Exercise set 3

December 28, 2017

To be handed in by January 1st, 2017, in the mailbox in Manchester building.

You are required to hand in solutions for **4 out of the following 6** exercises.

Exercise 1: Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C})$. Let $f(z) = \frac{az+b}{cz+d}$ be the corresponding Möbius transformation. Show that if f is not the identity, then it has one or two fixed points in $\mathbb{C} \cup \{\infty\}$. Show that if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU_2(\mathbb{C})$, f has exactly two fixed points.

Exercise 2: The aim of the exercise is to prove that Möbius transformations send lines and circles to lines and circles.

1. Show that any transformation of the form $f(z) = az$ or $f(z) = z + b$ for $a \in \mathbb{C}^*, b \in \mathbb{C}$ sends lines to lines, and circles to circles.
2. Show that the inversion map $T(z) = 1/z$ sends line which go through 0 to lines which go through 0, and other lines to circles. Show that T sends circles which do not go through 0 to circles, and circles which go through 0 to lines. [You may use that a line in $\mathbb{C}$ is a set of the form $\{z = x + iy \mid Ax + By = C\}$ for $A, B \in \mathbb{R}$, and a circle is a set of the form $\{z = x + iy \mid x^2 + y^2 + Ax + By = C\}$ for $A, B, C \in \mathbb{R}$]
3. Conclude.

Exercise 3:

1. Find the image of the disk bounded by the circle $|z - 2| = 2$ under the map $f(z) = \frac{z}{2z-8}$
2. Construct a Möbius transformation which takes the set $\{z \mid |z| < 1\}$ to the set $\{z = x + iy \mid y > 0\}$.

Exercise 4: Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. Show that f maps the real line to the real line iff we can take a, b, c, d to be **real numbers**.
2. Show that it maps the upper half plane $\mathbb{H} = \{z = x + iy \mid y > 0\}$ to itself iff a, b, c, d are real numbers and satisfy $ad - bc > 0$.

Exercise 5: (You may use the conclusions of Exercise 2). Suppose that Γ is a circle or straight line in the complex plane, containing distinct points z_1, z_2, z_3.

1. If T denotes the unique Möbius transformation sending the points z_1, z_2, z_3 to $0, 1, \infty$ respectively show that any further (distinct) point z_4 lies on Γ if and only if $T(z_4)$ is real.
2. Deduce that four distinct points in the complex plane lie on a circle or straight line if and only if their cross-ratio is real.

Exercise 6: We consider the following Riemannian metric g on $\mathbb{R}^2$:

$$\frac{\mathrm{d}x^2}{(1+x^2)^2} + \frac{\mathrm{d}y^2}{(1+y^2)^2}$$

1. Show that for any $a, b \in \mathbb{R}$, the horizontal path $\gamma : [0,1] \to \mathbb{R}^2$ parametrized by $\gamma(t) = (at + b, 0)$ has length $l_g(\gamma)$ at most π with respect to the Riemannian metric g.

2. Show that the same holds for vertical paths of the form $\gamma(t) = (0, at + b)$.

3. Deduce that in the metric d_g on $\mathbb{R}^2$ induced by the Riemanninan metric g, the distance between two points is at most 2π.