Exercise set 4

This is the fourth and last exercise of the course. To be handed in by February 26th, 2017, directly to me or via email to perin@math.huji.ac.il.

Solutions in english or typeset much appreciated, but of course hebrew and handwritten are fine (please write as clearly as possible).

You are required to hand in solutions for 4 out of the following 6 exercises.

Exercise 1: Let T_0 be the tree with vertices $\{u, \bar{x}, \bar{y}, \bar{z}\}$ and edges $\{u\bar{x}, u\bar{y}, u\bar{z}\}$. We denote by $T(a, b, c)$ the geometric realization of T_0 which gives lengths a, b, c to $u\bar{x}, u\bar{y}, u\bar{z}$ respectively. Let X be a geodesic metric space.

1. Show that for each geodesic triangle Δ in X with vertices x, y, z, there exists a unique triple (a, b, c) such that there exists a map $f_\Delta : \Delta \to T(a, b, c)$ which sends x, y, z to $\bar{x}, \bar{y}, \bar{z}$ respectively, and restricts to an isometry on each side of Δ.
2. Prove that X is hyperbolic if and only if there exists δ_1 such that for any geodesic triangle Δ, the diameter of $f_\Delta^{-1}(u)$ is at most δ_1.

Exercise 2: Let $(G, S = (s_1, \dots, s_k))$ be a marked group. Suppose (G, S) is the limit of a sequence of free marked groups in $\mathcal{G}_k$.

1. Let $T = (t_1, \dots, t_m)$ be another generating set for G. Show that (G, T) is also the limit of a converging sequence of free marked groups.
2. Let G_0 be a finitely generated subgroup of G. Show that G_0 is also a limit group.

Exercise 3:

1. Show that finite groups are isolated points in $\mathcal{G}_k$. Finiteness is thus an open property. Is it closed?
2. Show that for any $j < k$, the set of points of $\mathcal{G}_k$ which admit a generating set of size at most j is open.
3. Show that if G admits a finite presentation $\langle S \mid R \rangle$, then (G, S) admits a neighborhood in $\mathcal{G}_k$ which consists of quotients of G.

Exercise 4: A subgroup H of a group G is said to be malnormal if for any $g \in G$, the intersection $H \cap gHg^{-1}$ is non-trivial iff $g \in H$. A group G is said to be CSA if every maximal abelian subgroup of G is malnormal.

1. Show that if G is CSA, it must be commutative-transitive.
2. Prove that limit groups are CSA.

Exercise 5: Recall that the language of groups consists of the symbols $(\cdot, ^{-1}, 1)$ - this means that a first order formula is equivalent to a formula of the form

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \ OR_{i=1}^{p} \ AND_{j=1}^{q_i} \ w_{ij}(x_1, y_n, \dots, x_n, y_n) \Delta_{ij} 1$$

with $\Delta_{ij} \in \{=, \neq\}$. The first-order theory of a group is the set of all first-order formulas satisfied by a group.

1. Show that the groups $\mathbb{Z}$ and $\mathbb{Z}^n$ for $n \neq 1$ have different first-order theory in the language of groups (hint - every element in $\mathbb{Z}$ is even or odd).

2. Show that a group which has the same first-order theory as the free group does not contain a maximal abelian subgroup isomorphic to $\mathbb{Z}^n$.

Exercise 6: (Yet another characterization of limit groups) Let G, H be two finitely generated groups. We say that a sequence of morphisms $f_n : G \to H$ converges if for any $g \in G$ there exists $n_g \in \mathbb{N}$ such that either (i) for all $n \geq n_g$, $f_n(g) = 1$ or (ii) for all $n \geq n_g$, $f_n(g) \neq 1$. We then denote $\underrightarrow{\mathrm{Ker}} f_n$ the set of elements $g \in G$ for which $f_n(g)$ is eventually always 1.

1. Show that G is a limit group if and only if there exists a free group $\mathbb{F}$ and a convergent sequence of morphisms $f_n : G \to \mathbb{F}$ such that $\underrightarrow{\mathrm{Ker}} f_n$ is trivial.

2. Show that if $\mathbb{F}$ is a free group and $f_n : G \to \mathbb{F}$ is a convergent sequence, then for any n large enough f_n factors through the quotient $G \to G/\underrightarrow{\mathrm{Ker}} f_n$.

3. Deduce that $G/\underrightarrow{\mathrm{Ker}} f_n$ is a limit group.