# Exercise set 2

December 5, 2017

To be handed in by December 7th, 2017, in the mailbox in Manchester building.

You are required to hand in solutions for **5 out of the following 7** exercises.

**Exercise 1:** Given distinct points $P, Q$ on the sphere $\mathbb{S}^2$, show that the set

$$\{x \in \mathbb{S}^2 \mid d^{\mathbb{S}^2}(P, x) = d^{\mathbb{S}^2}(x, Q)\}$$

is a great circle.

**Exercise 2:** Let $\Delta$ be a spherical triangle (with side lengths strictly smaller than $\pi$).

1. Show that $\Delta$ is contained in a closed hemisphere $H$.
2. We take as a fact that $\Delta$ divides the sphere in two regions: the one contained in $H$, which we call the interior of $\Delta$, and another one, which we call the exterior of $\Delta$ and denote $E(\Delta)$. Denote by $\alpha, \beta, \gamma$ the angles of $E(\Delta)$, that is, the exterior angles of $\Delta$. Prove that the area of $E(\Delta)$ is equal to $\alpha + \beta + \gamma - \pi$.

**Exercise 3:** (Gauss-Bonnet for convex spherical $n$-gons) Let $\Pi$ be a spherical $n$-gon, that is, a set $P_1, \ldots, P_n$ of pairwise distinct points on $\mathbb{S}^2$, together with spherical segments $[P_i, P_{i+1}]$ which meet only in their endpoints.

1. Assume that $\Pi$ is contained in some open hemisphere $H$, and that for each $i$, the interior angle $\alpha_i$ at each vertex $P_i$ is contained in the open interval $(-\pi, \pi)$.

   We take as a fact that $\Pi$ divides the sphere in two region, only one of which is contained in $H$. Prove that the area of this region is given by

$$\alpha_1 + \ldots + \alpha_n - (n-2)\pi$$

2. (bonus) Are the states of Colorado and Wyoming spherical $n$-gons on the surface of the earth?

**Exercise 4:** Show that any two spherical triangles which have the same interior angles necessarily have the same lengths of sides. Compare with the euclidean case.

**Exercise 5:** Let $ABC$ and $A'B'C'$ be two spherical triangles so that the interior angles at $A$ and $A'$ are both $\alpha$, at $B$ and $B'$ are both $\beta$, and at $C$ and $C'$ are both $\gamma$. Show that there is an isometry of $\mathbb{S}^2$ sending $A$ to $A'$, $B$ to $B'$ and $C$ to $C'$ (you may use the conclusions of the previous exercise).

**Exercise 6:**

1. Show that given any $\alpha, \beta \in ]0, \pi[$ and $t \in ]0, \pi[$ there exists a spherical triangle $ABC$ with angles at $A, B$ respectively given by $\alpha, \beta$, and length of $AB$ given by $t$.
2. Compute the value of the angle $\gamma_t$ at $C$ of this triangle in terms of $t, \alpha, \beta$. What are $\lim_{t \to 0} \gamma_t$ and $\lim_{t \to \pi} \gamma_t$?
3. Show that given $\alpha, \beta, \gamma$ in $]0, \pi/2[$ with $\alpha + \beta + \gamma > \pi$, there exists a spherical triangle whose interior angles are $\alpha, \beta, \gamma$.

**Exercise 7:** Let $(X, d)$ be a metric space. Recall that if $\gamma : [a, b] \to X$ is a path in $X$, the length of $\gamma$ is defined to be

$$l(\gamma) = \sup \left\{ l_{\mathcal{D}}(\gamma) = \sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})) \middle| \mathcal{D} : t_0 = a < t_1 < \ldots < t_N = b \text{ a partition of } [a, b] \right\}$$

if it is not infinite.

Suppose that for any pair of points $P, Q$ in $X$, there is at least one path $\gamma : [a, b] \to X$ with $\gamma(a) = P$ and $\gamma(b) = Q$ for which the length $l(\gamma)$ of $\gamma$ is well defined. We define $d' : X \times X \to \mathbb{R}$ to be given by

$$d'(P, Q) = \inf\{l(\gamma) \mid \gamma : [a, b] \to X \text{ a path in } X \text{ with } \gamma(a) = P, \gamma(b) = Q\}$$

1. Show that $d'$ is a metric.
2. Show by an example that $d$ and $d'$ may be distinct.