calculus is one of the most developed domains in diffeology. We begin first by the definition of a differential form on a diffeological space.

28. DIFFERENTIAL FORMS. — Let $U \subset \mathbf{R}^n$, we denote by $\Lambda^k(\mathbf{R}^n)$ the vector space of linear k -forms on \mathbf{R}^n , $k \in \mathbf{N}$. We call *smooth k -form* on U any smooth map $a : U \rightarrow \Lambda^k(\mathbf{R}^n)$.

Now, let X be a diffeological space.

DEFINITION. *We call differential k -form on X any map α that associates, with every plot $P : U \rightarrow X$, a smooth k -form $\alpha(P)$ on U which satisfies the compatibility condition:*

$$\alpha(P \circ F) = F^*(\alpha(P)),$$

for all smooth parametrizations F in U .

The set of k -forms on X is denoted by $\Omega^k(X)$. Note that $\Omega^0(X) = C^\infty(X, \mathbf{R})$.

Note also that one can consider an n -domain U as a diffeological space, in this case a differential k -form α is immediately identified with its value $a = \alpha(1_U)$ on the identity; $\alpha \in \Omega^k(U)$ and $a \in C^\infty(U, \Lambda^k(\mathbf{R}^n))$.

The two main operations on the differential forms on a diffeological space are:

²⁵The concept of reflexive space has been suggested by Yael Karshon, and we established the equivalence with Frölicher spaces together with Augustin Batubenge and Jordan Watts, in a seminar at Toronto in 2010.

- (1) **THE PULLBACK.** Let $f \in C^\infty(X', X)$ be a smooth map between diffeological spaces, let $\alpha \in \Omega^k(X)$. Then $f^*(\alpha) \in \Omega^k(X')$ is the k -form defined by

$$[f^*(\alpha)](P') = \alpha(f \circ P'),$$

for all plots P' in X' .

- (2) **THE EXTERIOR DERIVATIVE.** Let $\alpha \in \Omega^k(X)$, its exterior derivative $d\alpha \in \Omega^{k+1}(X)$ is defined by

$$[d\alpha](P) = d[\alpha(P)],$$

for all plots P in X .

Then, we got a de Rham complex $\Omega^*(X)$, with an endomorphism d that satisfies

$$d \circ d = 0, \quad \text{and} \quad \begin{cases} Z_{\text{dR}}^*(X) = \ker(d: \Omega^*(X) \rightarrow \Omega^{*+1}(X)) \\ B_{\text{dR}}^*(X) = d(\Omega^{*-1}(X)) \subset Z_{\text{dR}}^*(X). \end{cases}$$

That defines a de Rham cohomology series of groups

$$H_{\text{dR}}^k(X) = Z_{\text{dR}}^k(X) / B_{\text{dR}}^k(X).$$

Note that this series begins with $k = 0$ for which $B_{\text{dR}}^0(X) = \{0\}$.

The first cohomology group $H_{\text{dR}}^0(X)$ is easy to compute. The differential df of a smooth function $f \in \Omega^0(X)$ vanishes if and only if f is constant on the connected components of X . Thus, $H_{\text{dR}}^0(X)$ is the real vector space generated by $\pi_0(X)$, that is, $\text{Maps}(\pi_0(X), \mathbf{R})$. \blacktriangleright

29. QUOTIENTING DIFFERENTIAL FORMS. — One of the main procedures on differential forms is *quotienting forms*. I mean the following: let X and X' be two diffeological spaces and let $\pi: X \rightarrow X'$ be a subduction. The following criterion [Sou84] identifies $f^*(\Omega^*(X'))$ into $\Omega^*(X)$.

PROPOSITION. *Let $\alpha \in \Omega^*(X)$. There exists $\beta \in \Omega^*(X')$ such that $\alpha = \pi^*(\beta)$ if and only if, for all pairs of plots P and P' in X , if $\pi \circ P = \pi \circ P'$ then $\alpha(P) = \alpha(P')$.*

That proposition helps us to compute $H_{\text{dR}}(T_\alpha)$ for example. Actually the criterion above has a simple declination for coverings.

PROPOSITION. *Let X be a diffeological space and $\pi: \tilde{X} \rightarrow X$ its universal covering. Let $\tilde{\alpha} \in \Omega^*(\tilde{X})$. There exists $\alpha \in \Omega^*(X)$ such that $\tilde{\alpha} = \pi^*(\alpha)$ if and only if, $\tilde{\alpha}$ is invariant by $\pi_1(X)$, that is, $k^*(\tilde{\alpha}) = \tilde{\alpha}$ for all $k \in \pi_1(X)$.*

Indeed, for the criterion above $\pi \circ P = \pi \circ P'$ if and only if, locally on each ball in the domain of the plots, there exists an element $k \in \pi_1(X)$ such that $P' = k \circ P$ on the ball.

Now, let us apply this criterion to any irrational torus $T = \mathbf{R}/\Gamma$, where Γ is a strict dense subgroup of \mathbf{R} . A 1-form $\tilde{a} \in \mathbf{R}$ writes $\tilde{a}(x) dx$. It is invariant by Γ if and only if $a(x) = a$ is constant. Let θ be the 1-form whose pullback is dx , then

$$\Omega^1(T) = \mathbf{R}\theta \quad \text{and} \quad H_{\text{dR}}^1(T) = \mathbf{R}.$$

Obviously, $H_{\text{dR}}^0(T) = \mathbf{R}$ and $H_{\text{dR}}^k(T) = \{0\}$ if $k > 1$. ►

30. PARASYMPLECTIC FORM ON THE SPACE OF GEODESICS. — It is well known that, if the space $\text{Geod}(M)$ of (oriented) *geodesic trajectories* (*a.k.a.* unparametrized geodesics) of a Riemannian manifold (M, g) is a manifold, then this manifold is naturally symplectic for the quotient of the presymplectic form defining the geodesic flow. A famous example is the geodesics of the sphere S^2 , for which the space of geodesics is also S^2 , equipped with the standard surface element²⁶. In this case, the mapping from the unit bundle US^2 to $\text{Geod}(S^2)$ is realized by the moment map of the rotations:

$$\ell: US^2 = \{(x, u) \in S^2 \times S^2 \mid u \cdot x = 0\} \rightarrow \text{Geod}(S^2) \text{ with } \ell(x, u) = x \wedge u.$$

Now, what about the space of geodesics of the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$, for example? It is certainly not a manifold because of the mix of closed and unclosed geodesics. And about the canonical symplectic structure, does it remain something from it? And what? That is exactly the kind of question diffeology is able to answer to.

