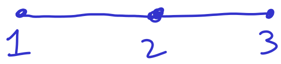
Expander graphs

Teacher: Ori Parzanchevski
parzan@math.huji.ac.il

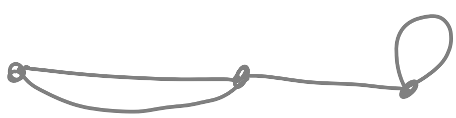
23/3/20

$G=(V,E)$ $\quad V$ set $\qquad E \subseteq \binom{V}{2}$ (unordered pairs of vertices)

e.g. $V=\{1,2,3\}$ $\quad E=\{\{1,2\},\{2,3\}\}$



(Later we might allow multiedges and loops



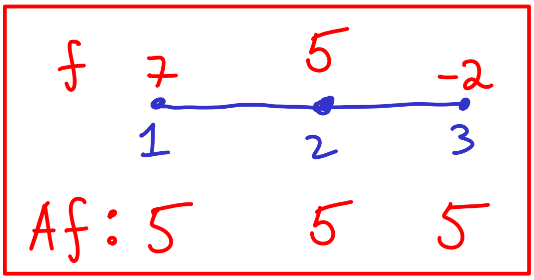
)

For a graph G, the adjacency operator $A = Adj$ (or A_G) is defined as follows:

$$A: \mathbb{R}^V \longrightarrow \mathbb{R}^V$$

$$(Af)(v) = \sum_{w\sim v} f(w)$$

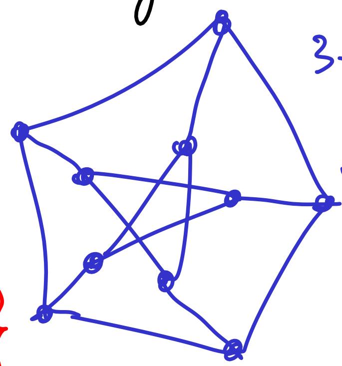
[$v \sim w$ means v and w are neighbors]



if we order the vertices, we can think of $\mathbb{R}^V$ as $\mathbb{R}^n$ $(n=|V|)$, and then A is represented by a matrix $A \in M_{n\times n}(\mathbb{R})$

$$A_{ij} = \begin{cases} 1 & \text{the } i\text{-th and } j\text{-th vertices are neighbors} \\ 0 & \text{otherwise} \end{cases}$$

For now, assume $|V| = n < \infty$ and that G is <u>k-regular</u>: every vertex has exactly k neighbors.


3-reg. graph

$\mathrm{Spec}(G) = \mathrm{Spec}(A_G)$

$\mathrm{Spec}(T) = \{\lambda \in \mathbb{C} \mid \exists v \neq 0 \text{ with } T(v) = \lambda v\}$

- $k \in \mathrm{Spec}(G)$ because $A\mathbb{1} = k\mathbb{1}$ (for k-regular)
- every $\lambda \in \mathrm{Spec}(A)$ is real. In fact:

<u>Reminder</u>: if T is a Hermitian operator on an inner product space ($T^* = T$), then the space has an O.N.B. of eigenvectors, with real eigenvalues. [for matrices, $(T^*)_{ij} = \overline{T_{ji}}$]

- every $\lambda \in \mathrm{Spec}$ has $|\lambda| \leq k$:

<u>Pf</u>: if $Af = \lambda f$, $\exists v_0 \in V$ with $|f(v_0)|$ max. ($|f(v_0)| \geq |f(v)|$ $\forall v \in V$). Assume $f(v_0) > 0$ (otherwise take $-f$). Then:

$$|\lambda f(v_0)| = |(Af)(v_0)| = \left|\sum_{w \sim v_0} f(w)\right| \leq \sum_{w \sim v_0} |f(w)| \leq k f(v_0)$$

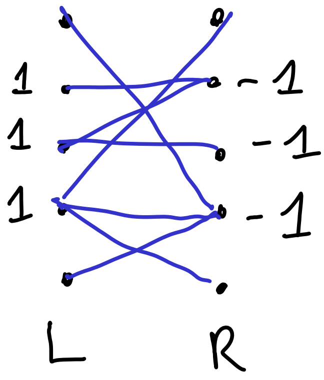
$\Rightarrow |\lambda| \leq k$ $\square$

So we have:

$$\mathrm{Spec}(A_G): \ k = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq -k$$

When is $-k \in \mathrm{Spec}(A)$?

Claim: iff G is bipartite: $V = R \sqcup L$ and $E \subseteq R \times L$

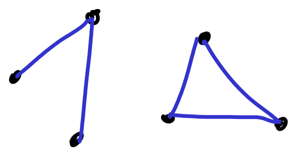


$\frac{1}{2}$ Pf: if G is bipartite take $f(v) = \begin{cases} 1 & v \in L \\ -1 & v \in R \end{cases}$

then $Af = -kf$.

Claim: $\lambda_2 = k$ iff G is disconnected

($V = V_1 \sqcup V_2$, with no $e \in V_1 \times V_2$)



$\frac{1}{2}$ Pf: if G is disconnected, $G = V_1 \sqcup V_2$, take

$f(v) = \begin{cases} 1 & v \in V_1 \\ 0 & v \notin V_1 \end{cases}$ $\left(f = \mathbb{1}_{V_1} \right)$. We still have $Af = kf$, and $f \notin \langle \mathbb{1} \rangle$, so k is an eigenvalue with multiplicity ≥ 2.

HW: $\lambda_2 = k \Rightarrow G$ is disconnected

Definition: G is an ε-expander if

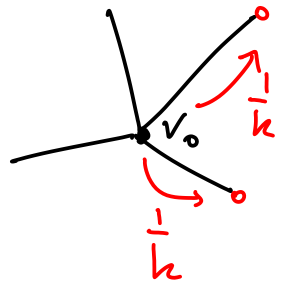
$$|\lambda_i| \leq \varepsilon \text{ for } i = 2, \dots, n \iff \lambda(G) \leq \varepsilon$$

Def: $\lambda(G) = \max\{|\lambda_2|, \dots, |\lambda_n|\}$.

Simple Random Walk (SRW)

24/3/20

Start in a vertex v_0, at every step move to a uniformly chosen random neighbor.



