Topological methods in the free group
Exercise set 1

November 12, 2017

To be handed in by November 23rd, 2017.

You are required to hand in solutions for **5 out of the following 7** exercises.

Exercise 1: Let S be a set. Consider the free group $F(S)$ on S. An element $g \in F(S)$ is a reduced word, in particular it is a finite sequence of elements of $S \cup S^{-1}$. We call length of g relative to S, and denote $l_S(g)$, the length of this sequence.

1. Show that any nontrivial element g admits at most $l_S(g)$ conjugates of minimal length, that is, of $\min\{l_S(g') \mid g' \text{ conjugate to } g\}$.

If g has minimal length in its conjugacy class, it is said to be **cyclically reduced**.

2. Show that g is cyclically reduced iff $l_S(g^k) = |k|\, l_S(g)$ for all $k \in \mathbb{Z} - \{0\}$.

Exercise 2: Consider a set S and $F(S)$ the free group on S. Let g, h be elements in $F(S)$ and n, m be integers.

- Show that if $g^n = h^n$ then $g = h$. [Hint: consider first the case where g is cyclically reduced - see exercise above].
- Show that if $g^m h^n = h^n g^m$ then g, h commute.

Exercise 3: Consider a set S and $F(S)$ the free group on S. Let g, h be elements in $F(S)$. The purpose of the exercise is to show that g, h commute iff they are powers of a common element.

1. Show the easy direction (if they are powers of a common element, then they commute).
2. Prove the result under the additional hypothesis that the first letter of h is not the inverse of the last letter in g, i.e. that there is no cancellation in the product gh.
3. Prove the result under the additional hypothesis that h is cyclically reduced.
4. Conclude.

Exercise 4: (Free groups are linear) Consider the set $S = \{A, B\}$ in $GL_2(\mathbb{R})$ where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$$

Prove that the subgroup H of $GL_2(\mathbb{R})$ generated by A, B is free on S (you may use the fact that π is not the root of any polynomial with coefficients in $\mathbb{Q}$).

Exercise 5: (Hopf property for free groups) We will see in the course that free groups are residually finite, i.e. that for any nontrivial element g of a free groups F, there exists a morphism $h : F \to G$ where G is a finite group, such that $h(g) \neq 1$.

Let F be a free group of finite rank k. We want to prove that F satisfies the Hopf property, namely, that any surjective morphism $p : F \to F$ is also injective.

1. Prove that there are only finitely many morphisms $F \to G$.

2. Suppose g is a nontrivial element in Ker$p$, let $h : F \to G$ be a morphism to a finite group such that $h(g) \neq 1$. Prove that the maps $h \circ p^n$ are pairwise distinct.

3. Conclude.

Exercise 6: (Generating set of smallest possible size is a basis) Let F be a free group of rank k.

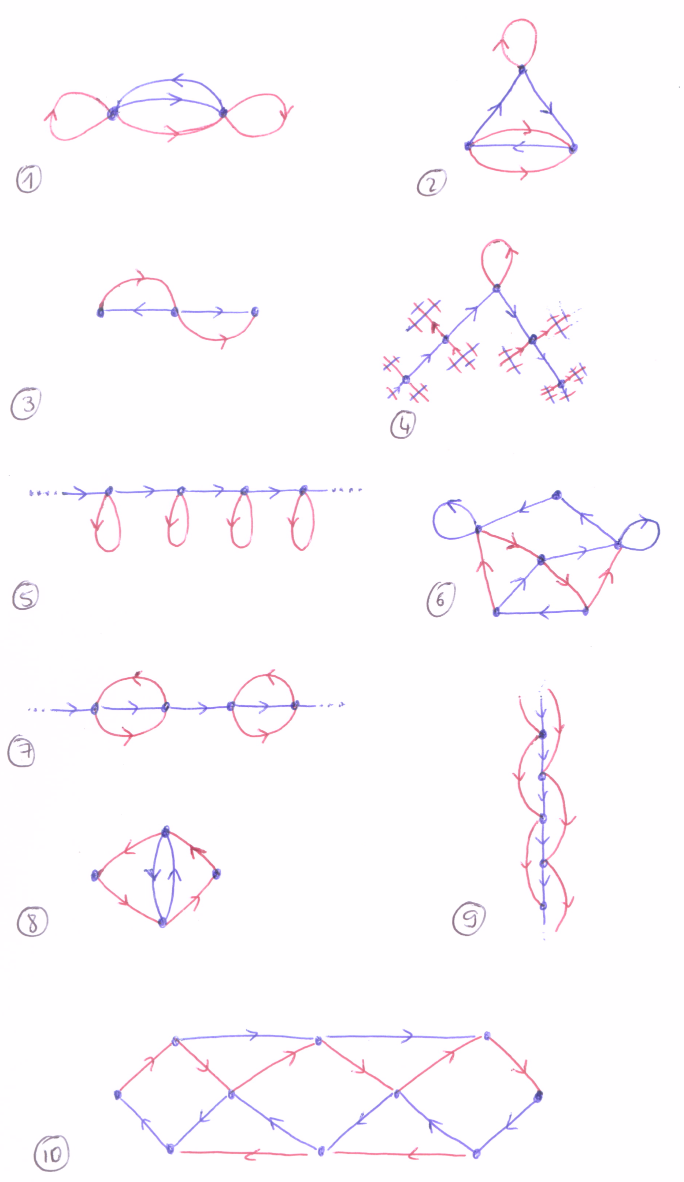
1. Prove that if $S \subseteq F$ is a generating set of size k, then S must be a basis for F. (You may use Hopf property of the free group, proved in the previous question).

2. Prove that G has no generating set of size strictly smaller than k.

Exercise 7: Let R_2 be the rose on two edges, one red and one blue, for which we choose an orientation. The colouring and orientation on each the following graphs gives a graph map to R_2, which maps red edges on the red edge of R_2, and blue edges on the blue edge of R_2, respecting the orientation.

1. Which of these maps are immersions? Which are coverings? (In graph 4, the two trees are infinite and regular)

2. Add edges to the graph which represent immersions to turn them into coverings.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