Notes on symplectic manifolds

The symplectic model: let us consider $\mathbb{R}^{2n}$, and let us denote by $(q,p) = (q^1 \ldots q^n, p^1 \ldots p^n)$ a point of $\mathbb{R}^{2n}$.

Let us remember that dq^i or dp^i is the constant linear 1-form defined on $\mathbb{R}^{2n}$ by:

$$dq^i \begin{pmatrix} u_q \\ u_p \end{pmatrix} = u_q^i \qquad dp^i \begin{pmatrix} u_q \\ u_p \end{pmatrix} = u_p^i$$

where $\begin{pmatrix} u_q \\ u_p \end{pmatrix} \in \mathbb{R}^{2n}$, $u_q = (u_q^1 \ldots u_q^n)$ $u_p = (u_p^1 \ldots u_p^n)$

Let us remember that $dp^i \wedge dq^i$ is the constant linear 2-form defined by:

$$dp^i \wedge dq^i (u, v) = u_p^i v_q^i - u_q^i v_p^i$$

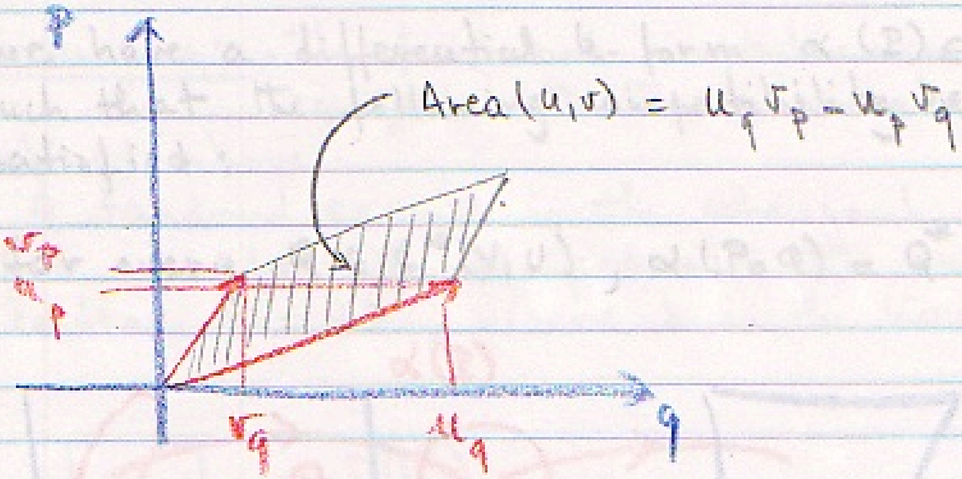
So the standard symplectic form of $\mathbb{R}^{2n}$ is defined by:

$$\omega_{can} = \sum_{i=1}^{n} dp^i \wedge dq^i$$

In other words, for every pair of vectors $u, v \in \mathbb{R}^{2n}$

$$\omega_{can}(u,v) = \sum_{i=1}^{n} u_p^i v_q^i - u_q^i v_p^i$$

Let us give a geometrical interpretation. For $n=1$, the value $\omega_{can}(u,v)$ is the area of the parallelogram defined by (u,v), see figure:



So for a general $u = \begin{pmatrix} u_q \\ u_p \end{pmatrix}$ $v = \begin{pmatrix} v_q \\ v_p \end{pmatrix}$, we have n such parallelograms:

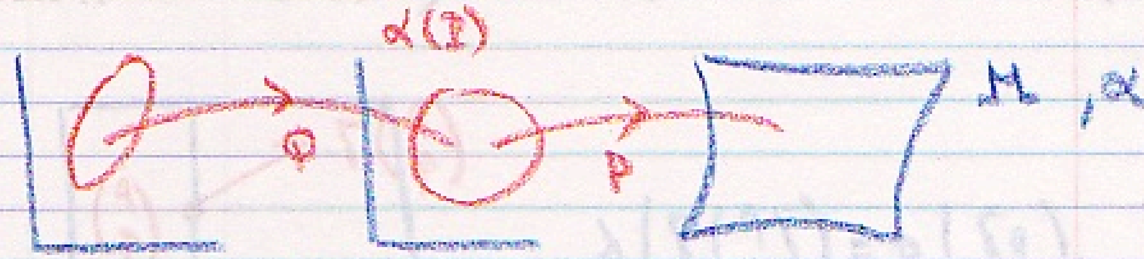
$$\begin{cases} \text{par}_i = \left[\begin{pmatrix} u_q^i \\ u_p^i \end{pmatrix}, \begin{pmatrix} v_q^i \\ v_p^i \end{pmatrix} \right] \\ \omega_{can}(u,v) = \sum_{i=1}^{n} \text{Area}(\text{par}_i) \end{cases}$$