We assume: G is k-regular, $|V| = n$

Denote X_t = the vertex we are in at time t. So: $X_0 = v_0$

$$X_1 \sim U\begin{pmatrix}\text{neighbors}\\ \text{of } v_0\end{pmatrix} \quad \circ \circ \circ$$

Denote $P_t^{(v_0)} : V \to \mathbb{R} \qquad P_t^{(v_0)}(v) = \mathrm{Prob}(X_t = v \mid X_0 = v_0)$

Claim: $P_{t+1} = \frac{1}{k} \cdot A P_t$

Pf: $$P_{t+1}(v) = \sum_{v \sim w} P_t(w) \cdot \overbrace{\mathrm{Prob}\left(X_{t+1} = v \mid X_t = w\right)}^{1/k}$$

$$= \frac{1}{k} \sum_{v \sim w} P_t(w) = \frac{1}{k}(AP_t)(v) \qquad \square$$

$\Rightarrow P_t = \left(\frac{A}{k}\right)^t P_0 \qquad$ (by induction)

Define: $M = \frac{1}{k} A \qquad$ (normalized adj. op. / Markov operator)

$M \to$ Markov, $\to$ Mean, $\to$ ממוצע

$\mu_i = \frac{\lambda_i}{k}$ so $\mathrm{Spec}\, M : 1 = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq -1$

We recall A and M have a ONB of eigenvectors

$f_1, \ldots, f_n : V \to \mathbb{R}, \quad \langle f_i, f_j \rangle = \delta_{ij} \; (\to \|f_i\| = 1)$

$\langle f, g \rangle = \sum_{v \in V} f(v) g(v) \qquad A f_i = \lambda_i f_i \; (\to M f_i = \mu_i f_i)$

for any $f: V \to \mathbb{R}$, $f = \sum_{i=1}^{n} \alpha_i f_i$

ONB $\to$ $\alpha_i = \langle f, f_i \rangle$

$$\to \boxed{\|f\|^2} = \Big\| \sum_{i=1}^{n} \underbrace{\alpha_i \cdot f_i}_{\text{orth. to each other}} \Big\|^2 \underset{\uparrow \text{ Pythagoras}}{=} \sum \| \alpha_i \cdot f_i \|^2$$

$$= \sum |\alpha_i|^2 \|f_i\|^2 = \boxed{\sum_{i=1}^{n} |\alpha_i|^2}$$

Parseval

Back to SRW: Write $P_0 = \sum_{i=1}^{n} \alpha_i f_i$ $\quad (\alpha_i = \langle P_0, f_i \rangle)$

and then $P_t = M^t P_0 = \sum_{i=1}^{n} \alpha_i M^t f_i = \sum_{i=1}^{n} \alpha_i \mu_i^t f_i$

$\underline{i=1}$: $\mu_1 = 1$, $f_1 = \frac{1}{\sqrt{n}}$ $\quad \left(\left\langle \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right\rangle = 1 \right)$

$\alpha_1 = \left\langle P_0, \frac{1}{\sqrt{n}} \right\rangle = \frac{1}{\sqrt{n}}$

$\Big\}\ \alpha_1 \mu_1^t f_1 = \frac{1}{n} = u$

uniform distribution

$u = \frac{1}{n}$

so: $$\|P_t - u\|^2 = \Big\| \sum_{i=2}^{n} \alpha_i \mu_i^t f_i \Big\|^2 \underset{\text{Pyt}}{=} \sum_{i=2}^{n} |\alpha_i|^2 |\mu_i|^{2t} \cancel{\|f_i\|^2}$$

if G was an ε-expander: $|\lambda_2|, \dots, |\lambda_n| \leq \varepsilon$

$\to |\mu_2|, \dots, |\mu_n| \leq \frac{\varepsilon}{k}$

$$\to \|P_t - u\|^2 \leq \left(\frac{\varepsilon}{k}\right)^{2t} \underbrace{\sum_{i=2}^{n} |\alpha_i|^2}_{\text{(Parseval)} \leq \|P_0\|^2} \leq \left(\frac{\varepsilon}{k}\right)^{2t}$$

$\underline{\text{Conc}}$: in a k-reg. ε-exp., $\|P_t - u\| \leq \left(\frac{\varepsilon}{k}\right)^t$

So SRW on expanders converges to uniform dist. rapidly.

Worst case: $\varepsilon = k$: G is disconnected $(\lambda_2 = k)$

G is bipartite $(\lambda_n = -k)$

$\Rightarrow \max\{|\mu_2| \ldots |\mu_n|\} = 1$. $\quad \|P_t - u\| \leq 1^t = 1$

Lower bound: $\|P_t - u\|^2 = \sum_{i=2}^{n} |\alpha_i|^2 |\mu_i|^{2t} \geq |\alpha_j|^2 \left(\frac{\lambda(G)}{k}\right)^{2t}$

where $|\mu_j|$ is maximal among $|\mu_2|, \ldots, |\mu_n|$ $(j=2 \text{ or } j=n)$

$\left(|\mu_j| = \frac{\lambda(G)}{k}\right)$. So if $\alpha_j \neq 0$ then we get

$$\left.\begin{array}{l} \|P_t - u\| \geq |\alpha_j| \left(\frac{\lambda(G)}{k}\right)^t = \Omega\left(\left(\frac{\lambda(G)}{k}\right)^t\right) \\ \text{We already know } \|P_t - u\| = O\left(\left(\frac{\lambda(G)}{k}\right)^t\right) \end{array}\right\} \Theta\left(\left(\frac{\lambda(G)}{k}\right)^t\right)$$

To get $\alpha_j \neq 0$, we take v_0 for which $f_j(v_0) \neq 0$,

so $\alpha_j = \langle P_0, f_j \rangle = f_j(v_0) \neq 0$.

Conc: $\exists$ starting point from which $\|P_t - u\| = \Theta\left(\left(\frac{\lambda(G)}{k}\right)^t\right)$.

Definition: G is a bipartite ε-expander if $\lambda_n = -k$ and $\max\{|\lambda_2|, |\lambda_{n-1}|\} \leq \varepsilon$

Claim: if G is a bipartite ε-expander, and P_t is the dist. of SRW with "lazy first step" (= at 1st step, stay in place with prob. $\frac{1}{2}$, move to each neighbor with prob. $\frac{1}{2k}$), then $\|P_t - u\| \leq \left(\frac{\varepsilon}{k}\right)^{t-1}$.

Examples

K_n - complete graph on n vertices

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 1 & 1 & \\ 1 & 1 & 0 & 1 & 1 & \\ \vdots & & & & & \end{bmatrix} = J - I \quad (J = \text{all one matrix})$$

$\operatorname{Spec} A = n-1, -1, -1, -1, \ldots, -1$

(by $\operatorname{Spec} J = n, 0, 0, \ldots, 0$ as $\operatorname{rank} J = 1 \to \operatorname{null} J = n-1$

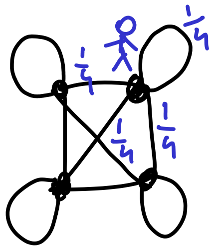
and $Bv = \lambda v \longrightarrow (B - I)v = Bv - v = \lambda v - v = (\lambda - 1)v$)

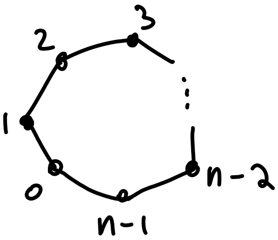
so K_n is a 1-expander (but that's cheating)

Even better: $\overline{K_n}$ = graph with $A = J$

This is the ultimate expander:

a 0-expander ($\operatorname{Spec} J = n, 0, 0, \ldots, 0$)

