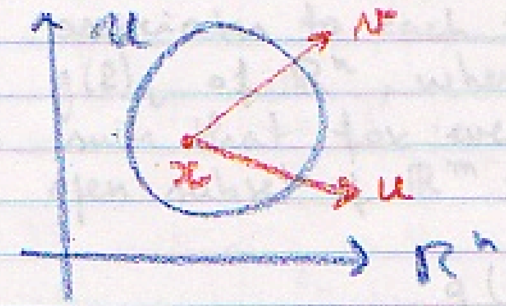
Geodesics, the general picture

Geometry is the art of measuring length and angles. Originally, angles and angles of the earth

$$\underbrace{\text{Geo}}_{\text{earth}} + \underbrace{\text{metry}}_{\text{measure}}$$

this is done by the mean of a scalar product g (see notes on the question)



$$\begin{cases} \|v\| = \sqrt{g_x(v,v)} \\ \|u\| = \|v\| = 1 \quad \cos(u,v) = g(u,v) \end{cases}$$

We have seen the definition of a scalar product on an open subset of $\mathbb{R}^n$. It is a map g which associates to each point x of the open subset $\mathcal{U}$ a scalar product g_x of $\mathbb{R}^n$ such that the coordinates g_{ij} of g, defined by:

$$g_{ij}(x) = g_x(e_i, e_j)$$

where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$, are smooth functions.

The question now is: how to define an euclidian product, also called a Riemmannian metric, on a manifold?

We use our approach of manifolds (described in the notes): it is first a smooth (diffeological) space, and second locally diffeomorphic to $\mathbb{R}^n$ (for some given n). So, we shall define a riemmannian metric g on a manifold M in 2-steps: first step we shall define g as a covariant 2-tensors (it is just a name for what we shall introduce), and after we shall ask this tensor to be positive definite on the charts of M. So:

a) A covariant 2-tensor on a smooth space X is a map which associates to each smooth parametrization $P : U \to X$ a map $g(P)$ defined on U which associates to each point $r \in U$ a bilinear map $g(P)_r$ of $\mathbb{R}^n$, where U is an open subset of $\mathbb{R}^n$, and such that for every $\varphi \in C^\infty(V, U)$, where V is an open subset of $\mathbb{R}^m$:

$$g(P \circ \varphi) = \varphi^*(g(P))$$

I remind that for a general covariant 2-tensor γ defined on U we have:

$$\varphi^*(\gamma)_s (v, v') = \gamma_{\varphi(s)} (D(\varphi)(s)(v), D(\varphi)(s)(v'))$$

where $s \in V$, $v, v' \in \mathbb{R}^m$.

b) If X is a manifold, we say that a covariant 2-tensor g defined on X is a riemmannian metric iff for any chart F of X, $g(F)$ is an euclidian metric defined on $\mathrm{dom}(F)$. That is, for each point $r \in \mathrm{dom}(F)$, $g(F)_r$ is an euclidian product on $\mathbb{R}^n$, where $\dim(X) = n$.

So, now let M be a manifold and let g be a riemmannian metric defined on M.

Let TM be the tangent space of M (defined in the notes). Let T^*M be the cotangent space of M.

Let $y \in TM$, for convenience we will write y as a pair $y = (x, v)$ where x is the base-point of y an v belongs to the tangent space at x of M, that is T_xM.

Let $a \in T^*M$, for convenience we will write a as a pair $a = (x, p)$ where x is the base point of a and p belongs to the cotangent space at x of M, that is $T_x^*M := (T_xM)^*$, the space $L(T_xM, \mathbb{R})$.

The metric g at the point $x \in M$ defines a linear isomorphism* from T_xM to T_x^*M by:

$$\begin{aligned} \hat{g}_x : T_xM &\to T_x^*M \\ v &\mapsto [v' \mapsto g_x(v, v')] \end{aligned}$$

So we have such a commutative diagram:

$$\begin{tikzcd} TM \arrow[rr, "\hat{g}"] \arrow[dr, "pr"'] & & T^*M \arrow[dl, "pr"] \\ & M & \end{tikzcd} \qquad \begin{cases} pr(x, v) = x \\ pr(x, a) = x \end{cases}$$

Now, we define the sphere bundle SM over M by restriction of TM on the unit vectors:

$$SM = \{(x,u) \in TM \mid g_x(u,u) = 1\} \subset TM$$

Now let us (re) introduce the Liouville form on T^*M

$a = (x,p) \in T^*M$, $\delta a \in T_a T^*M$ $\quad \lambda(\delta a) = p(\delta x)$, with $\delta x = D(pr)(a)(\delta a)$.

