Topological methods in the free group
Exercise set 3

January 25, 2018

To be handed in by February 23rd, 2017.

You are required to hand in solutions for **4 out of the following 6** exercises.

Exercise 1: Let H be a finitely generated subgroup of the free group $\mathbb{F}_n$ of rank n, which we see as the fundamental group of the rose R_n. Let Δ_H be the core graph of H, v its base vertex, and $j : \Delta_H \to R_n$ the corresponding embedding. Suppose $f : \Delta_H \to \Delta_H$ is an automorphism of the immersion, i.e. that $j \circ f = j$.

Prove that for any path p in Δ_H joining v to $f(v)$, the element $g = [j(p)]$ is in the normalizer of H (i.e., $ghg^{-1} \in H$ for any $h \in H$).

Exercise 2: Count the number subgroups of index 3 of $\mathbb{F}(a, b)$ which do not contain the elements a, ab.

Exercise 3: Let $\mathbb{F}(a, b)$ be the free group on two generators.

1. Is $H = \langle aba^{-2}, a^2ba^{-1}, a^3b, a^4, a^3b^{-1}\rangle$ normal in $\mathbb{F}(a, b)$?
2. Give a generating set for the intersection in $\mathbb{F}(a, b)$ of the subgroups $H_1 = \langle b, aba^{-1}\rangle$ and $H_2 = \langle b^{-1}ab, ba^{-1}, ba\rangle$.

Exercise 4: Show that a normal subgroup of odd index of $\mathbb{F}(a_1, \ldots, a_n)$ which doesn't contain a_i cannot contain a_i^2.

Exercise 5: Prove Hannah Neumann's conjecture in the case where the two subgroups have finite index, namely, show that if H_1, H_2 are finite index subgroups of a free group $\mathbb{F}$, then

$$\mathrm{rk}(H_1 \cap H_2) - 1 \leq (\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1)$$

[Hint: follow the proof of Howson's theorem remembering what you know about core graphs of finite index subgroups]

Exercise 6: Let H be a finitely generated subgroup of the free group $\mathbb{F}(a_1, \ldots, a_n)$ such that for any $g \in \mathbb{F}$, there exists $k \in \mathbb{N}$ with $g^k \in H$. Prove that H must have finite index in $\mathbb{F}_n$.