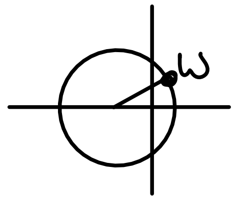


$C_n =$ 

$V = \{v_k \mid k \in \mathbb{Z}/n\}$

Spec $A = 2, \quad \ldots\ldots \quad \boxed{, -2}$ iff $2|n$

Let $\omega = e^{\frac{2\pi i}{n}} = \sqrt[n]{1}$

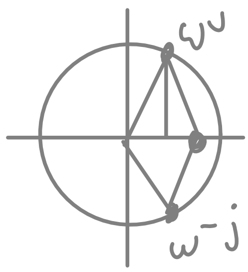


For $0 \leq j \leq n-1$, define $f_j(v_k) = \omega^{jk}$

f_j is an e.func: $(Af_j)(v_k) = f_j(v_{\underbrace{k-1}_{\text{mod } n}}) + f_j(v_{\underbrace{k+1}_{\text{mod } n}})$

$$= \omega^{j(k-1)} + \omega^{j(k+1)} = \omega^{jk}(\omega^j + \omega^{-j}) = f_j(v_k)\left(2\cos\left(\frac{j 2\pi}{n}\right)\right)$$

so f_j has e.val $2\cos\left(\frac{j}{n}\cdot 2\pi\right)$



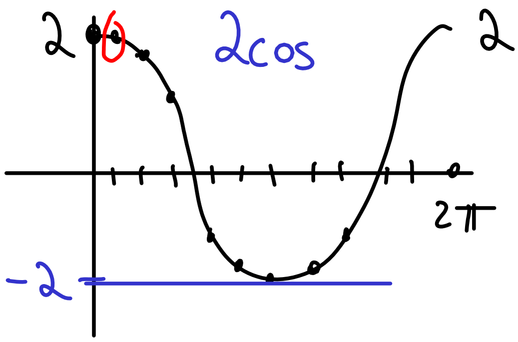
we got n e.funcs. they are

indep:

$$\begin{array}{c|cccc} & v_0 & v_1 & v_2 & \cdots \\ f_0 & 1 & 1 & 1 & \cdots \\ f_1 & 1 & \omega & \omega^2 & \cdots \\ f_2 & 1 & \omega^2 & \omega^4 & \cdots \\ \vdots & 1 & \omega^3 & \omega^6 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{array}$$

Van der Monde matrix

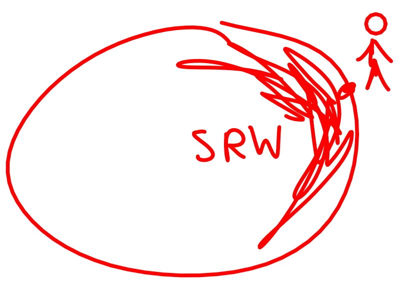
ex: Det $\neq 0$



So we found all e.v.

$\lambda(C_n) \geq 2\cos\left(\frac{2\pi}{n}\right) \xrightarrow{n \to \infty} 2$

so C_n are bad expanders:



Projective Planes

30/3/20

Let q prime, define a graph $\mathbb{P}^2\mathbb{F}_q$:

(proj. plane over q)

V = lines & Planes in $\mathbb{F}_q^3$

$$E = \left\{ \{l, p\} \;\middle|\; \begin{matrix} l \text{ line, } p \text{ plane} \\ l \subset p \end{matrix} \right\}$$

$$\# \text{lines in } \mathbb{F}_q^3 = \frac{q^3 - 1}{q - 1} \begin{matrix} \leftarrow \text{nonzero vectors} \\ \leftarrow \text{every line has } p-1 \text{ gens.} \end{matrix}$$

$$= q^2 + q + 1$$

line in $\mathbb{F}_3^3$: $\begin{Bmatrix} (0,0,0) \\ (1,0,0) \\ (2,0,0) \end{Bmatrix}$

planes in $\mathbb{F}_q^3$ = same, as lines $\underset{1:1}{\overset{\perp}{\rightleftarrows}}$ planes

so $|V| = 2(q^2+q+1)$

$\deg(l) = \#\{p \mid l \subset p\}$

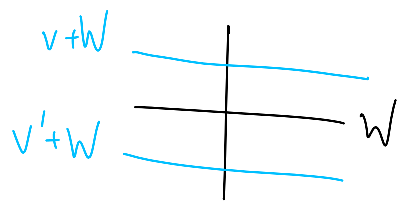
$$0 \subset l \subset ? \subset \mathbb{F}_q^3$$

4th isom. theorem: $\exists$ correspondence between subgps of G/N to sgps of G containing N ($N \trianglelefteq G$)

Likewise: between subspaces of V/W and subspaces of V cont. W. In our case: subspaces of $\mathbb{F}_q^3/l$

(Reminder: $V/W = \{v + W \mid v \in V\}$, $(v+W) + (v'+W) = v + v' + W$

$\alpha(v+W) = \alpha v + W$)



$\mathbb{F}_q^3/\ell$ is of dim $2 \to \cong \mathbb{F}_q^2$.

$\boxed{\mathbb{F}_q^2 \text{ has } \frac{q^2-1}{q-1} \text{ proper subspaces } = q+1}$

so every ℓ is contained in $q+1$ planes, and by $\perp$, every plane contains $q+1$ lines

another argument...

So $G = \mathbb{P}^2\mathbb{F}_q$ is $(q+1)$-regular, on $2(q^2+q+1)$ verts.

<u>Goal</u>: G is a bipartite $\sqrt{q}$-expander.

<u>Lemma</u>: For a bip. graph, every <u>nonzero</u> e.val λ comes with a twin $-\lambda$ e.val.

<u>Reason</u>:
$$A = \left[\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right]$$
for some B (could be non-square, for non-reg. graphs)

If $Av = \lambda v$, write $v = \left(\frac{w}{u}\right)$, so
$$\left(\frac{\lambda w}{\lambda u}\right) = A\left(\frac{w}{u}\right) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} Bu \\ B^T w \end{pmatrix} \leftrightarrow \left\{\begin{matrix} Bu = \lambda w \\ B^T w = \lambda u \end{matrix}\right\}$$

Take $v' = \left(\frac{w}{-u}\right)$. Then
$$Av' = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} w \\ -u \end{pmatrix} = \begin{pmatrix} -Bu \\ B^T w \end{pmatrix} = \begin{pmatrix} -\lambda w \\ \lambda u \end{pmatrix} = -\lambda v'$$

Careful: if $u=0$, then $v'=v \to \lambda = -\lambda \to \lambda = 0$
if $w=0$, then $v' = -v$

<u>Observation</u>: if $A = \mathrm{Adj}(G)$, $A_{v,w} = \begin{cases} 1 & v \sim w \\ 0 & \text{otherwise} \end{cases}$

then $(A^\ell)_{v,w} = \#$ paths of length ℓ from v to w.