let us denote by ϖ the pullback

$$\varpi = \hat{g}^*(\lambda)$$

So, for $y = (x,v) \in TM$ and $\delta y \in T_y TM$ we have

$$\boxed{\varpi(\delta y) = g(v, \delta x)} \quad \text{with } \delta x = D(pr)(y)(\delta y)$$

let us denote by ω the exterior derivative of ϖ:

$$\omega = d\varpi$$

That is for $y = (x,v) \in TM$, $\delta y, \delta' y \in T_y TM$ we have

$$\boxed{\omega(\delta y, \delta' y) = \delta[g(v, \delta' x)] - \delta'[g(v, \delta x)]}$$

let us explicit this definition: let $s \mapsto y_s = (x_s, v_s)$ be a path in TM such that:

$$y_0 = y \quad \left.\frac{dy_s}{ds}\right|_{s=0} = \delta y \quad , \quad v_s \in T_{x_s} M \text{ for all } s.$$

as well let $s' \mapsto y_{s'}$ be a path such that

$$y_0 = y \qquad \left.\frac{d\,y_{s'}}{ds'}\right|_{s'=0} = \delta' y \quad , v_{s'} \in T_{x_{s'}} M \text{ for all } s'.$$

We have by definition:

$$\delta x = \left.\frac{d\,x_s}{ds}\right|_{s=0} \quad \text{and} \quad \delta' x = \left.\frac{d\,x_{s'}}{ds'}\right|_{s'=0}$$

δx and $\delta' x$ belong to $T_x M$ where $y_0 = y = (x, v)$.

We can find a map $(s, s') \mapsto y_{s,s'}$ such that:

$$y_s = y_{s,0} \quad y_{s'} = y_{0,s'} \quad \delta y = \left.\frac{\partial}{\partial s} y_{s,0}\right|_{s=0} \quad \delta' y = \left.\frac{\partial}{\partial s'} y_{0,s'}\right|_{s'=0}$$

(use a chart at the point y and prove it). Then

$$\delta x = \left.\frac{\partial}{\partial s} x_{s,0}\right|_{s=0} \quad \text{and} \quad \delta' x = \left.\frac{\partial}{\partial s} x_{s,s'}\right|_{s'=0}$$

Now, we can rewrite the value of ω:

$$\omega(\delta y, \delta' y) = \frac{\partial}{\partial s}\left\{ g_{x_{s,0}}\left(v_{s,0}, \left.\frac{\partial}{\partial s'} x_{s,s'}\right|_{s'=0}\right)\right\}$$

$$- \frac{\partial}{\partial s'}\left\{ g_{x_{0,s'}}\left(v_{0,s'}, \left.\frac{\partial}{\partial s} x_{s,s'}\right|_{s=0}\right)\right\}$$

But we cannot go further without introducing a chart and computing in terms of local coordinates.

So, let F be some chart of M around the point x let $\{g_{ij}\}_{i,j=1}^{n}$ be the coordinates of the metric g in the chart F. That is, $x = F(\underline{x})$, $\underline{x} = (x^1, \dots, x^n)$ and:

$$g_{ij}(\underline{x}) = g_{F(x)}\left(D(F)(\underline{x})(e_i), D(F)(\underline{x})(e_j) \right),$$

where e_i are the canonical basis vectors and $x = F(\underline{x})$. Let us introduce the coordinates of the vector v:

$$v = \sum_{i=1}^{n} v^i D(F)(\underline{x})(e_i) \quad , \underline{v} = (v^1, \dots, v^n).$$

Actually $D(F)(\underline{x}) : \mathbb{R}^n \longrightarrow T_{F(\underline{x})}M$ in a canonical basis of TM, associated to the chart F. So we get:

$$\omega(\delta y, \delta' y) = \sum_{i,j=1}^{n} \frac{\partial}{\partial s}\left\{ g_{ij}(\underline{x}_{s,0})\, v^i_{s,0} \frac{\partial x^j_{s,s'}}{\partial s'}\bigg|_{s'=0} \right\}$$

$$- \frac{\partial}{\partial s'}\left\{ g_{ij}(\underline{x}_{0,s'})\, v^i_{0,s'} \frac{\partial x^j_{s,s'}}{\partial s}\bigg|_{s=0} \right\}$$

$$= \sum_{i,j=1}^{n} \frac{\partial g_{ij}(\underline{x})}{\partial x^k} \frac{\partial x^k}{\partial s}\bigg|_{s=s'=0} v^i_{0,0} \frac{\partial x^j_{0,s'}}{\partial s'}\bigg|_{s'=0}$$

