Exercise set 4

January 15, 2018

To be handed in by February 11th, 2018, in the mailbox in Manchester building.

You are required to hand in solutions for **4 out of the following 6** exercises.

Exercise 1: The aim of this exercise is to prove that the distance function

$$d_{\mathbb{H}^2}(P,Q) = \inf\{l_{\mathbb{H}^2}(\gamma) \mid \gamma : [a,b] \to \mathbb{H}^2 \text{ piecewise smooth curve with } \gamma(a) = P, \gamma(b) = Q\}$$

induced on $\mathbb{H}^2$ by the Riemannian metric we defined satisfies $d(P,Q) = 0$ iff $P = Q$. Let $\gamma : [a,b] \to \mathbb{H}^2$ be a piecwise smooth curve joining distinct points P and Q. Write $\gamma(t) = (x(t), y(t))$.

1. Suppose that for all $t \in [a,b]$ we have $y(t) \leq 2y(a)$. Show that $l_{\mathbb{H}^2}(\gamma) \geq d_{\mathbb{R}^2}(P,Q)/2y(a)$, where $d_{\mathbb{R}^2}$ denotes the usual Euclidean metric.
2. Suppose now that there exists $t_0 \in [a,b]$ such that $y(t_0) = 2y(a)$, wlog assume that t_0 is minimal for this property, and denote $\gamma_1 = \gamma \mid_{[a,t_0]}$. Show that $l_{\mathbb{H}^2}(\gamma_1) \geq 1/2$.
3. Conclude.

Exercise 2: A new isometry of $\mathbb{H}^2$.

1. Show that the map $z \mapsto -\bar{z}$ sends $\mathbb{H}^2$ to itself, and that it is an isometry of Riemannian metrics from $\mathbb{H}^2$ to itself.
2. Show that it is not of the form $z \mapsto \frac{az+b}{cz+d}$ for some matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL_2(\mathbb{R})$.

Exercise 3:

1. Let $a \in \mathbb{R}, r \in \mathbb{R}^+$ and $\alpha, \beta \in (0, \pi)$. Without computation, show that the distance in $\mathbb{H}^2$ between the points $x = a + re^{i\alpha}$ and $y = a + re^{i\beta}$ depends only on α and β.
2. Show that

$$d_{\mathbb{H}^2}(x,y) = \left[\ln \left(\frac{\sin t}{1 + \cos t} \right) \right]_{t=\alpha}^{t=\beta}$$

Exercise 4: Consider the hyperbolic line $L = \{(0,y) \mid y > 0\}$. Let $p = (a,b) \in \mathbb{H}^2$ be such that $a \neq 0$.

1. By applying the triangle inequality in the triangle formed by the points $(a,b), (-a,b)$ and $(0,y)$ for some $y > 0$, show that the shortest path from (a,b) to a point of L is the arc of the euclidean half circle with center at the origin going through (a,b).
2. (Bonus) Let $d > 0$. Show that the set of points at distance d of L is not a hyperbolic line.

Exercise 5: (Triangles in $\mathbb{H}^2$ are thin) Let $a > 0$. Define $L_1 = L_1(a) = \{(0,y) \mid y > 0\}$, $L_2 = L_2(a) = \{(x,y) \mid (x-a)^2 + y^2 = a, y > 0\}$ and $L_3 = L_3(a) = \{(2a, y) \mid y > 0\}$.

1. Using the conclusions of Exercise 3.2 and Exercise 4.1, show that the distance $d(L_1, p)$ for a point p on L_2 increases monotonically to infinity as p moves towards $(2a, 0)$ on the line L_2.
2. Show that for any point p of L_2 we have $d(p, L_1 \cup L_3) \leq \ln(1 + \sqrt{2})$ (you can start by proving that this distance is maximal for the point $p = (a,a)$).

3. Let $p_0q_0r_0$ be a triangle in $\mathbb{H}^2$ (whose sides are segments of hyperbolic lines). Show that there exists $a \geq 0$ such that $p_0q_0r_0$ is isometric to a triangle pqr which lies inside the region bounded by lines $L_1(a), L_2(a), L_3(a)$ and such that $p, q \in L_2(a)$.

4. Deduce that for any triangle in $\mathbb{H}^2$, any point on one of the sides of the triangle lies within a distance at most $(\ln(1+\sqrt{2}))$ from one of the other two sides.

Exercise 6: Consider in $\mathbb{H}^2$ the hyperbolic circle centered at $a + ib$ with radius r, i.e. the set

$$C = \{z \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(z, a+ib) = r\}$$

1. Show that C is the Euclidean circle with centre $a + ib(e^r + e^{-r})/2$ and radius $b(e^r - e^{-r})/2$ (you may want to reduce to the case where $a = 0$).

2. Deduce that any Euclidean circle is also a hyperbolic circle.