$\underline{\text{Induct. basis}}$: $(A^2)_{v,w} = \sum_{u\in V} \underbrace{A_{v,u}A_{u,w}}_{\substack{1 \text{ if } v\sim u\sim w \\ 0 \text{ otherwise}}} = \#\,2\text{-paths } v\sim *\sim w$

complete the proof yourself!

Back to $\mathbb{P}^2\mathbb{F}_q$: $(A^2)_{p,\ell} = (A^2)_{\ell,p} = 0$ $\left(\begin{matrix}\text{bipartite} \Rightarrow \\ \text{no } \ell\sim *\sim p\end{matrix}\right)$

$$(A^2)_{\ell,\ell'} = \#\{p \mid \ell\sim p\sim \ell'\} = \begin{cases} 1 & \ell\neq \ell' \\ q+1 & \ell=\ell' \end{cases}$$

every 2 lines span a unique plane

q+1 planes containing ℓ

$$(A^2)_{p,p'} = \#\{\ell \mid p\sim \ell\sim p'\} = \begin{cases} 1 & p\neq p' \\ q+1 & p=p' \end{cases}$$

($p\cap p'$ is a line)

$$\text{so } A^2 = \left(\begin{array}{c|c} \begin{matrix} q+1 & 1 & 1 \\ 1 & q+1 & 1 \\ 1 & 1 & \ddots \end{matrix} & 0 \\ \hline 0 & \begin{matrix} q+1 & 1 & 1 \\ 1 & q+1 & \ddots \\ \vdots & \vdots & \end{matrix} \end{array}\right)$$

31/3/20

$$A^2 = \left(\begin{array}{c|c} J+qI & 0 \\ \hline 0 & J+qI \end{array}\right) \quad \text{where } J = \text{all one mat.}$$

$\mathrm{Spec}\, J_n = (n,0,0,\dots)$ (complete graph w/ loops)

$\mathrm{Spec}\left(J_{q^2+q+1} + qI\right) = \left(q^2+2q+1, q, q, q, \dots\right)$

$\mathrm{Spec}\left(A^2\right) = \left(q^2+2q+1, q^2+2q+1, q, q, q, \dots\right)$

$\xrightarrow{\text{Lemma}}$

$\mathrm{Spec}(A) = \left(q+1, -(q+1), \sqrt{q}, -\sqrt{q}, \sqrt{q}, -\sqrt{q}, \dots\right)$

So G is a $(q+1)$-regular bip. $\sqrt{q}$-expander.

$$A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right) \longrightarrow \operatorname{Spec} A = \operatorname{Spec} B \sqcup \operatorname{Spec} C$$

Challenge: Fix k, create k-reg expanders with $n \to \infty$.

> **Def**: A family of k-reg graphs is a family of expanders if $\exists \varepsilon < k$ s.t. all are ε-expanders.

How good can expanders be? (After fixing k)

Take a k-reg ε-expander with n vertices.

$$tr(A^2) = \sum_v (A^2)_{v,v} = |V| \cdot \deg = nk$$

OTOH $A^2 \sim \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix} \to tr(A^2) = \sum_{i=1}^n \lambda_i^2 \leq k^2 + (n-1)\varepsilon^2$

$$\to \varepsilon \geq \sqrt{\frac{nk-k^2}{n-1}} = \sqrt{k} \cdot \sqrt{\frac{n-k}{n-1}} \xrightarrow{n \to \infty} \sqrt{k}$$

so for large n, ε cannot drop below $\sqrt{k}$.

(Proj. planes achieve this!

bipartite: $\sqrt{\frac{nk-2k^2}{n-2}} = \sqrt{k}\sqrt{\frac{n-2k}{n-2}} \longrightarrow \sqrt{k}$)

> Alon-Boppana Thm: ε cannot drop below $2\sqrt{k-1}$.
> This is tight - there are ∞ families of $2\sqrt{k-1}$-exp.

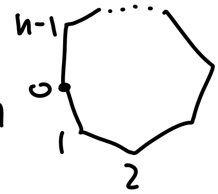
Cayley Graphs: Take group G, $S \subseteq G$,

Construct $\boxed{\Gamma = Cay(G, S)}$: $Vert(\Gamma) = G$

$$\text{Edges}(\Gamma) = \left\{ (g, gs) \;\middle|\; \begin{matrix} g \in G \\ s \in S \end{matrix} \right\}$$

this is a <u>directed</u> graph. We will assume $S^{-1} = S$ $\left(\{ s^{-1} \mid s \in S \} = S \right)$. Then we get an <u>undirected</u> graph: for $(g, gs) \in E$ we also have $(gs, gss^{-1}) = (gs, g) \in E$, so $A^T = A$.

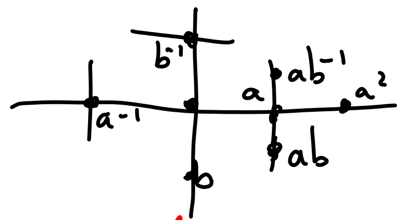
<u>Example</u>: $G = \mathbb{Z}/n$, $S = \{1, -1\} \longrightarrow \Gamma = C_n$



$G = \mathbb{Z}$ $\quad S = \{1, -1\} \to \Gamma =$

$G = F_k = F(x_1, \dots, x_k)$, $S = \{x_1, \dots, x_k, x_1^{-1}, \dots, x_k^{-1}\}$

(free group on k letters) $\quad \Gamma = T_{2k}$



Thm: Abelian groups don't give expanders:

If $\{\Gamma_i\}_{i=1}^{\infty}$ is a family of k-reg Cayley $\Gamma_i = \text{Cay}(G_i, S_i)$ graphs of abelian gps and $n_i = |V_{\Gamma_i}| = |G_i| \to \infty$ then $\lambda(G_i) \longrightarrow k$.

<u>Thm</u>: Expanders have logarithmic diameter:

$$\text{diam}(G) \leq \lfloor \log_{k/\varepsilon} n \rfloor + 1$$

<u>Pf</u>: n, k, ε as allways. P_t = random walk. We know $\| P_t - \frac{\mathbb{1}}{n} \|^2 \leq \left(\frac{\varepsilon}{k} \right)^{2t}$. Take $S_t(v_0) = \left\{ w \;\middle|\; \begin{matrix} \exists \text{ path of} \\ \text{len } t \text{ from} \\ v_0 \text{ to } w \end{matrix} \right\}$

$\operatorname{supp} P_t \subseteq S_t(v_0) \longrightarrow$

$$\|P_t - \tfrac{1}{n}\|^2 \geqslant \sum_{w \notin S_t(v_0)} \left(P_t - \tfrac{1}{n}\right)(w)^2 = \sum_{w \notin S_t(v_0)} \frac{1}{n^2} \; - \; |V \setminus S_t(v_0)| \cdot \frac{1}{n^2}$$

$\Rightarrow |V \setminus S_t(v_0)| \leqslant n^2 \left(\frac{\varepsilon}{k}\right)^{2t} \rightarrow$ if $n^2 \left(\frac{\varepsilon}{k}\right)^{2t} < 1$

then $S_t(v_0) = V$ so $\operatorname{diam}(G) \leq t$

We get this at $t = \lfloor \log_{k/\varepsilon} n \rfloor + 1$. $\square$

Now look at Cayley graphs:

A t-path in Γ looks like: $g - gs_1 - gs_1s_2 - \cdots - gs_1s_2\ldots s_t$

with $s_i \in S$.

