Moment map : the simplest case

The moment map has been explained in other lecture notes of this course. Here we shall see the simplest case of moment map.

Let (M,ω) be a presymplectic manifold. Let G be a Lie group acting symplectically on M. Let $g \mapsto g_M$ be the action, $g \in G$.

$$\forall g \in G \qquad g_M^* \omega = \omega$$

Let us assume that there exists a 1-form α such that

$$\omega = d\alpha$$

Let us assume moreover that α is also invariant by G:

$$\forall g \in G \qquad g_M^* \alpha = \alpha .$$

So, the action of G on M is hamiltonian and one moment $\Psi : M \to \mathcal{G}^*$ is given by:

$$\forall Z \in \mathcal{G} \qquad \Psi(m).Z = \alpha(Z_M(m))$$

when $m \in M$ and Z_M denotes the fundamental field associated to Z

Proof : This is an application of the Cartan formula we have:

$$\underbrace{L_{Z_\mu}\alpha}_{\text{Lie derivative}} = \underbrace{d\alpha(Z_\mu)}_{\text{contraction of } d\alpha \text{ and } Z_\mu} + d(\underbrace{\alpha(Z_\mu)}_{\text{contraction of } \alpha \text{ and } Z_\mu})$$

but $L_{Z_\mu}\alpha = 0$, becaun $g_\mu^*\alpha = \alpha$, and $d\alpha = \omega$. so,

$\omega(Z_\mu) + d(\alpha(Z_\mu)) = 0$ thus $\omega(Z_\mu) = -d(\alpha(Z_\mu))$,

thus $m \longmapsto \alpha(Z_\mu(m))$ is a solution of the moment map equation. $\square$

Application: This apply for the action of the group of isometries on the unit tangent bundle of a riemmannian manifold.