Now let us define a symplectic manifold: we now that a manifold is a smooth (diffeological) space M, such that for every point $x \in M$ there exist a local diffeomorphism F from $\mathbb{R}^N$ to M with $x \in F(\text{dom}(F))$. Where $\text{dom}(F)$ is de domain of definition of F, $F(\text{dom}(F)) =: \text{Val}(F)$.

Now, generally speaking a differential k-form on a smooth space is defined by its values on the domains of the smooth parametrizations. That is, for every smooth parametrization $P: U \to M$

we have a differential k-form $\alpha(P) \in \Omega^k(\mathrm{dom}(P))$ such that the following compatibility relation is satisfied:

for every $Q \in C^\infty(V, U)$, $\alpha(P \circ Q) = Q^*(\alpha(P))$.



$\alpha(P \circ Q) = Q^*(\alpha(P))$

A symplectic manifold is a manifold M, together with a 2-form ω such that there exist a family of charts $\{F_i\}_{i \in J}$ satisfying:

$$\omega(F_i) = \omega_{can} \restriction \mathrm{dom}(F_i)$$

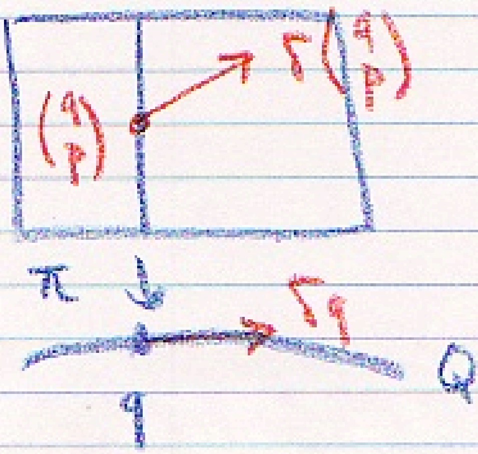
where ω_{can} is the canonical symplectic form on $\mathbb{R}^{2n}$, for some n. So $\dim(M) = 2n$ is even. Such a form ω is called symplectic.

the charts F_i are called Darboux charts. This name comes from the first and most important theorem in symplectic geometry:

Theorem (Darboux) Let M be a manifold, let ω be a 2-form on M. the form ω is symplectic if and only if ω is closed: $d\omega = 0$, and non degenerate: $\ker \omega_x = \{0\}$, for all $x \in M$.

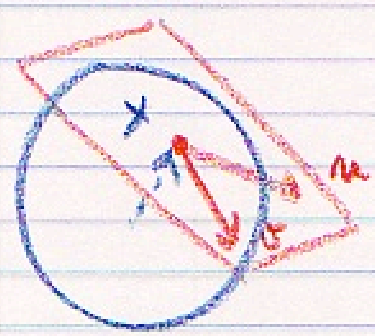
Examples: • The first and simplest example is $\mathbb{R}^{2n}$ equipped with the canonical 2-form ω_{can} defined above.

• A standard example is the cotangent space T^*Q of a manifold Q, equipped with the exterior differential $\omega = d\lambda$, where λ is the Liouville form.



$$\lambda\left(\xi\binom{q}{p}\right) = p(\xi q)$$

• The Sphere S^2 equipped with the Area form:



$$\text{Area}_x(u,v) = \det|x\ u\ v| = \langle x, u \times v\rangle$$

But, from all the even dimensional spheres $S^2, S^4, S^6, \ldots$ only the sphere S^2 is symplectic. If we can find closed 2-forms on the other spheres $S^4, S^6, S^8 \ldots$ They always have a singularity, that is there is always a point x where the 2-form degenerates.