So $S_t(e) = \{ s_1 s_2 \cdot \ldots \cdot s_t \mid s_i \in S \}$. If G is abelian

then

$$\begin{tikzcd} & s_2 \arrow[dr, dash] & \\ e \arrow[ur, dash] \arrow[dr, dash] & & s_1 s_2 = s_2 s_1 \\ & s_1 \arrow[ur, dash] & \end{tikzcd}$$

. So all that matter is the number of times each $s \in S$ appears in $s_1 \cdot s_2 \cdot \ldots \cdot s_t$.

So write $S = \{s_1, s_2, \ldots, s_k\}$ and then

$$S_t(e) = \{ s_1^{n_1} s_2^{n_2} \cdot \ldots \cdot s_k^{n_k} \mid \underbrace{n_1 + n_2 + \ldots + n_k = t}_{\binom{t+k-1}{k-1} \text{ choices}} \}$$

$$\Rightarrow |S_t(e)| \leq \binom{t+k-1}{k-1} \leq t^k$$

$\Rightarrow$ if $t < \sqrt[k]{n}$ then $S_t(e) \neq V$. But we saw that at $t = \lfloor \log_{k/\varepsilon}(n) \rfloor + 1$ we have $S_t(e) = V$

if $\varepsilon<k$ then for n large enough $\lfloor \log_{k/\varepsilon}(n)\rfloor+1<\sqrt[k]{n}$, so a ∞-family of abelian Cay. graphs cannot be ε-exp. for any $\varepsilon<k$.

Thm (Margulis): pick $d\geq 3$, and $S\subseteq SL_d(\mathbb{Z})=G$ s.t. $\langle S\rangle=G$ $\left(\text{e.g.: } S=\left\{\begin{pmatrix}1 & & \pm1\\ & \ddots & \\ & & 1\end{pmatrix}\right\}\right)$

Then $\Gamma_p=\mathrm{Cay}(SL_d(\mathbb{F}_p), S \bmod p)$ is a family of expanders.

Idea: (1) $SL_d(\mathbb{Z})$ have property (T) (Kazhdan) $d\geq 3$

(2) Property (T) $\longrightarrow$ expanders (Margulis)

Def: $T:V\to V$, V inner prod. space (fin. dim.)

$$\|T\|=\sup_{v\neq 0}\frac{\|T(v)\|}{\|v\|}=\max_{\|v\|=1}\|T(v)\|$$

idea: by def any v satisfies $\|T(v)\|\leq\|T\|\cdot\|v\|$

Claim: If T is self adjoint, then $\|T\|=\max\{|\lambda| \mid \lambda\in \mathrm{spec}\,T\}$

Pf: if $|\lambda|$ is max and $Tv=\lambda v$ $(v\neq 0)$, then

$$\|T\|\geq\frac{\|Tv\|}{\|v\|}=\frac{\|\lambda v\|}{\|v\|}=|\lambda|.$$

OTOH, T self adj $\to\exists$ O.N.B. v_i, $Tv_i=\lambda v_i$, and

then $\|T(v)\|^2=\|T(\sum\langle v,v_i\rangle v_i)\|^2=\|\sum\langle v,v_i\rangle\lambda_i v_i\|^2$

$$\overset{\text{Pyth.}}{=}\sum|\langle v,v_i\rangle|^2|\lambda_i|^2\leq|\lambda|^2\sum|\langle v,v_i\rangle|^2\underset{\text{Parseval}}{=}|\lambda|^2\|v\|^2 \quad\square$$

Doesn't hold in general: $A = \begin{pmatrix} 2 & 100 \\ 0 & 1 \end{pmatrix}$

$\operatorname{Spec} A = \{1, 2\}$, but

$$\|A\| \geq \frac{\left\|A\begin{pmatrix}0\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}0\\1\end{pmatrix}\right\|} = \left\|\begin{pmatrix}100\\1\end{pmatrix}\right\| \approx \sqrt{10001} \approx 100$$

$\boxed{20/4/20}$

Example: back to SRW $\quad P_t = M^t P_0 \quad M = \frac{A}{k}$

$$\|P_t - u\| = \|M^t(P_0 - u)\| \leq \|M\| \, \|M^{t-1}(P_0 - u)\|$$
$$\leq \dots \leq \underbrace{\|M\|^t}_{1} \underbrace{\|P_0 - u\|}_{\sim 1} \sim 1$$

More carefully:

$$\mathbb{R}^V = \text{constant} \oplus Z = \langle \mathbb{1} \rangle \oplus \mathbb{1}^\perp$$

orth. dir. sum

$$Z = Z(G) = \left\{ f : V_G \to \mathbb{R} \;\middle|\; \sum_{v \in V} f(v) = 0 \right\}$$

and A, M preserve the decomp: $A(\langle \mathbb{1} \rangle) \subseteq \langle \mathbb{1} \rangle$

$A(Z) \subseteq Z$

$$\|P_t - u\| = \|M^t \underbrace{(P_0 - u)}_{\text{in } Z}\| = \|(M|_Z)^t (P_0 - u)\|$$

$$\leq \underbrace{\|M|_Z\|^t}_{\max\{|\lambda| \,|\, \lambda \in \operatorname{Spec} M|_Z\} = \frac{\lambda(G)}{k}} \underbrace{\|P_0 - u\|}_{\leq 1} \leq \left(\frac{\lambda(G)}{k}\right)^t$$

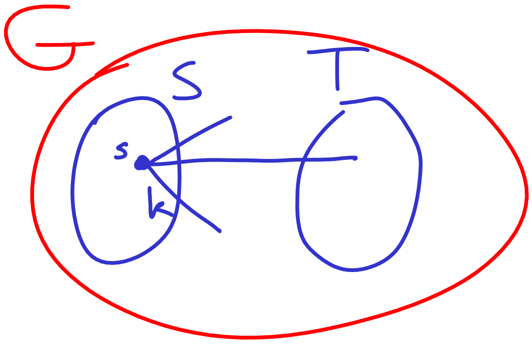
Expander Mixing Lemma: ($|V| = n$, k-reg ε-expander)

For $S, T \subseteq V$,

$$\left| |E(S,T)| - \frac{k|S||T|}{n} \right| \leq \varepsilon \sqrt{|S||T|}$$

where $E(S,T)$ are edges between S and T.

Heuristics: every egde leaving $s \in S$ has "prob" $\frac{|T|}{n}$ to go to T

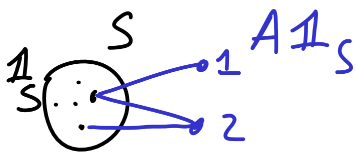


so we expect s to have $k\frac{|T|}{n}$ neighbors in T, so in total we expect $|S| \cdot k \cdot \frac{|T|}{n}$ edges from S to T.