$$+ g_{ij}(\underline{x}) \frac{\partial v^i_{s,0}}{\partial s}\bigg|_{s=0} \frac{\partial x^j_{0,s'}}{\partial s'}\bigg|_{s'=0} + \cancel{g_{ij}(\underline{x})\, v^i_{0,0} \frac{\partial^2 x^j_{s,s'}}{\partial s \partial s'}\bigg|_{s=s'=0}}$$

$$- \frac{\partial g_{ij}(\underline{x})}{\partial x^k} \frac{\partial x^k}{\partial s'}\bigg|_{s=s'=0} v^i_{0,0} \frac{\partial x^j_{s,0}}{\partial s}\bigg|_{s=0}$$

$$- g_{ij}(\underline{x}) \frac{\partial v^i_{0,s'}}{\partial s'}\bigg|_{s'=0} \frac{\partial x^j_{s,0}}{\partial s}\bigg|_{s=0} - \cancel{g_{ij}(\underline{x})\, v^i_{0,0} \frac{\partial^2 x^j_{s,s'}}{\partial s' \partial s}\bigg|_{s=s'=0}}$$

We introduce the notations:

$$\partial_k g_{ij} = \frac{\partial}{\partial x^k} g_{ij}(\underline{x})$$

$$\delta x^k = \left.\frac{\partial x^k_{s,0}}{\partial s}\right|_{s=0} \qquad \delta' x^k = \left.\frac{\partial x^k_{0,s'}}{\partial s'}\right|_{s'=0}$$

$$\delta v^i = \left.\frac{\partial v^i_{s,0}}{\partial s}\right|_{s=0} \qquad \delta' v^i = \left.\frac{\partial v^i_{0,s'}}{\partial s'}\right|_{s'=0}$$

And we use the Einstein convention that the same index up and down on two variable are summed, for example:

$$\sum_{i,j=1}^{n} g_{ij}\, v^i \delta x^j \text{ write } g_{ij}\, v^i \delta x^j$$

or

$$\sum_{k=1}^{n} g_{ik}\, v^k \text{ writes } g_{ik}\, v^k$$

So, we get:

$$\omega(\delta y, \delta' y) = g_{ij}\, \delta v^i \delta' x^k + \partial_k g_{ij}\, \delta x^k v^i \delta' x^j$$
$$- g_{ij}\, \delta' v^i \delta x^k - \partial_k g_{ij}\, \delta' x^k v^i \delta x^j$$

We introduce now the so-called Christoffel symbols and we shall discuss their signification later. For now they are just a series of real functions defined on the domain of the chart F and defined by:

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}\left(\partial_k g_{j\ell} + \partial_j g_{k\ell} - \partial_\ell g_{jk}\right)$$

they satisfy the symmetry:

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

and we shall use the following property:

$$\partial_j g_{kl} = \Gamma^m_{jk} g_{ml} + \Gamma^m_{jl} g_{km}$$

So we rewrite the expression of ω:

$$\begin{aligned}\omega(\delta y, \delta' y) = {} & g_{ij}\, \delta v^i \delta' x^j + \left(\Gamma^m_{ki} g_{mj} + \cancel{\Gamma^m_{kj} g_{mi}}\right) \delta x^k v^i \delta' x^j \\ & - g_{ij}\, \delta' v^i \delta x^j - \left(\Gamma^m_{ki} g_{mj} + \cancel{\Gamma^m_{kj} g_{mi}}\right) \delta' x^k v^i \delta' x^j\end{aligned}$$

$$\left(\Gamma^m_{kj} = \Gamma^m_{jk}\right)$$

So:

$$\begin{aligned}\omega(\delta y, \delta' y) &= g_{ij}\, \delta v^i \delta' x^j + g_{mj} \Gamma^m_{ki}\, \delta x^k v^i \delta' x^j \\ &\quad - g_{ij}\, \delta' v^i \delta x^j - g_{mj} \Gamma^m_{ki}\, \delta' x^k v^i \delta x^j \\ &= g_{ij}\, \delta v^i \delta' x^j + g_{ij} \Gamma^i_{km}\, \delta x^k v^m \delta' x^j \quad (i \leftrightarrow m) \\ &\quad - g_{ij}\, \delta' v^i \delta x^j - g_{ij} \Gamma^i_{km}\, \delta' x^k v^m \delta x^j \quad (i \leftrightarrow m) \\ &= g_{ij}\left(\delta v^i + \Gamma^i_{km} v^m \delta x^k\right) \delta' x^i \\ &\quad - g_{ij}\left(\delta' v^i + \Gamma^i_{km} v^m \delta' x^k\right) \delta x^i\end{aligned}$$