The geodesics of T^2 are the characteristics of the differential $d\lambda$ of the Liouville 1-form λ on UT^2 , associated with the ordinary Euclidean product:

$$\lambda(\delta y) = u \cdot \delta x, \quad \text{with } y = (x, u) \in UT^2 \quad \text{and} \quad \delta y \in T_y(UT^2).$$

And then,

$$\text{Geod}(T^2) = \left\{ \text{pr}\{x + tu\}_{t \in \mathbf{R}} \times \{u\} \subset T^2 \times S^1 \mid (x, u) \in UT^2 \right\}.$$

The direction of the geodesic $\text{pr}_2: (\text{pr}\{x + tu\}_{t \in \mathbf{R}}, u) \mapsto u$ is a natural projection on S^1

$$\begin{array}{ccc} UT^2 & \xrightarrow{\pi} & \text{Geod}(T^2) \\ \text{pr}_2 \searrow & & \swarrow \text{pr}_2 \\ & S^1 & \end{array}$$

The fiber $\text{pr}_2^{-1}(u) \subset \text{Geod}(T^2)$, is the torus T_u of all lines with slope u . We have seen that, depending if the slope is rational or not, we get a circle or an irrational torus. As we claimed, $\text{Geod}(T^2)$ equipped with the quotient diffeology of UT^2

²⁶For a judicious choice of constant.

is not a manifold. However, there exists on $\text{Geod}(T^2)$ a closed 2-form ω such that $d\lambda = \pi^*(\omega)$ [PIZ16-b]. We say that $\text{Geod}(T^2)$ is *parasymplectic*²⁷

Actually, this example is just a special case of the general situation [PIZ16-c].

THEOREM. *Let M be a Riemannian manifold. Let $\text{Geod}(M)$ be the space of geodesics, defined as the characteristics of the canonical presymplectic 2-form $d\lambda$ on the unit bundle UM . Then, there exists a closed 2-form ω on $\text{Geod}(M)$ such that $d\lambda = \pi^*(\omega)$.*

This result is a direct application of the criterion above on quotienting forms. It is strange that we had to wait so long to clarify this important point which should have been one of the first results in diffeology. ►

31. DIFFERENTIAL FORMS ON MANIFOLDS WITH CORNERS. — We claimed previously that the smooth maps $f : K_n \rightarrow \mathbf{R}$, where K_n is the n -dimensional corner, are the restrictions of smooth functions defined on some open neighbourhood of K_n in \mathbf{R}^n . We have more [GIZ16-a] :

PROPOSITION. *Let $\omega \in \Omega^k(K^n)$ be a differential k -form on K^n . Then, there exists a smooth k -form $\bar{\omega}$ defined on some open neighbourhood of $K^n \subset \mathbf{R}^n$ such that $\omega = \bar{\omega} \upharpoonright K^n$.*

This proposition has then a corollary (*op. cit.*) :

THEOREM. *Let M be a smooth manifold. Let $W \subset M$, equipped with the subset diffeology, be a submanifold with boundary and corners. Any differential form on W is the restriction of a smooth form defined on an open neighbourhood.*

That closes the discussion about the compatibility between diffeology and manifolds with boundary and corners for any question relative to de Rham complex. ►

32. THE PROBLEM WITH THE DE RHAM HOMOMORPHISM. — Let us focus on 1-forms and precisely the 1-forms on T_α , to take an example. The integration on paths defines the first de Rham homomorphism. Let ε be a closed 1-form on T_α (actually any 1-form since they are all closed). Consider the map

$$\gamma \mapsto \int_\gamma \varepsilon = \int_0^1 \varepsilon(\gamma)_x(1) dx,$$

where $\gamma \in \text{Paths}(T_\alpha)$. Because ε is closed and because the integral depends only on the fixed end homotopy class of γ , restricted to $\text{Loops}(T_\alpha)$, this integral defines a homomorphism from $\pi_1(T_\alpha)$ to \mathbf{R} . And since the integral on a loop of

²⁷The meaning of the word *symplectic* in diffeology is still under debate. That is why I use the wording *parasymplectic* to indicate a closed 2-form.

a differential df vanishes, the integral depends only on the cohomology class of ε . Thus, we get the first de Rham homomorphism²⁸:

$$h: H_{\text{dR}}^1(T_\alpha) \rightarrow \text{Hom}(\pi_1(T_\alpha), \mathbf{R}) \quad \text{defined by} \quad \varepsilon \mapsto [\ell \mapsto \int_\ell \varepsilon].$$

Now, since $H_{\text{dR}}^1(T_\alpha) = \mathbf{R}$ and $\text{Hom}(\pi_1(T_\alpha), \mathbf{R}) = \mathbf{R}^2$, the de Rham homomorphism cannot be an isomorphism as it is the case for Euclidean domains, or more generally ordinary manifolds. Actually, it is precisely given by

$$h: a \mapsto [(n, m) \mapsto a(n + \alpha m)],$$

where $a \in H_{\text{dR}}^1(T_\alpha)$ is represented by adt on \mathbf{R} , and $\pi_1(T_\alpha) = \mathbf{Z} + \alpha\mathbf{Z} \subset \mathbf{R}$.

This hiatus is specific to diffeology, versus differential geometry. It is however still true that, for any diffeological space, the first de Rham homomorphism is injective, and we can interpret geometrically its cokernel²⁹.