Pf: for $f: V \to \mathbb{R}$, write $f = f^{1} + f^{Z}$ for the decomp. into $\langle \mathbb{1} \rangle \oplus Z$.

First, observe $\langle A\mathbb{1}_S, \mathbb{1}_T \rangle = \sum_{v \in T} (A\mathbb{1}_S)(v)$

$$= \sum_{v \in T} \# \left\{ \begin{matrix} \text{neighbors of} \\ v \text{ in } S \end{matrix} \right\} = |E(S,T)|$$



OTOH, $\langle A\mathbb{1}_S, \mathbb{1}_T \rangle = \langle A(\mathbb{1}_S^1 + \mathbb{1}_S^Z), \mathbb{1}_T^1 + \mathbb{1}_T^Z \rangle$

$$= \langle A\mathbb{1}_S^1, \mathbb{1}_T^1 \rangle + \langle A\mathbb{1}_S^Z, \mathbb{1}_T^Z \rangle$$

($AZ \subseteq Z$, $A\mathbb{1} \subseteq \langle \mathbb{1} \rangle$, and $\mathbb{1} \perp Z \Rightarrow$ the other summands vanish)

$$\mathbb{1}_S^{\mathbb{1}} = \mathrm{Proj}_{\mathbb{1}}(\mathbb{1}_S) = \left\langle \mathbb{1}_S, \frac{\mathbb{1}}{\sqrt{n}} \right\rangle \cdot \frac{\mathbb{1}}{\sqrt{n}} = \frac{|S|}{n} \cdot \mathbb{1}$$

$$\mathbb{1}_T^{\mathbb{1}} = \frac{|T|}{n} \cdot \mathbb{1}$$

so $\left\langle A \mathbb{1}_S^{\mathbb{1}}, \mathbb{1}_T^{\mathbb{1}} \right\rangle = \left\langle k \frac{|S|}{n} \mathbb{1}, \frac{|T|}{n} \mathbb{1} \right\rangle = \frac{k|S||T|}{n}$.

$$\left\langle A \mathbb{1}_S^{Z}, \mathbb{1}_T^{Z} \right\rangle \underset{\text{C.S.}}{\leq} \|A \mathbb{1}_S^Z\| \, \|\mathbb{1}_T^Z\| = \|A|_Z \mathbb{1}_S^Z\| \, \|\mathbb{1}_T^Z\|$$

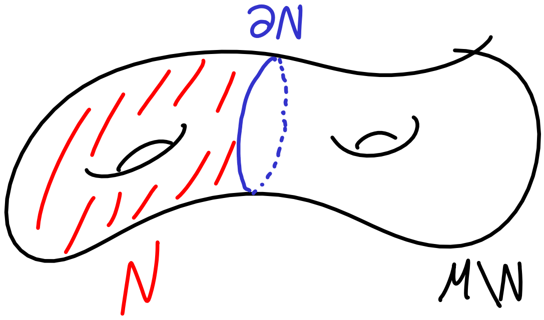
$$\leq \underbrace{\|A|_Z\|}_{=\lambda(G)} \underbrace{\|\mathbb{1}_S^Z\|}_{\leq \|\mathbb{1}_S\| = \sqrt{|S|}} \|\mathbb{1}_T^Z\| \leq \lambda(G)\sqrt{|S||T|}$$

$\square$

Note: if $S \cap T \neq \emptyset$, you need to count edges in $S \cap T \times S \cap T$ twice (they contribute twice to $\sum_{v \in T} \#\left\{ \begin{matrix} \text{neighbors of} \\ v \text{ in } S \end{matrix} \right\}$).

Cheeger inequalities

For a Rimmanian manifold (geometric shape) the Cheeger constant $h(M)$ measures bottlenecks:



$$h(M) = \inf_{N \subseteq M} \frac{\mathrm{vol}(\partial N)}{\mathrm{vol}(N)\,\mathrm{vol}(M \setminus N)}$$

Classic Cheeger ineq. (Cheeger–Buser, Maz'ya) relate $h(M)$ to the spectral theory of M

In graphs, we define the Cheeger constant

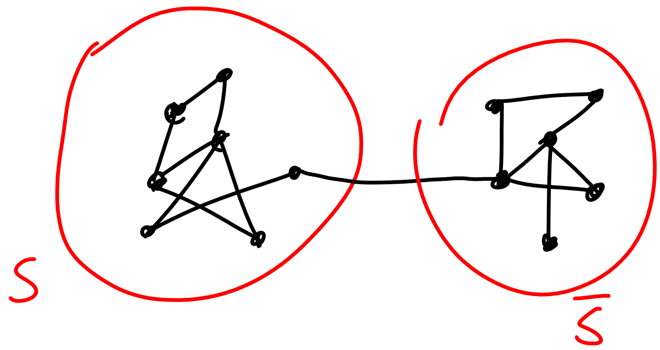
$$h'(G) = \min_{\emptyset \neq S \subsetneq V} \frac{|\partial S| \cdot n}{|S||\bar{S}|} \quad \text{or} \quad h(G) = \min_{\emptyset \neq S \subseteq V} \frac{|\partial S|}{\min(|S|, |\bar{S}|)}$$

where $\bar{S} = V \setminus S$, $\partial S = E(S, \bar{S})$

Note: $\boxed{h(G) \leq h'(G) \leq 2h(G)}$ $\left(\text{if } |S| \leq \frac{n}{2},\ \frac{n}{2} \leq |\bar{S}| \leq n \right)$

$\boxed{21/4/20}$

example:



low Cheeger const.

$S, \bar{S}$ large

$|\partial S| = 1$

$h(G) = 0 \iff G$ is disconnected.

Rayleigh quotient: if $T: V \to V$ V inn. prod. sp.

we define for $v \in V$ the Rayleigh q. of v w.r.t. T

$$R_v = \frac{\langle Tv, v \rangle}{\langle v, v \rangle}$$

if v is a T-eigenvector, $Tv = \lambda v$, then $R_v = \lambda$.

Assume that T is self-adjoint with

ONB of e.vecs $v_1, \dots, v_n$, e.vals $\lambda_1 \geq \dots \geq \lambda_n$, then if $v = \sum_{i=1}^{n} \alpha_i v_i$ then

$$R_v = \frac{\langle \Sigma \alpha_i \lambda_i v_i, \Sigma \alpha_i v_i \rangle}{\langle \Sigma \alpha_i v_i, \Sigma \alpha_i v_i \rangle} = \frac{\sum |\alpha_i|^2 \lambda_i}{\sum |\alpha_i|^2} \rightarrow \text{weighted avg. of } \lambda_i \text{ with weight } \frac{|\alpha_i|^2}{\sum |\alpha_i|^2}$$

in particular, $\lambda_1 \geq R_v \geq \lambda_n \ \forall v$, and also

if $v \perp v_1$ then $\lambda_2 \geq R_v \geq \lambda_n$, and so on...

