Exercises set 3

This is the third exercise of the course. To be handed in by January 29th, directly to me or via email to perin@math.huji.ac.il.

Solutions in english or typeset much appreciated, but of course hebrew and handwritten are fine (please write as clearly as possible).

You are required to hand in solutions for 5 out of the following 7 exercises.

Exercise 1: (Maximal possible growth function) Let G be a group endowed with a finite generating set S. Show that the cardinality of the ball of radius n in (G, d_S) is at most $1 + k\Sigma_{0 \leq m < n}(k-1)^m$ where $k = 2|S|$. Show that this bound is attained for all n iff the group is free on S.

Exercise 2: (Finding free subgroups of amalgamated products) Let G be the amalgamated product $A *_C B$, and let T be the corresponding tree on which G acts with two orbits of vertices and one orbit of edges.

1. Show that an element $g \in G$ lies in a conjugate of A or B if and only if it fixes a vertex v in T.
2. Let H be a subgroup of G which does not intersect any conjugate of A or B non-trivially. Show that H is a free group.

Exercise 3: (Finite order elements in amalgamated products) Let $G = A *_C B$ be an amalgamated product. By using the corresponding action on a tree, show that if $g \in G$ has finite order (i.e. $g^m = 1$ for some $m \in \mathbb{N}$), then g lies in a conjugate of A or in a conjugate of B.

Exercise 4: (Actions of elements of the free group on its Cayley graph) Consider the action of $F(a, b)$ on its Cayley graph T.

1. Let $g = baab^{-1}$. Draw (a piece of) T, and draw the paths joining $g^{-2}, g^{-1}, 1, g, g^2$. Show that there exists a line in T which is preserved by g, and on which g acts as a translation with translation length 2.
2. Show that for any nontrivial element $g \in F(a, b)$, there is a line A_g preserved by g on which g acts by translation. What is the translation length $l(g)$?
3. Show that for any point $x \in T$ we have $d(x, g \cdot x) = 2d(x, A_g) + l(g)$, and deduce that A_g is exactly the set of points minimally displaced by g.

Exercise 5: (Types of isometries of trees) Let G be a group acting on (the geometric realization of) a tree T (by isometries). We call translation length of an element g of G the number

$$l(g) = \inf\{d(x, g \cdot x) \mid x \in T\}$$

1. Show that this inf is a min.

We call "axis of g", and denote by $\mathrm{Ax}(g)$, the set $\{x \in T \mid d(x, g \cdot x) = l(g)\}$.

2. Suppose $l(g) = 0$. Show that the set $\mathrm{Ax}(g)$ is a subtree of T.
3. Suppose $l(g) > 0$. Show that $\mathrm{Ax}(g)$ is a line, on which g acts by a translation of length $l(g)$.(Hint: remember Exercise 4).
4. Show that for any point $y \in T$ we have $d(y, g \cdot y) = 2d(y, \mathrm{Ax}(g)) + l(g)$.

Exercise 6: (Isometries of the hyperbolic plane) In this exercise, you can think of the hyperbolic upper half plane model $\mathbb{H}^2$ either with real coordinates ($\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$) or with complex coordinates ($\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$) according to convenience.

1. Show that the following transformations are isometries of $\mathbb{H}^2$: reflection through the y-axis ($(x, y) \mapsto (-x, y)$ or $z \to \bar{z}$), horizontal translation ($(x, y) \mapsto (x + a, y)$ or $z \mapsto z + a$), dilation by a factor $\lambda \in \mathbb{R}$ ($(x, y) \mapsto (\lambda x, \lambda y)$ or $z \mapsto \lambda z$), and reflection in the unit (half) circle ($(x, y) \mapsto (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ or $z \mapsto 1/\bar{z}$).

2. Show that any map of the form $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{R}$ satisfying $ad - bc = 1$ is an isometry of $\mathbb{H}^2$. (Hint: write it as a composition of the isometries above).

3. Shows that this gives a morphism from $\mathrm{SL}_2(\mathbb{R})$ to the group of isometries of $\mathbb{H}^2$.

4. What is its kernel?

Remark: *In fact, the image of $SL_2(\mathbb{Z})$ by this morphism is exactly the group of all orientation preserving isometries of $\mathbb{H}^2$.*

Exercise 7: (Geodesic segments in $\mathbb{H}^2$)

1. Show that the path with the shortest length joining the points $U = (0, u)$ et $V = (0, v)$ in $\mathbb{H}^2$ is unique, and that it is the vertical straight line segment.

2. Show that any orientation preserving isometry of $\mathbb{H}^2$ (see exercise and remark above) sends the x-axis to another vertical line or to a half circle centered on the y-axis (you may want to consider the image of the point iy when $y \to 0$ and $y \to \infty$ to begin with). Show that any half circle can be obtained in this way.

3. Deduce that the shortest path in $\mathbb{H}^2$ between two points P, Q is an arc of the unique circle whose center is on the x-axis which goes through P and Q.