Let $\tilde{T}_\alpha (= \mathbf{R})$ denote the universal covering of T_α . Consider then a homomorphism ρ from $\pi_1(T_\alpha)$ to $(\mathbf{R}, +)$. Then, build the associated bundle $\text{pr}: \tilde{T}_\alpha \times_\rho \mathbf{R} \rightarrow T_\alpha$, where $\pi_1(T_\alpha)$ acts diagonally on the product $\tilde{T}_\alpha \times \mathbf{R}$. That is, for all $(x, t) \in \tilde{T}_\alpha \times \mathbf{R}$ and all $k \in \pi_1(T_\alpha)$, $k: (x, t) \mapsto (x + k, t + \rho(k))$. Let $\text{class}: \tilde{T}_\alpha \times \mathbf{R} \rightarrow \tilde{T}_\alpha \times_\rho \mathbf{R}$ be the projection.

$$\begin{array}{ccc} \tilde{T}_\alpha \times \mathbf{R} & \xrightarrow{\text{class}} & \tilde{T}_\alpha \times_\rho \mathbf{R} \\ \text{pr}_1 \downarrow & & \downarrow \text{pr} \\ \tilde{T}_\alpha & \xrightarrow{\pi} & T_\alpha \end{array}$$

And the right down arrow, $\text{pr}: \text{class}(x, t) \mapsto \pi(x)$ is a principal $(\mathbf{R}, +)$ fiber bundle for the action $s: \text{class}(x, t) \mapsto \text{class}(x, t + s)$. This principal fiber bundle has a natural connection induced by the connection of the universal covering. Pick a path γ in T_α and a point \tilde{x} over $x = \gamma(t)$, there exists a unique lifting $\tilde{\gamma}$ such that $\tilde{\gamma}(t) = \tilde{x}$. Then, we define the horizontal lifting $\bar{\gamma}$ of γ passing through $\text{class}(\tilde{x}, s)$ by $\bar{\gamma}(t') = \text{class}(\tilde{\gamma}(t'), t' - t + s)$. This connection is, by construction, flat. Indeed, the subspace $\{\text{class}(\tilde{x}, 0) \mid \tilde{x} \in \tilde{T}_\alpha\}$ is a reduction of the principal fiber bundle $\text{pr}: \tilde{T}_\alpha \times_\rho \mathbf{R} \rightarrow T_\alpha$ to the group $\pi_1(T_\alpha)/\ker(\rho)$. Now, in [PIZ13, §8.30] we prove that a homomorphism $\rho: \pi_1(T_\alpha) \rightarrow \mathbf{R}$ gives a trivial fiber bundle pr if and only if ρ is the de Rham homomorphism of a closed 1-form ε . And eventually, we prove that:

²⁸See [PIZ13, §6.74] for the general construction and for the justifications needed.

²⁹For differential forms in higher degree this is still a work in progress.

PROPOSITION. *The cokernel of the de Rham homomorphism is equivalent to the set of equivalence classes of $(\mathbf{R}, +)$ -principal bundle over T_α , equipped with a flat connection. This result is actually general for any diffeological space X .*

Note that this analysis put on an equal footing: the surjectivity of the first de Rham homomorphism, and the triviality of principal $(\mathbf{R}, +)$ -principal bundles over manifolds. That deserved to be noticed. ►

33. THE CHAIN-HOMOTOPY OPERATOR. — The *Chain-Homotopy Operator* K is a fundamental construction in differential calculus [PIZ13, §6.83]. It is related in particular to integration of closed differential forms, homotopic invariance of de Rham cohomology, moment map in symplectic geometry, as we shall see in the following.

Let X be a diffeological space, there exists a smooth linear operator

$$K: \Omega^p(X) \rightarrow \Omega^{p-1}(\text{Paths}(X)) \quad \text{with} \quad p \geq 1,$$

that satisfies

$$K \circ d + d \circ K = \hat{1}^* - \hat{0}^*,$$

where $\hat{0}, \hat{1}: \text{Paths}(X) \rightarrow X$ are defined by $\hat{t}(\gamma) = \gamma(t)$.

Explicitly, let α be a p -form of X , with $p > 1$, and $P: U \rightarrow \text{Paths}(X)$ be an n -plot. The value of $K\alpha$ on the plot P , at the point $r \in U$, evaluated on $(p-1)$ vectors $(v)_{i=2}^p = (v_2) \dots (v_p)$ of \mathbf{R}^n , is given by

$$K\alpha(P)_r(v)_{i=2}^p = \int_0^1 \alpha \left(\left(\begin{pmatrix} t \\ r \end{pmatrix} \mapsto P(r)(t) \right)_{\left(\begin{smallmatrix} t \\ r \end{smallmatrix} \right)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_i \end{pmatrix}_{i=2}^p \right) dt.$$

For $p = 1$,

$$K\alpha: \gamma \mapsto \int_\gamma \alpha = \int_0^1 \alpha(\gamma)_t dt$$

is the usual integration along the paths.

Note that there is no equivalent to this operator in ordinary differential geometry since there is no concept of differential forms on the space of paths, even for a manifold. Of course there are a few bypasses, but no one is as direct or efficient than the operator K , as we shall see now. ►

34. HOMOTOPIC INVARIANCE OF DE RHAM COHOMOLOGY. — Consider an homotopy $t \mapsto f_t$ in $\text{Paths}(C^\infty(X, X'))$, where X and X' are two diffeological spaces.

PROPOSITION. *Let $\alpha \in \Omega^k(X')$ and $d\alpha = 0$. Then, $f_1^*(\alpha) = f_0^*(\alpha) + d\beta$, with $\beta \in \Omega^{k-1}(X)$.*

In other words, the de Rham cohomology is homotopic invariant:

$$\text{class}(f_1^*(\alpha)) = \text{class}(f_0^*(\alpha)) \in H_{\text{dR}}^k(X).$$

Let us prove it rapidly. Consider the smooth map $\phi: X \rightarrow \text{Paths}(X')$ defined by $\phi(x) = [t \mapsto f_t(x)]$. Take the pullback of the chain-homotopy identity, $\phi^*(K(d\alpha) + d(K\alpha)) = \phi^*(\hat{1}^*(\alpha)) - \phi^*(\hat{0}^*(\alpha))$. That is, $\phi^*(d(K\alpha)) = d(\phi^*(K\alpha)) = (\hat{1} \circ \phi)^*(\alpha) - (\hat{0} \circ \phi)^*(\alpha)$. But, $\hat{t} \circ \phi = f_t$, thus $f_1^*(\alpha) - f_0^*(\alpha) = d\beta$ with $\beta = \phi^*(K\alpha)$.

That is one of the most striking uses of this chain-homotopy operator. And that proves at the same time how one can take advantage of diffeology, even in a traditional course on differential geometry. ►

35. INTEGRATION OF CLOSED 1-FORMS. — Consider a manifold M and a closed 1-form α on M . We know that if α is integral, that is, its integral on every loop is a multiple of some number called the *period*, then there exists a smooth function f from M to the circle S^1 such that $\alpha = f^*(\theta)$, where θ is the canonical length element. This specific construction has an ultimate generalization in diffeology which avoids the integral condition. And that is what diffeology is for, indeed.

