Exercise set 1

November 14, 2017

To be handed in by November 16th, 2017, in the mailbox in Manchester building.

You are required to hand in solutions for **6 out of the following 8** exercises.

Exercise 1: Suppose f is an isometry of the Euclidean space $\mathbb{R}^n$ which fixes all the points of a hyperplane $H = \{x \mid \langle x|u\rangle = c\}$ (where $u \in \mathbb{R}^n$ is a unit vector and $c \in \mathbb{R}$). The aim of the exercise is to show that f is the reflection R_H in H or the identity.

1. Show this first in the case where $c = 0$.
2. In the general case, show that $T_{-cu} \circ f \circ T_{cu}$ is an isometry which fixes the hyperplane $H_0 = \{x \mid \langle x|u\rangle = 0\}$.
3. Conclude.

Exercise 2: (Compositions of reflections in "hyperplanes" in $\mathbb{R}^2$)

1. Let v be a unit vector, let $L = \{x \mid \langle x|u\rangle = 0\}$ and $L' = \{x \mid \langle x|u\rangle = c\}$ be parallel lines. Show that the composition $R_{L'} \circ R_L$ is a translation.
2. Let v, w be two linearly independent unit vectors in $\mathbb{R}^2$. They define two distinct lines $L = \mathrm{Span}(v), L' = \mathrm{Span}(w)$ through the origin.
   (i) Show that there is an orthogonal matrix A such that $R_{L'} \circ R_L(x) = Ax$ for all x and $\det(A) = 1$.
   (ii) By looking at the image of v, prove that $R_{L'} \circ R_L$ is a rotation around the origin of angle 2α, where α is the angle between v and w (i.e., $\cos^{-1}(\langle v|w\rangle) = \alpha$).

Exercise 3: Suppose f is an isometry of $\mathbb{R}^n$ defined by $f(x) = Ax + b$ for some orthogonal matrix A and some $b \in \mathbb{R}^n$.

Show that f is the reflection in a hyperplane H if and only if there exists a unit (column) vector u such that $A = I - 2uu^t$ and $b = \alpha u$ for some $\alpha \in \mathbb{R}$. (Note that if u is a column vector, u^t is a row vector and the product uu^t is a square matrix).

Exercise 4:

1. Let f be an isometry of $\mathbb{R}^n$. Show that if $y_1, \ldots, y_m \in \mathbb{R}^n$ we have that
$$f\left(\frac{1}{m}\sum_{j=1}^{m} y_j\right) = \frac{1}{m}\sum_{j=1}^{m} f(y_j)$$
2. Let G be a finite subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ (i.e. a finite set of isometries of $\mathbb{R}^n$ such that if $f, h \in G$ then $f \circ h \in G$ and $f^{-1} \in G, h^{-1} \in G$). Show that there is a point of $\mathbb{R}^n$ fixed by all the elements of G. (Hint: fix a point $u \in \mathbb{R}^n$, and consider the average of the set of images of u by the elements of G).

Exercise 5: Let C be a cube centered on the origin, and let $Sym(C)$ be the set of isometries of $\mathbb{R}^3$ sending C to itself. Denote by $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$ the set of the four diagonals of C (a diagonal is a segment joining 2 opposite vertices - see picture).

1. Suppose $f \in Sym(C)$. Show that it sends a diagonal to a diagonal. Show that this gives a bijection from $\mathcal{D}$ to itself.

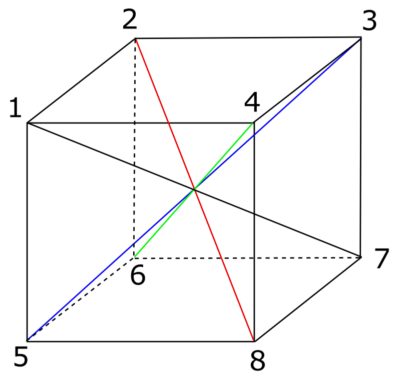

Figure 1: The four diagonals d_1, d_2, d_3, d_4 of the cube

2. List all the possible bijections $b : \mathcal{D} \to \mathcal{D}$, and for each bijection b, give an isometry of $\mathbb{R}^3$ which induces b on the diagonals.

Exercise 6: Given $u, v \in \mathbb{R}^2$ with $v \neq 0$ we write $L_(u, v) = \{x \in \mathbb{R}^2 \mid x = u + tv \text{ for some } t \in \mathbb{R}\}$ for the line going through u and directed by v.

1. Show that if $u' \in L_{(u,v)}$ and $\alpha \in \mathbb{R} - \{0\}$ then $L_{(u,v)} = L_{(u',v)} = L_{(u,\alpha v)}$.
2. Show that $L_{(u,v)}$ and $L_{(u',v')}$ are parallel if and only if $u - u' \notin \mathrm{Span}(v, v')$.
3. Show that the fifth postulate of Euclid holds in the Euclidean plane $\mathbb{R}^2$, i.e. that given a line L and a point p not in L, there is a unique line parallel to L which contains p.

Exercise 7: (The shortest path between two points in $\mathbb{R}^2$ is a line segment) Let γ be a piecewise continuously differentiable curve $[0, 1] \to \mathbb{R}^2$ defined by $\gamma(t) = (x(t), y(t))$ which satisfies $\gamma(0) = (0, 0)$ and $\gamma(1) = (0, 1)$. [piecewise continuously differentiable means that the interval $[0, 1]$ can be divided in finitely many pieces, so that on each piece γ is differentiable with continuous derivative].

1. Prove that the length $l(\gamma)$ of γ satisfies $l(\gamma) \geq \int_0^1 |y'(t)|\, dt$.
2. Prove that if γ has shortest possible length among all the piecewise continuously differentiable curves joining $(0, 0)$ to $(0, 1)$, then it is of the form $\gamma(t) = (0, y(t))$, where $y : [0, 1] \to [0, 1]$ is increasing.

Exercise 8: Show that the curve $\Gamma : [0, 1] \to \mathbb{R}^2$ defined by $\gamma(t) = (t, t\sin(1/t))$ for $t > 0$ and $\gamma(0) = (0, 0)$ does not have finite length.

[Hint: use the definition of length given by partitions of $[0, 1]$, not the integral].