Exercise set 2

This is the second exercise of the course. To be handed in by January 2nd, 2017, directly to me or via email to perin@math.huji.ac.il.

Solutions in english or typeset much appreciated, but of course hebrew and handwritten are fine (please write as clearly as possible).

You are required to hand in solutions for **6 out of the following 8** exercises.

Exercise 1: Show that an n-regular tree is quasi-isometric to an m-regular tree if $m, n \geq 3$.

Exercise 2: Let G, H be groups with finite generating sets S, T respectively. Give a characterization of group homomorphisms $f : G \to H$ for which $f : (G, d_S) \to (H, d_T)$ is a quasi isometry.

Exercise 3: Let X and Y be two metric spaces. Prove or disprove the following statements.

1. If X and Y are homeomorphic, they are quasi-isometric.
2. If X and Y are quasi-isometric, they are homeomorphic.

Exercise 4:

1. For $p \geq 1$, let d_p be the metric on $\mathbb{R}^2$ induced by the p-norm $\|(x, y)\|_p = (x^p + y^p)^{1/p}$. Show that all the metric spaces $(\mathbb{R}^2, d_p)$ are quasi-isometric.
2. Find a metric d on $\mathbb{R}^2$ so that $(\mathbb{R}^2, d)$ is not quasi-isometric to $(\mathbb{R}^2, d_2)$. Can you find one which induces the standard topology and one which doesn't?

Exercise 5: The purpose of this exercise is to prove "by hand" that $\mathbb{R}$ and $\mathbb{R}^2$ are not quasi-isometric. Suppose thus that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a (C, D)-quasi-isometric embedding.

1. Show that a continuous map from a circle to a line identifies a pair of antipodal points.
2. Let $a \geq 1$. Let $x_0, \ldots, x_{2n}$ be evenly spaced points on the circle of radius a with n large enough so that $d(x_i, x_{i+1}) < 1$. Show that there exists i with $d(\phi(x_i), \phi(x_{n+i})) \leq 2C + 2D$.
3. Conclude.

Exercise 6: Consider the group $BS(1, 2)$ given by the following presentation:

$$\langle a, t \mid tat^{-1} = a^2 \rangle$$

1. Show that $BS(1, 2)$ is torsion free.
2. Prove that the morphism $\mathbb{Z} \to BS(1, 2)$ defined by sending the generator of $\mathbb{Z}$ to a is not a quasi isometric embedding.

Exercise 7: Recall that we defined the subgroup H of $SL_2(\mathbb{Z})$ to be the subgroup generated by $\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, and that we showed it is free on $\{\alpha, \beta\}$.

1. Show that H consists of matrices of the form $\begin{pmatrix} 4j+1 & 2k \\ 2l & 4m+1 \end{pmatrix}$ for $j, k, l, m \in \mathbb{Z}$ whose determinant is 1.
2. Consider the map $p : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/2\mathbb{Z})$ given by reducing the coefficients of a matrix mod 2. Let $\Gamma(2)$ be the kernel of p. Show that H has finite index in $\Gamma(2)$.
3. Deduce that $SL_2(\mathbb{Z})$ is quasi-isometric to a tree.

Exercise 8: Let G, H be finitely generated infinite groups. Prove that $e(G \times H) = 1$.