Let α be a closed 1-form on a connected diffeological space X . Consider the equivalence relation on $\text{Paths}(X)$ defined by

$$\gamma \sim \gamma' \quad \text{if} \quad \text{ends}(\gamma) = \text{ends}(\gamma') \quad \text{and} \quad \int_{\gamma} \alpha = \int_{\gamma'} \alpha.$$

The quotient $\mathfrak{X}_{\alpha} = \text{Paths}(X)/\sim$ is a groupoid for the addition³⁰ $\text{class}(\gamma) + \text{class}(\gamma') = \text{class}(\gamma \vee \gamma')$, when $\gamma(1) = \gamma'(0)$. Because the integral of α on γ does not depend on its fixed-endpoints homotopy class, the groupoid \mathfrak{X}_{α} is a covering groupoid, a quotient of the Poincaré groupoid \mathfrak{X} . Let $F_{\alpha}: \mathfrak{X}_{\alpha} \rightarrow \mathbf{R}$ be $F_{\alpha}(\text{class}(\gamma)) = K\alpha(\gamma) = \int_{\gamma} \alpha$.

$$\begin{array}{ccccccc} \text{Paths}(X) & \xrightarrow{\text{class}} & \mathfrak{X} & \xrightarrow{\text{class}} & \mathfrak{X}_{\alpha} & \xrightarrow{F_{\alpha}} & \mathbf{R} \\ & \searrow \text{ends} & \downarrow & \swarrow & & & \downarrow \text{pr} \\ & & X \times X & \xrightarrow{f_{\alpha}} & T_{\alpha} = \mathbf{R}/P_{\alpha} & & \end{array}$$

³⁰Actually this addition is defined on the stationary paths, but since the space of stationary paths is a deformation retract of the space of paths, that does not really matter.

The function F integrates α on \mathfrak{X}_α , in the sense that, thanks to the chain-homotopy identity, $dF_\alpha = \hat{1}^*(\alpha) - \hat{0}^*(\alpha)$. Let $o \in X$ be a point defined as an *origin* of X , and define X_α to be the subspace of \mathfrak{X}_α made of classes of paths with origin o . Let $\pi = \hat{1} \upharpoonright X_\alpha$, then $dF = \pi^*(\alpha)$, with $F = F_\alpha \upharpoonright X_\alpha$. The covering X_α of X is the smallest covering where the pullback of α is exact. We call it the integration covering of α . Its structure group P_α is the *group of periods* of α , that is,

$$P_\alpha = \left\{ \int_\ell \alpha \mid \ell \in \text{Loops}(X) \right\}.$$

PROPOSITION. *If the group of periods P_α is a strict subgroup of \mathbf{R} , then $\text{pr}: \mathbf{R} \rightarrow T_\alpha = \mathbf{R}/P_\alpha$ is a covering, and there exists a smooth map $f: X \rightarrow T_\alpha$ such that $f^*(\theta) = \alpha$, with θ being the projection of dt on T_α by pr .*

This proposition [PIZ13, §8.29] is the ultimate generalization of the integration of integral 1-forms in differential geometry, allowed by diffeology. We can notice that the usual condition of second countability for manifolds is in fact a sufficient precondition. The real obstruction, valid for manifolds as well as for general diffeological spaces, is that the group of periods P_α is discrete in \mathbf{R} . And that is equivalent to be a strict subgroup of \mathbf{R} . ►

SYMPLECTIC DIFFEOLOGY

During the '90s a lot of symplectic-like geometry situations have been explored, essentially in infinite dimensional spaces, but not only, with more or less success. What missed to these attempts of generalization was a uniform framework of concepts and vocabulary, precise definitions framing the context of their studies. Each example came with its own heuristic and context, for example the nature of the moment maps were not clearly stated: for the case of the moment of imprimitivity [Zie96] it was a function with values Dirac delta functions (distributions), for another example involving the connections of a torus bundle [Dnl99] it was the curvature of the connection. And Elisa Prato's quasifolds [Pra01], for which the moment map is defined on a space that is not a legitimate manifold but a singular quotient, adds up to these infinite dimensional examples.

What we shall see now is how diffeology is the missing framework, where all these examples find their places, are treated on an equal footing, and give what we are waiting from them.

The general objects of interest will be arbitrary closed 2-forms on diffeological spaces, for which we introduce this new terminology.

DEFINITION. We call *parasymplectic space*³¹ any diffeological space X , equipped with a closed 2-form ω .

Next, we consider the group of *symmetries* (or *automorphisms*) of ω , denoted by $\text{Diff}(X, \omega)$. The pseudogroup $\text{Diff}_{\text{loc}}(X, \omega)$ of local symmetries will play some role too. Then, to introduce the *moment map* for any group of symmetries G , we need to precise some vocabulary and notations³².

DEFINITION. We shall call *momentum*³³ of a diffeological group G any left-invariant 1-form. We denote by \mathcal{G}^* its space of momenta, that is,

$$\mathcal{G}^* = \{\varepsilon \in \Omega^1(G) \mid L(g)^*(\varepsilon) = \varepsilon, \text{ for all } g \in G\}.$$

The set \mathcal{G}^* is obviously a real vector space³⁴.

36. THE MOMENT MAP. — Let (X, ω) be a parasymplectic space and G be a diffeological group. A *symmetric action* of G on (X, ω) is a smooth morphism $g \mapsto g_X$ from G to $\text{Diff}(X, \omega)$, where $\text{Diff}(X, \omega)$ is equipped with the functional diffeology. That is,

$$\text{for all } g \in G, \quad g_X^*(\omega) = \omega.$$

Now, to grab the essential nature of the moment map, which is a map from X to \mathcal{G}^* , we need to understand it in the simplest possible case. That is, when ω is exact $\omega = d\alpha$, and when α is also invariant by G , $g_X^*(\alpha) = \alpha$. In these conditions, the moment map is given by

$$\mu: X \rightarrow \mathcal{G}^* \quad \text{with} \quad \mu(x) = \hat{x}^*(\alpha),$$

where $\hat{x}: G \rightarrow X$ is the *orbit map* $\hat{x}(g) = g_X(x)$. We check immediately that, since α is invariant by G , $\hat{x}^*(\alpha)$ is left invariant by G , and therefore $\mu(x) \in \mathcal{G}^*$.