(BTW, $\lambda_1 = \max\limits_{v \neq 0} R_v$, $\lambda_2 = \max\limits_{v \perp v_1} R_v$, ...)

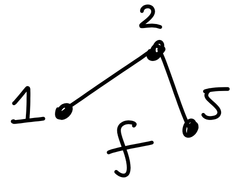
Goal: prove DCI: $\frac{h(G)^2}{2k} \leq k - \lambda_2 \leq h'(G)$ (for k-reg. graph)

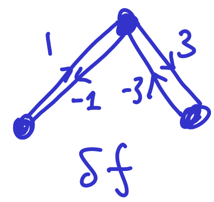
again, we see $\lambda_2 = k \Leftrightarrow h = 0$ ($\Leftrightarrow$ disconnected).

<u>Laplacian</u> Given a (finite) graph G, for $f: V \to \mathbb{R}$ we define a function on directed edges

$$(\delta f)(\overset{v}{\bullet} \longrightarrow \overset{w}{\bullet}) = f(w) - f(v)$$

δ = "differential"





We always have $(\delta f)(v \to w) = -(\delta f)(w \to v)$.

Define the space of flows on G:

$$\Omega(G) = \left\{ f : \text{directed edges} \longrightarrow \mathbb{R} \;\middle|\; f(wv) = -f(vw) \ \ \forall \{v,w\} \in E \right\}$$

We got $\delta : \mathbb{R}^V \to \Omega(G)$.

$\dim \Omega(G) = |E|$ (even though there are $2|E|$ dir. edges)

$\Omega(G)$ doesn't come with a natural basis.

it has a natural inner product: $\langle f, f' \rangle = \sum_{e \in E} f(e) f'(e)$ same direction

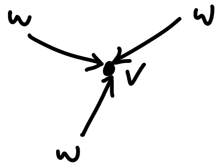
$f, f' \in \Omega$

well defined because $(-1)^2 = 1$.

Def: the Laplacian of G is $\Delta := \delta^* \delta : \mathbb{R}^V \to \mathbb{R}^V$

$$\begin{tikzcd} \mathbb{R}^V \arrow[rr] \arrow[dr, "\delta"'] & & \mathbb{R}^V \\ & \Omega \arrow[ur, "\delta^*"'] & \end{tikzcd}$$

$$\boxed{\begin{aligned} (\delta f)(wv) &= f(v) - f(w) \\ (\delta^* F)(v) &= \sum_{v \sim w} F(wv) \end{aligned}}$$



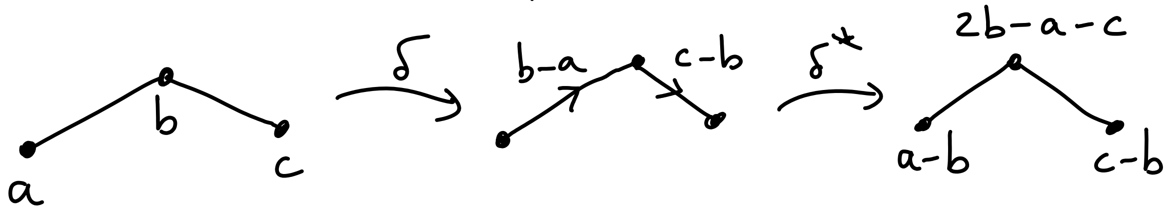
$$\langle \delta 1_v, F \rangle \overset{?}{=} \langle 1_v, \delta^* F \rangle = \sum_w 1_v(w)(\delta^* F)(w) = (\delta^* F)(v)$$

$$\sum_e (\delta 1_v)(e) F(e) = \sum_{v \sim w} \underbrace{(\delta 1_v)(wv)}_{1} F(wv) = \sum_{v \sim w} F(wv)$$

ex: choose basis for $\Omega(G)$, compute δ, δ^* as matrices.

---

example:



we see that $\Delta = D - A$ (prove this!)

If G is k-regular, $D = kI$, and Δ, A have the same spectral properties: $\operatorname{Spec} \Delta = \{k - \lambda \mid \lambda \in \operatorname{Spec} A\}$

If G is not regular, Δ is interesting.

For any $T : V \to W$ (V, W inn. prod. sp.),

① $T^* T$ is self-adjoint with non-negative spectrum

② $\ker T^* T = \ker T$

Pf: ① $(T^*T)^* = T^*T$ (self adj)

if $\underset{\|v\|=1}{T^*Tv = \lambda v}$, $\lambda = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \geq 0$.

② if $Tv = 0 \to T^*Tv = 0$. If $T^*Tv = 0$ then

$0 = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \to Tv = 0$. □

So, Δ has nonnegative spectrum, and $\ker\Delta = \ker\delta$

$\ker\delta = \{f : V \to \mathbb{R} \mid \delta f \equiv 0\}$ = locally const. funcs.

$\to$ mult. of 0 in Spec of Δ = # conn. comp.

(q: can you find #con. comp. from Spec A ?)

Note: The Rayleigh quotient of Δ is

$$R_f = \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\langle \delta^*\delta f, f\rangle}{\langle f, f\rangle} = \frac{\langle \delta f, \delta f\rangle}{\langle f, f\rangle} = \frac{\|\delta f\|^2}{\|f\|^2}$$

Write: Spec Δ : $0 = \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$

if G k-reg $\gamma_i = k - \lambda_i$

Thm (Discrete Cheeger Inequalities):

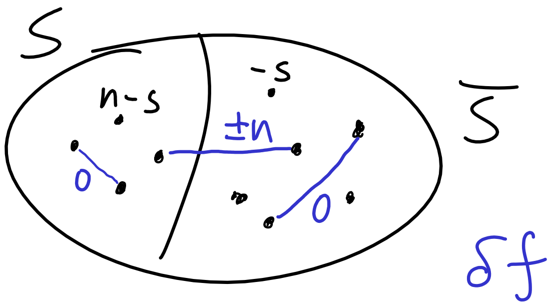
If G (finite) has max deg K, then

$$\boxed{\frac{h(G)^2}{2K} \leq \gamma_2 \leq h'(G)}$$

Pf: easy direction: find $S \subseteq V$ with $\frac{|\partial S| n}{|S||\bar{S}|} = h'(G)$

Define $(n-s)\mathbb{1}_S - s\,\mathbb{1}_{\bar{S}} = f$

$s = |S|$ so $f \perp \mathbb{1}$



Because $f \perp \mathbb{1}$ = the e.vec of γ_1, we have

$$\gamma_2 \leq R_f = \frac{\|\delta f\|^2}{\|f\|^2} = \frac{|\partial S| n^2}{s(n-s)n} = h'(G)$$

$$\|f\|^2 = s(n-s)^2 + (n-s)(-s)^2 = s(n-s)n$$

$$\|\delta f\|^2 = \sum_e (\delta f)(e)^2 = |\partial S| n^2$$

$\blacksquare$

Hard direction: we want γ_2 small $\to$ G has sparse cut

$\gamma_1 \to$ constant

γ_2 tries to be locally const & orthogonal to $\mathbb{1}$

if f is a normalized γ_2-e.vec: $\gamma_2 = R_f = \|\delta f\|^2$ small

so we want to order V by $f(v)$ and cut at some point (zero = avg f?, median f?)