So, we introduce the symbols $\hat{\delta}$ and $\hat{\delta}'$:

$$\hat{\delta} v^i = \delta v^i + \Gamma^i_{jk}\, \delta x^j v^k$$
$$\hat{\delta}' v^i = \delta' v^i + \Gamma^i_{jk}\, \delta' x^j v^k$$

Thus the form ω writes symply:

$$\omega(\delta y, \delta' y) = g_{ij}\, \hat{\delta} v^i \delta' x^j - g_{ij}\, \hat{\delta}' v^i \delta x^j$$

Now, we show in the technical appendice that the $\hat{\delta} v^1, \dots \hat{\delta} v^n$ are the coordinates of a vector $\hat{\delta} v$ tangent to M at x.

$$\hat{\delta} v = D(F)(\underline{x}) \begin{pmatrix} \hat{\delta} v^1 \\ \vdots \\ \hat{\delta} v^n \end{pmatrix} \in T_x M$$

Thus, the form ω write finally:

$$\boxed{\omega(\delta y, \delta' y) = g(\hat{\delta} v, \delta' x) - g(\hat{\delta}' v, \delta x)}$$

The map defined above by:

$$\Phi : \delta(x, v) \longmapsto (\delta x, \hat{\delta} v)$$

is an identification between $T_y TM$ and $T_x M \times T_x M$

the vector $\hat{\delta} v$ is called the covariant differential of v by δx. But we do not developp the concept of covariant differential more than necessary here.

Now let us consider the restriction of ω to the sphere bundle: let $y = (x, u) \in SM$, that is $g(u,u) = 1$, and $\delta y = \delta(x, u) \simeq (\delta x, \hat{\delta} u) \in T_x M \times T_x M$. But since $g(u,u) = 1$ we have:

$$g(u, \hat{\delta} u) = \frac{1}{2} \delta[g(u,u)] = 0$$

Thus $\hat{\delta} u$ is orthogonal to u. Then the equation of the kernel of $\omega | SM$ writes:

$$(\delta x, \hat{\delta} u) \in \ker(\omega | SM) \iff g(\hat{\delta} u, \delta' x) - g(\hat{\delta}' u, \delta x) = 0 \text{ for all}$$
$$\delta' x \in T_x M \text{ and } \hat{\delta}' u \in \ker \bar{u}$$

where $\bar{u} = [v \mapsto g(u, v)] \in T_x^* M$.

So the system becomes, introducing a Lagrange multiplier associated to the constraint on $\hat{\delta} u$:

$$g(\hat{\delta} u, \delta' x) - g(\hat{\delta}' u, \delta x) = \alpha \, g(u, \hat{\delta}' u) \quad \forall \, \hat{\delta}' u, \delta' x$$

We get:

$$\delta x = \alpha u \quad \text{and} \quad \hat{\delta} u = 0$$

That is:

$$\delta y = \delta(x, u) \simeq (\delta x, \hat{\delta} u) \in \ker(\omega | SM) \iff \begin{cases} \delta x = \alpha u, \ \alpha \in \mathbb{R} \\ \hat{\delta} u = 0 \end{cases}$$

or $\ker(\omega | SM) = \mathbb{R} \cdot \begin{pmatrix} u \\ 0 \end{pmatrix}$

with the identification given by the covariant differential.

These equations define on TM a 1-dimensional foliation called the geodesic foliation or geodesic flow. More precisely the geodesic flow is the flow associated to the vector field defined by

$$\Phi\left(\xi\begin{pmatrix} x \\ u \end{pmatrix}\right) = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

The equation $\hat{\delta} u$ writes in coordinates:

$$\hat{\delta} u^i = \delta u^i + \Gamma^i_{jk}\, u^j \delta x^k = 0 \qquad u^i = \delta x^i$$

that is if $\delta x = \dfrac{dx}{ds}$:

$$\boxed{\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0}$$

this equation is know as the equation of geodesics.

The integral curves of the distribution $y \mapsto \ker(\omega|SM)$ are the oriented, unparametrized geodesics of the metric g.

the image of a geodesic $\gamma \subset SM$, integral curve of the distribution $(x,u) \mapsto \ker(\omega|SM)$, on M by the projection $(x,u) \mapsto x$, is called a geodesic trajectory.