But, as we know, not all closed 2-forms are exact, and even if they are exact, they do not necessarily have an invariant primitive. We shall see now, how we can generally come to a situation, so close to the simple case above, that modulo some minor subtleties we can build a good moment map in all cases.

Let us consider now the general case, with X connected. Let K be the chain-homotopy operator, defined previously. Then, the differential 1-form $K\omega$, defined on $\text{Paths}(X)$, satisfies $d[K\omega] = (\hat{1}^* - \hat{0}^*)(\omega)$, and $K\omega$ is invariant by G

³¹The quality of being *symplectic* or *presymplectic* will be discussed and get a precise meaning. The word *parasymplectic* seemed free and appropriate to denote a simple closed 2-form.

³²Remember that a diffeological group is a group which is a diffeological space such that the multiplication and the inversion are smooth.

³³Plural *momenta*.

³⁴It is also a diffeological vector space for the functional diffeology, but we shall not discuss that point here.

[PIZ13, §6.84]. Considering $\bar{\omega} = (\hat{1}^* - \hat{0}^*)(\omega)$ and $\bar{\alpha} = K\omega$, we are in the simple case: $\bar{\omega} = d\bar{\alpha}$ and $\bar{\alpha}$ invariant by G . We can apply the construction above and then we define the *Paths Moment Map* by

$$\Psi: \text{Paths}(X) \rightarrow \mathcal{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(K\omega),$$

where $\hat{\gamma}: G \rightarrow \text{Paths}(X)$ is the orbit map $\hat{\gamma}(g) = g_X \circ \gamma$ of the path γ .

The paths moment map is additive with respect to the concatenation,

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma').$$

And it is equivariant by G , which acts by composition on $\text{Paths}(X)$, and by coadjoint action on \mathcal{G}^* . That is, for all $g, k \in G$ and $\varepsilon \in \mathcal{G}^*$,

$$\text{Ad}(g): k \mapsto gkg^{-1} \quad \text{and} \quad \text{Ad}_*(g): \varepsilon \mapsto \text{Ad}(g)_*(\varepsilon) = \text{Ad}(g^{-1})^*(\varepsilon).$$

Then, we define the *Holonomy* of the action of G on X as the subgroup

$$\Gamma = \{\Psi(\ell) \mid \ell \in \text{Loops}(X)\} \subset \mathcal{G}^*.$$

The group Γ is made of (closed) Ad_* -invariant momenta. But $\Psi(\ell)$ depends only on the homotopy class of ℓ , then Γ is a homomorphic image of $\pi_1(X)$, more precisely its abelianized.

DEFINITION. If $\Gamma = \{0\}$ we say that the action of G on (X, ω) is Hamiltonian. The holonomy Γ is the obstruction for the action of the group G to be Hamiltonian.

Now, we can push forward the paths moment map on \mathcal{G}^*/Γ , as suggested by the commutative diagram:

$$\begin{array}{ccc} \text{Paths}(X) & \xrightarrow{\Psi} & \mathcal{G}^* \\ \text{ends} \downarrow & & \downarrow \text{class} \\ X \times X & \xrightarrow{\phi} & \mathcal{G}^*/\Gamma \end{array}$$

And we get then the *Two-Points Moment Map*:

$$\phi(x, x') = \text{class}(\Psi(\gamma)) \in \mathcal{G}^*/\Gamma, \quad \text{for any path } \gamma \text{ such that } \text{ends}(\gamma) = (x, x').$$

The additivity of Ψ becomes the Chasles' cocycle condition on ϕ :

$$\phi(x, x') + \phi(x', x'') = \phi(x, x'').$$

The group Γ is invariant by the coadjoint action. Thus, the coadjoint action passes to the quotient group \mathcal{G}^*/Γ , and ϕ is a natural group valued moment map, equivariant for this quotient coadjoint action.

Because X is connected, there exists always a map

$$\mu: X \rightarrow \mathcal{G}^*/\Gamma \quad \text{such that} \quad \psi(x, x') = \mu(x') - \mu(x).$$

The solutions of this equation are given by

$$\mu(x) = \psi(x_0, x) + c,$$

where x_0 is a chosen point in X and c is a constant. These are the *One-Point Moment Maps*. But these moment maps μ are *a priori* no longer equivariant. Their variance introduces a 1-cocycle θ of G with values in \mathcal{G}^*/Γ such that

$$\mu(g(x)) = \text{Ad}_*(g)(\mu(x)) + \theta(g), \quad \text{with} \quad \theta(g) = \psi(x_0, g(x_0)) + \Delta c(g),$$

where Δc is the coboundary due to the constant c in the choice of μ . We say that the action of G on (X, ω) is *exact* when the cocycle θ is trivial. Defining

$$\text{Ad}_*^\theta(g): \nu \mapsto \text{Ad}_*(g)(\nu) + \theta(g), \quad \text{then} \quad \text{Ad}_*^\theta(g g') = \text{Ad}_*^\theta(g) \circ \text{Ad}_*^\theta(g').$$

The cocycle property of θ , that is, $\theta(g g') = \text{Ad}_*(g)(\theta(g')) + \theta(g)$, makes Ad_*^θ an action of G on the group \mathcal{G}^*/Γ . This action is called the *affine action*. For the affine action, the moment map μ is equivariant.

$$\mu(g(x)) = \text{Ad}_*^\theta(g)(\mu(x)).$$

This construction extends to the category $\{\text{Diffeology}\}$, the moment map for manifolds introduced by Souriau in [Sou70]. When X is a manifold and the action of G is Hamiltonian, they are the standard moment maps he defined there. The remarkable point is that none of the constructions brought up above involves differential equations, and there is no need of considering a possible Lie algebra either. That is a very important point. The momenta appear as invariant 1-forms on the group, naturally, without intermediary, and the moment map as a map in the space of momenta.

Note that the group of automorphisms $G_\omega = \text{Diff}(X, \omega)$ is a legitimate diffeological group. The above constructions apply and give rise to universal objects: *universal momenta* \mathcal{G}_ω^* , *universal path moment map* Ψ_ω , *universal holonomy* Γ_ω , *universal two-points moment map* ϕ_ω , *universal moment maps* μ_ω , *universal Souriau's cocycles* θ_ω .

A *parasymplectic action* of a diffeological group G is a smooth morphism $h: G \rightarrow G_\omega$, and the objects, associated with G , introduced by the above moment maps constructions, are naturally subordinate to their universal counterparts.

