Poincaré's group and moments

Let E be a Minkowski space, i.e. a vector space of dimension 4 equipped with an hyperbolic metric of signature $+---$.

We call <u>Lorentz's frame</u> any basis S such that

$$X = S\begin{pmatrix} x \\ t \end{pmatrix} \quad \text{and} \quad \bar{S}S = \begin{pmatrix} -\mathbb{1}_3 & 0 \\ 0 & 1 \end{pmatrix}$$

in other words $g(\delta X, \delta X) = (\delta t)^2 - \|\delta x\|^2$. We know that any Lorentz matrix L of the restricted Lorentz's group can be written:

$$L = \exp\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{matrix} b \in \mathbb{R}^3 \\ A \in SO(3) \end{matrix}$$

More precisely the map $(b, A) \mapsto L$ is a diffeomorphism (not an isomorphism!) from $\mathbb{R}^3 \times SO(3)$ onto the restricted Lorentz group Lor.

An element Λ of the Lie algebra lor of Lor is a matrix

$$\Lambda = \begin{pmatrix} j(\omega) & \beta \\ \bar{\beta} & 0 \end{pmatrix} \quad \omega \in \mathbb{R}^3, \beta \in \mathbb{R}^3$$

an element a of the restricted Poincaré group Poi writes

$$a = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \quad L \in \mathrm{Lor}, \ C \in \mathbb{R}^4$$

where E is identified with the subspace of $\begin{pmatrix} X \\ 1 \end{pmatrix} \subset \mathbb{R}^5$.

So an element of $\mathfrak{Poi}$ the Lie algebra of Poi write

$$Z = \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \quad \Lambda \in \mathfrak{lor},\ \Gamma \in \mathbb{R}^4$$

writing
$$\Gamma = \begin{pmatrix} \gamma \\ \varepsilon \end{pmatrix} \quad \varepsilon \in \mathbb{R},\ \gamma \in \mathbb{R}^3$$
we get for Z the following expression:

$$Z = \begin{pmatrix} j(\omega) & \beta & \gamma \\ \bar{\beta} & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$$

[compare with an element of gal].

An element $\mu \in \mathfrak{Poi}^*$ can be identified to a pair:

$$\mu = (M, P) \text{ with } \bar{M} = -M \text{ and } P \in \mathbb{R}^4$$

transposed for g.

thanks to $\mu(Z) = -\frac{1}{2} T_r (M.\Lambda) - \bar{P}.\Gamma$

In terms of matrices that gives:

$$M = \begin{pmatrix} j(\ell) & g \\ \bar{g} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} p \\ E \end{pmatrix} \quad \begin{cases} \ell, g, p \in \mathbb{R}^3 \\ E \in \mathbb{R} \end{cases}$$

and then $\mu(Z) = \langle \ell, \omega \rangle - \langle g, \beta \rangle + \langle p, \gamma \rangle - \varepsilon E$

[Note the similarity with the galilean version].

Note: we shall not prove that but the cohomology group $H^1(\mathrm{Poi}, \mathfrak{Poi}^*)$ is zero.

Consequence: For a presymplectic manifold (M, ω) equipped with an hamiltonian action of Poi with moment Ψ there exists an additive constant such that

$$\Psi(a_M(m)) = Ad_*(a)(\Psi(m))$$

for every $a \in$ Poi and $m \in M$. So there exists a natural way to fix the arbitrary constant involved in the definition of the moment map. This is a big difference with the galilean case.

Comparing Gal and Poi and Moments:

The groups Gal and Poi are both subgroups of the group of affin transformations of $\mathbb{R}^5$ with $X = \begin{pmatrix} x \\ t \\ 1 \end{pmatrix}$

The intersection $\tilde{G} = \text{Gal} \cap \text{Poi}$ is still a Lie group. And if Gal and Poi act on a same presymplectic manifold (M, ω), which is the case for a system of a free particle as we have seen above, the moment maps through the action of $\tilde{G}$ can help to identify some common components in the galilean and Einsteinian cases. That is:

$$a \in \tilde{G} \qquad a = \begin{pmatrix} A & 0 & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}$$

$$z \in \tilde{\mathfrak{g}} \qquad z = \begin{pmatrix} j(\omega) & 0 & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$$

So $\tilde{\mu}(Z) = \langle l, \omega \rangle + \langle p, \gamma \rangle - E\varepsilon \quad \forall Z \in \tilde{g}$

in consequence the relativistic elements l, p, E can be called again:

l : angular momentum
p : kinetic momentum
E : energy

and the 4-vector

$$P = \begin{pmatrix} p \\ E \end{pmatrix}$$: energy momentum vector.