Topological methods in the free group
Exercise set 1

December 12, 2017

To be handed in by December 17th, 2017.

You are required to hand in solutions for **4 out of the following 6** exercises.

Exercise 1:

1. Find a basis for each one of the following subgroups of $\mathbb{F}(a,b)$ and $\mathbb{F}(a,b,c)$ respectively: $H_1 = \langle a^3bab, a^2, aba\bar{b}^2\rangle$, $H_2 = \langle bc^2\bar{b}, \bar{b}a^2c, acac, c\bar{b}\rangle$
2. Decide whether the following elements belong to H_1: $baba, a\bar{b}, a^{-1}ba\bar{b}^2$.
3. Decide whether the following elements belong to H_2: $aca^{-1}b, bc^{-2}, acb^{-1}a$.

Exercise 2: Let e, e' be an admissible pair of edges in a connected graph Γ. Let $f : \Gamma \to \Gamma/[e = e']$ be the corresponding folding map. Prove that the induced morphism f_* is an isomorphism iff $\tau(e) \neq \tau(e')$.

Exercise 3: Describe a finite procedure which given a finite set of elements $g, h_1, \ldots, h_k$ of the free group $F(a_1, \ldots, a_n)$, decides whether some conjugate of the element g belongs to the subgroup H of $F(a_1, \ldots, a_n)$ generated by $h_1, \ldots, h_k$.

Exercise 4: Let Δ be a connected graph. Let $f : \tilde{\Delta} \to \Delta$ be a covering map such that $\tilde{\Delta}$ is a tree.

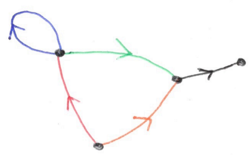
1. (Universal property of universal cover) Suppose $g : \Gamma \to \Delta$ is a covering map with Γ connected. Show that there exists a map $h : \tilde{\Delta} \to \Gamma$ such that $f = g \circ h$.
2. (Uniqueness of universal cover) Prove that if $f_1 : \tilde{\Delta}_1 \to \Delta$ and $f_2 : \tilde{\Delta}_2 \to \Delta$ are covering maps such that $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ are trees, then there is an isomorphism $h : \tilde{\Delta}_1 \to \tilde{\Delta}_2$ (i.e. a bijective graph map).

Exercise 5: Let H, K be finitely generated subgroups of $\mathbb{F}(a_1, \ldots, a_n)$. Denote by Δ_H, Δ_K the graphs representing H, K obtained by the usual folding procedure, together with the orientation and coloring of the edges giving the immersion in R_n.

Show that $H \leq J$ iff there exists a graph map $\Delta_H \to \Delta_K$ which respects the basepoint, the colouring of the edges and their orientation.

Exercise 6:

1. Draw the universal cover of the following graph (in as regular a way as possible), with colours and arrows on th edges to indicate their image by the covering map:



2. Let R_2 denote the rose with two petals, whose fundamental group we identify with $\mathbb{F}(a,b)$. Draw the covering $f : \Gamma \to R_2$ such that $f_*(\pi_1(\Gamma, u)) = \langle a \rangle$.