We shall illustrate this construction by two examples in the next paragraphs. More examples can be found in [PIZ13, §9.27 \cdots 9.34] ►

37. **EXAMPLE 1 : THE MOMENT OF IMPRIMITIVITY** — Consider the cotangent space T^*M of a manifold M , equipped with the standard symplectic form $\omega = d\lambda$, where λ is the Liouville form. Let G be the Abelian group $C^\infty(M, \mathbf{R})$. Consider the following action of G on T^*M by

$$f : (x, a) \mapsto (x, a - df_x),$$

where $x \in M$, $a \in T_x^*M$ and df_x is the differential of f at the point x . Then, the moment map is given by:

$$\mu : (x, a) \mapsto d[f \mapsto f(x)] = d[\delta_x],$$

where δ_x is the Dirac distribution $\delta_x(f) = f(x)$.

Since δ_x is a smooth function on $C^\infty(M, \mathbf{R})$, its differential is a 1-form. Let us check that this 1-form is invariant:

Let $h \in C^\infty(M, \mathbf{R})$, $L(h)^*(\mu(x)) = L(h)^*(d[\delta_x]) = d[L(h)^*(\delta_x)] = d[\delta_x \circ L(h)]$, but $\delta_x \circ L(h) : f \mapsto \delta_x(f + h) = f(x) + h(x)$. Then, $d[\delta_x \circ L(h)] = d[f \mapsto f(x) + h(x)] = d[f \mapsto f(x)] = \mu(x)$.

We see that in this case the moment map identifies with a function with values distributions, but still has the definite formal statute of a map into the space of momenta of the group of symmetries.

Moreover this action is Hamiltonian and exact. This example, generalized to diffeological space, is developed in [PIZ13, Exercise 147]. ►

38. **EXAMPLE 2 : THE 1-FORMS ON A SURFACE** — Let Σ be a closed surface, oriented by a 2-form Surf . Consider $\Omega^1(\Sigma)$, the infinite dimensional vector space of 1-forms on Σ , equipped with the functional diffeology. Let ω be the antisymmetric bilinear map defined on $\Omega^1(\Sigma)$ by

$$\omega(\alpha, \beta) = \int_{\Sigma} \alpha \wedge \beta,$$

for all α, β in $\Omega^1(\Sigma)$ ³⁵. With this bilinear form is naturally associated a differential 2-form ω on $\Omega^1(\Sigma)$, defined by

$$\omega(P)_r(\delta r, \delta' r) = \int_{\Sigma} \frac{\partial P(r)}{\partial r}(\delta r) \wedge \frac{\partial P(r)}{\partial r}(\delta' r),$$

³⁵Since the exterior product $\alpha \wedge \beta$ is a 2-form of Σ , there exists $\phi \in C^\infty(\Sigma, \mathbf{R})$ such that $\alpha \wedge \beta = \phi \times \text{Surf}$. By definition, $\int_{\Sigma} \alpha \wedge \beta = \int_{\Sigma} \phi \times \text{Surf}$.

for all n -plots $P: U \rightarrow X$, for all $r \in U$, δr and $\delta' r$ in \mathbf{R}^n . Moreover, ω is the differential of the 1-form λ on $\Omega^1(\Sigma)$:

$$\omega = d\lambda \quad \text{with} \quad \lambda(P)_r(\delta r) = \frac{1}{2} \int_{\Sigma} P(r) \wedge \frac{\partial P(r)}{\partial r}(\delta r).$$

Define now the action of the additive group $C^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$ by

$$f: \alpha \mapsto \alpha + df.$$

Then, $C^\infty(\Sigma, \mathbf{R})$ acts by symmetries on $(\Omega^1(\Sigma), \omega)$, for all f in $C^\infty(\Sigma, \mathbf{R})$, $f^*(\omega) = \omega$.

The moment map of $C^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$ is given (up to a constant) by

$$\mu: \alpha \mapsto d \left[f \mapsto \int_{\Sigma} f \times d\alpha \right].$$

The moment map is invariant by the action of $C^\infty(\Sigma, \mathbf{R})$, that is exact, and Hamiltonian. And here again, the moment map is a function with values distributions. Now, $\mu(\alpha)$ is fully characterized by $d\alpha$. This is why we find in the literature on the subject, that the moment map for this action is the exterior derivative (or curvature, depending on the authors) $\alpha \mapsto d\alpha$. As we see again on this example, diffeology gives a precise meaning by procuring a unifying context. One can find the complete computation of this example in [PIZ13, §9.27]. \blacktriangleright

39. SYMPLECTIC MANIFOLDS ARE COADJOINT ORBITS — Because symplectic forms of manifolds have no local invariants, as we know thanks to Darboux's theorem, they have a huge group of automorphisms. This group is big enough to be transitive [Boo69], then the universal moment map identifies the symplectic manifold with its image, which, by equivariance, is a coadjoint orbit (affine or not) of its group of symmetries. In other words, coadjoint orbits are the universal models of symplectic manifolds.

Precisely, let M be a connected Hausdorff manifold, let ω be a closed 2-form on M . Let $G_\omega = \text{Diff}(M, \omega)$ be its group of symmetries and \mathcal{G}_ω^* its space of momenta. Let Γ_ω be the holonomy, and μ_ω be a universal moment map with values in $\mathcal{G}_\omega^*/\Gamma_\omega$. We have then:

THEOREM. *The form ω is symplectic, that is non-degenerate, if and only if:*

- (1) *The group G_ω is transitive on M .*
- (2) *The universal moment map $\mu_\omega: M \rightarrow \mathcal{G}_\omega^*/\Gamma_\omega$ is injective.*

This theorem is proved in [PIZ13, §9.23], but let us make some comments on the keys elements. Consider the closed 2-form $\omega = (x^2 + y^2) dx \wedge dy$, one can show

that it has an injective universal moment map μ_ω . But its group G_ω is not transitive since ω is degenerate in $(0, 0)$, and only at that point. Thus, the transitivity of G_ω is necessary.

Let us give some hint about the sequel of the proof. Assume ω symplectic. Let $m_0, m_1 \in M$ and p be a path connecting these points. For all $f \in C^\infty(M, \mathbf{R})$ with compact support let

$$F: t \mapsto e^{t \operatorname{grad}_\omega(f)}$$

be the exponential of the symplectic gradient of the f . Then, F is a 1-parameter group of automorphisms, and its value on $\Psi_\omega(p)$ is:

$$\Psi_\omega(p)(F) = [f(m_1) - f(m_0)] \times dt.$$

Now, if $\mu_\omega(m_0) = \mu_\omega(m_1)$, then there exists a loop ℓ in M such that $\Psi_\omega(p) = \Psi_\omega(\ell)$. Applied to the 1-plot F we deduce $f(m_1) = f(m_0)$ for all f , Therefore $m_0 = m_1$, and μ_ω is injective.

Conversely, let us assume that G_ω is transitive, and μ_ω injective. By transitivity, the rank of ω is constant. Now, let us assume that ω is degenerate, that is, $\dim(\ker(\omega)) > 0$. Since the distribution $\ker(\omega)$ is integrable, given two different points m_0 and m_1 in a characteristic there exists a path p connecting these two points and drawn entirely in the characteristic. That is, such that $d p(t)/dt \in \ker(\omega)$ for all t . But that implies $\Psi_\omega(p) = 0$ [PIZ13, §9.20]. Hence, $\mu_\omega(m_0) = \mu_\omega(m_1)$. But we assumed μ_ω injective. Thus, ω is non degenerate, that is, symplectic. \blacktriangleright

40. HAMILTONIAN DIFFEOMORPHISMS — Let (X, ω) be a parasymplectic diffeological space. We have seen above that the universal moment map μ_ω takes its values into the quotient $\mathcal{G}_\omega^*/\Gamma_\omega$, where the holonomy $\Gamma_\omega \subset \mathcal{G}_\omega^*$ is a subgroup made of closed Ad_* -invariant 1-forms on G . This group Γ_ω is the very obstruction for the action of the group of symmetries $G_\omega = \operatorname{Diff}(X, \omega)$ to be Hamiltonian.

PROPOSITION. *There exists a largest connected subgroup $\operatorname{Ham}(X, \omega) \subset G_\omega$ with vanishing holonomy. This group is called the group of Hamiltonian Diffeomorphisms. Every hamiltonian smooth action on (X, ω) factorizes through $\operatorname{Ham}(X, \omega)$.*

Hence, the moment map $\bar{\mu}_\omega$ with respect to the action of $H_\omega = \operatorname{Ham}(X, \omega)$ takes its values in the vector space of momenta \mathcal{H}_ω^* . One can prove also that the Hamiltonian moment map $\bar{\mu}_\omega$ is still injective when X is a manifold. It is probably an embedding in the diffeological sense [PIZ13, §2.13], but that has still to be proved.

Maybe the most interesting is how this group is built. Let G_ω° be the neutral component of G_ω , and $\pi: \tilde{G}_\omega^\circ \rightarrow G_\omega^\circ$ be its universal covering³⁶. For all $\gamma \in \Gamma_\omega$, let $F_\gamma \in \text{Hom}^\infty(\tilde{G}_\omega^\circ, \mathbf{R})$ be the primitive of $\pi^*(\gamma)$, that is, $dF_\gamma = \pi^*(\gamma)$ and $F_\gamma(1_{\tilde{G}_\omega^\circ}) = 0$. Next, let

$$F: \tilde{G}_\omega^\circ \rightarrow \mathbf{R}^{\Gamma_\omega} = \prod_{\gamma \in \Gamma_\omega} \mathbf{R} \quad \text{defined by} \quad F(\tilde{g}) = (F_\gamma(\tilde{g}))_{\gamma \in \Gamma_\omega}.$$

The map F is a smooth homomorphism, where $\mathbf{R}^{\Gamma_\omega}$ is equipped with the product diffeology. Then [PIZ13, §9.15],

$$\text{Ham}(X, \omega) = \pi(\ker(F)).$$

That definition gives the same group of Hamiltonian diffeomorphisms when X is a manifold [PIZ13, §9.16].

Now, let P_γ be the group of periods of $\gamma \in \Gamma_\omega$, that is,

$$P_\gamma = F(\pi_1(G_\omega^\circ, 1_{G_\omega^\circ})) = \left\{ \int_\ell \gamma \mid \ell \in \text{Loops}(G_\omega^\circ, 1) \right\}.$$

Then, let

$$\mathbf{P}_\omega = \prod_{\gamma \in \Gamma_\omega} P_\gamma \quad \text{and} \quad \mathbf{T}_\omega = \prod_{\gamma \in \Gamma_\omega} T_\gamma \quad \text{with} \quad T_\gamma = \mathbf{R}/P_\gamma.$$

Each homomorphism F_γ projects onto a smooth homomorphism $f_\gamma: G_\omega^\circ \rightarrow T_\gamma$, and the homomorphism F projects onto a smooth homomorphism $f: G_\omega^\circ \rightarrow \mathbf{T}_\omega$, according to the following commutative diagram of smooth homomorphisms:

$$\begin{array}{ccc} \tilde{G}_\omega^\circ & \xrightarrow{F} & \mathbf{R}^{\Gamma_\omega} \\ \pi \downarrow & & \downarrow \text{pr} \\ G_\omega^\circ & \xrightarrow{f} & \mathbf{T}_\omega \end{array}$$

with $\ker(\pi) = \pi_1(G_\omega^\circ, 1_{G_\omega^\circ})$, $\ker(\text{pr}) = \mathbf{P}_\omega = F(\ker(\pi))$. We get then

$$\text{Ham}(X, \omega) = \ker(f),$$

an alternative, somewhat intrinsic, definition of the group of Hamiltonian diffeomorphisms as the kernel of the *Holonomy Homomorphism* f . ►

³⁶For universal covering of diffeological groups see [PIZ13, §7.10].

IN CONCLUSION

We have proposed in this text to switch from the rigid category $\{\text{Manifolds}\}$ to the flexible category $\{\text{Diffeology}\}$, which is now well developed. This is a category closed under every set theoretic operation, complete, co-complete and Cartesian closed, which includes on an equal footing: manifolds, singular quotients and infinite dimensional spaces. An ideal situation already, from the pure point of view of categoricians, and mainly the reason for the success of the theory in that discipline: in “Model Category Structures”, see for example [ChrWu14] [Kih19], or in “Differentiable Homotopy Theory”³⁷ [IwaNob19] etc.³⁸.

But of course, the success of diffeology lies first and foremost in its very strength in geometry. The geometer will find pleasant and useful the flexibility of Diffeology, to extend in a unique formal and versatile framework, different constructions in various fields, without inventing each time a heuristic framework that momentarily satisfies its needs. For example, the construction of the moment map and the integration of any closed 2-form on any diffeological space [PIZ13, §8.42], are the prerequisite for an extension of symplectic geometry on spaces that are not manifolds, but that burst into mathematical physics these last decades with the problems of quantization and field theory.

Then, beyond all these circumstances and technicalities, what diffeology has to offer on a more formal or conceptual level? The answer lies partly in Felix Klein’s Erlangen Program [Kle72].

«The totality of all these transformations we designate as the *principal group* of space-transformations; *geometric properties are not changed by the transformations of the principal group*. And, conversely, *geometric properties are characterized by their remaining invariant under the transformations of the principal group*.

.../...

As a generalization of geometry arises then the following comprehensive problem:

Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group.»

³⁷Opposed to intrinsic Geometric Homotopy Theory, we described previously in this paper.

³⁸The Japanese school is very productive in these fields these days, using diffeology as a tool or a general framework. One can find other papers on the subject, already published or not, for examples [SYH18, Kih19-b, Har19, HaSh19, Kur19]

As we know, these considerations are regarded by mathematicians as the modern understanding of the word/concept of *Geometry*. A geometry is given as soon a space and a group of transformations of this space are given³⁹.

Consider Euclidean Geometry, defined by the group of Euclidean transformations, our *principal group*, on the Euclidean space. We can interpret, for example, the *Euclidean distance* as the invariant associated with the action of the Euclidean group on the set of pairs of points. We can superpose a pair of points onto another pair of points, by an Euclidean transformation, if and only if the distance between the points is the same for the two pairs⁴⁰. Hence, *geometric properties* or *geometric invariants* can be regarded as the orbits of the *principal group* in some spaces built on top of the principal space, and also as fixed/invariant points, since an orbit is a fixed point in the set of all the subsets of that space. In brief, what emerges from these considerations suggested by Felix Klein's principle is this:

CLAIM. — A geometry is associated/defined by a *principal group* of transformations of some space, according to Klein's statement. The various geometric properties/invariants are described by the various actions of the principal group on spaces built on top of the principal space: products, set of subsets etc. Each one of these properties, embodied as orbits, stabilizers, quotients etc., captures a part of this geometry.

Now, how diffeology fits in this context?

- One can regard a diffeological space as the collection of the plots which gives its structure. That is the *passive approach*.

- Or we can look at the space through the action of its group of diffeomorphisms⁴¹: on itself, but also on its powers or parts or maps etc. That is the *active approach*.

This dichotomy appears already for manifolds, where the change of coordinates (transition functions of an atlas) is the passive approach. The active approach, as the action of the group of diffeomorphisms, is often neglected and there are a few reasons for that. Among them, the group of diffeomorphisms is not a Lie group *stricto sensu*, it does not fit in the category {Manifolds}, and that creates a psychological issue. A second reason: its action on the manifold itself is

³⁹Jean-Marie Souriau reduces the concept of geometry to the group itself [Sou03]. But this is an extreme point of view I am not sure to share, for some reasons.

⁴⁰We could continue with the case of triangles and other elementary constructions: circles, parallels etc. and a comparison between Euclidean and Symplectic Geometry for example, from a strict Kleinian point of view. See the discussion in [PIZ02].

⁴¹We consider more precisely the action of the pseudo-group of local diffeomorphisms.

transitive⁴², there are no immediate invariants, one have first to consider some secondary/subordinate spaces to make appear the first invariants.

These obstacles, psychological or real, vanish in diffeology. First of all, the group of diffeomorphisms is naturally, a diffeological group. And here is an example where the action of the group of diffeomorphisms, the principal group in the sense of Klein, on the space captures a good preliminary image of its geometry⁴³.

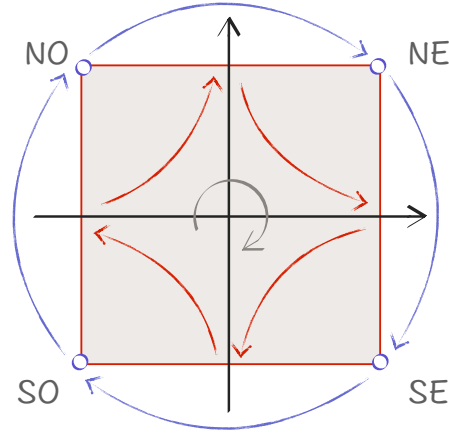


Figure 5 — A Diffeomorphism Of The Square.

Consider the square $Sq = [-1, +1]^2$, equipped with the subset diffeology. Its decomposition in orbits, by its group of diffeomorphisms, is the following:

- (1) The 4-corners-orbit, in blue on the figure.
- (2) The 4-edges-orbit, in red in the figure.
- (3) The interior-orbit, in gray in the figure.

Any diffeomorphism preserves separately the interior of the square and its border, that is a consequence of the D-topology⁴⁴. But the fact that a diffeomorphism of the square cannot map a corner into the interior of an edge is a typical smooth property, see [GIZ16-b, GIZ17-a], and some comments on more general stratified spaces [GIZ17-b].

On this simple example we can experiment Klein's principle, for the group of diffeomorphisms of a non-transitive diffeological space. The square being naturally an object of the theory, there is no need of heuristic extension here⁴⁵.

⁴²Generally manifolds are regarded connected, Hausdorff and second countable.

⁴³And not just of its topology.

⁴⁴A diffeomorphism is in particular an homeomorphism for the D-topology. We can use that, or homotopy.

⁴⁵See article (26).

CLAIM. — *Hence, considering the group of diffeomorphisms of a diffeological space as its principal group, we can look at diffeology as the formal framework that makes Differential Geometry, the geometry — in the sense of Felix Klein — of the group of diffeomorphisms*⁴⁶.

That principle, in the framework of Diffeology, can be regarded as the definitive expression of Souriau's point of view, developed in his paper «Les Groupes Comme Universaux» [Sou03].

Because diffeology is a such large and stable category, that encompasses satisfactorily so many various situations, from singular quotients to infinite dimension, mixing even these cases [PIZ16], one can believe that this interpretation of diffeology fulfils its claim, and answers in some sense the second part of the title of this book/conference.

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⁴⁶It is possible to weaken this claim by considering instead the (larger) pseudogroup of local diffeomorphisms.

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