we will take this f and show that for some t,

$$S = \{v \mid f(v) \leq t\} \text{ gives } \frac{|\partial S|}{\min(|S|,|\bar{S}|)} \leq \sqrt{2K\gamma_2}.$$

$$(\text{so} \to \quad h(G)^2 \leq 2K\gamma_2)$$

Goal: $\frac{h^2}{2K} \le \gamma_2$ where γ_2 = second e.v. of $\Delta = \delta^*\delta = D - A$, K = max deg

$h = \min_{\substack{S \subseteq V \\ 0<|S|\le \frac{n}{2}}} \frac{|\partial S|}{|S|}$. also: $R_f = \frac{\langle \Delta f, f\rangle}{\langle f,f\rangle} = \frac{\|\delta f\|^2}{\|f\|^2}$.

27/4/20

For $f: V \to \mathbb{R}$ define $L_t^f = \{v \in V \mid f(v)^2 > t\}$

<u>Lemma</u>: for $0 \ne f: V \to \mathbb{R}$, there exist $0 < t < \max(f(v)^2)$ s.t. $\frac{|\partial L_t^f|}{|L_t^f|} \le \sqrt{2K R_f}$ ($= \sqrt{2K\gamma_2}$ for f = e.vec of γ_2 so if $|L_t^f| \le n/2$, we're done)

<u>Pf</u>: Assume $\|f\| = 1$, write $M = \max f(v)^2$.

$$1 = \|f\|^2 = \sum_v f(v)^2 = \sum_v \int_0^M \mathbb{1}_{[0, f(v)^2]}(t)\, dt$$

$$= \int_0^M \sum_v \mathbb{1}_{[0, f(v)^2]}(t)\, dt = \int_0^M \#\{v \mid f(v)^2 > t\}\, dt = \int_0^M |L_t^f|\, dt$$

On the other hand, $e \in \partial L_t^f$ iff $f(e^\pm)^2 \le t < f(e^\mp)^2$ so

$$\int_0^M |\partial L_t^f|\, dt = \int_0^M \sum_{e\in E} \mathbb{1}_{[f(e^\pm)^2, f(e^\mp)^2]}(t)\, dt$$

$$= \sum_{e\in E} \int_0^M \mathbb{1}_{[f(e^\pm)^2, f(e^\mp)^2]}(t)\, dt = \sum_{e\in E} |f(e^+)^2 - f(e^-)^2|$$

(we would be happier to get $\sum_e |f(e^+) - f(e^-)|^2 = \|\delta f\|^2$
$\to$ check what the Cheeger ineq. would have been)

So, $\int_0^M |\partial L_t^f|\, dt = \sum_e |f(e^+)^2 - f(e^-)^2| = \sum_e |f(e^+)-f(e^-)||f(e^+)+f(e^-)|$

$$\underset{C.S.}{\leq} \underbrace{\sqrt{\sum_e |f(e^+)-f(e^-)|^2}}_{\|\delta f\|} \sqrt{\sum_e |f(e^+)+f(e^-)|^2}$$

$$\leq \|\delta f\| \sqrt{2\sum_e f(e^+)^2 + f(e^-)^2} \leq \sqrt{R_f\, 2K \sum_{v\in V} f(v)^2} = \sqrt{2KR_f}$$

In total we got $\int_0^M |\partial L_t^f|\, dt \leq \sqrt{2KR_f} = \int_0^M \sqrt{2KR_f}\, |L_t^f|\, dt$

so at some $0<t<M$ we must have $|\partial L_t^f| \leq \sqrt{2KR_f}\, |L_t^f|$ ∎

<u>Prop.</u>: if $f \perp \mathbb{1}$ ($f \in \mathbb{Z}$) then for some $t \in \mathbb{R}$,

either $S = \{v \mid f(v) > t\}$ or $S = \{v \mid f(v) < t\}$

satisfies $\frac{|\partial S|}{|S|} \leq \sqrt{2KR_f}$ and $|S| \leq \frac{n}{2}$.

(Cheeger follows by taking $f =$ e.vec of γ_2)

<u>Pf</u>: Denote $m = \text{median}(f)$, $f^+(v) = \max(f(v)-m, 0)$

$f^-(v) = \max(m - f(v), 0)$

so, $f^+ - f^- = f - m\mathbb{1}$, $f^+ \perp f^-$.

★ $\|\delta f\|^2 \geq \|\delta f^+\|^2 + \|\delta f^-\|^2$: for every $e \in E$, if

$f(e^+), f(e^-) \geq m$ (or $\leq m$), $f^+ = f - m$ on e^+, e^-

$f^- = 0$ on e^+, e^-

so $\delta f(e)^2 = \delta f^+(e)^2 + \underbrace{\delta f^-(e)^2}_{0}$

if $f(e^-) < m < f(e^+)$ then

$$(\delta f)(e)^2 = |f(e^+) - f(e^-)|^2 = |f(e^+) - m + m - f(e^-)|^2$$

$$= |\delta f^+(e) + \delta f^-(e)|^2 \geq \delta f^+(e)^2 + \delta f^-(e)^2 .$$

Now:

$$R_f \underset{\substack{\uparrow \\ \text{by Rayleigh} \\ \Delta \mathbb{1} = 0}}{\geq} R_{f - m\mathbb{1}} = \frac{\|\delta(f - m\mathbb{1})\|^2}{\|f^+ - f^-\|^2} \underset{\substack{\uparrow \\ f^+ \perp f^-}}{\overset{\substack{\delta\mathbb{1}=0 \\ \downarrow}}{=}} \frac{\|\delta f\|^2}{\|f^+\|^2 + \|f^-\|^2}$$

$$\overset{\star}{\geq} \frac{\|\delta f^+\|^2 + \|\delta f^-\|^2}{\|f^+\|^2 + \|f^-\|^2} \geq \min\{R_{f^+}, R_{f^-}\}$$

So now, applying the first Lemma to f^+ (and f^-)

we got for some t $\quad \dfrac{|\partial L_t^{f^+}|}{|L_t^f|} \leq \sqrt{2KR_{f^+}}$

for some t' $\quad \dfrac{|\partial L_{t'}^{f^-}|}{|L_{t'}^{f^-}|} \leq \sqrt{2KR_{f^-}}$

one of these $\leq \sqrt{2KR_f}$

and both $|L_t^{f^+}|$ and $|L_{t'}^{f^-}|$ are $\leq \frac{n}{2}$. $\square$

$\boxed{4/5/20}$

Cor: $\underbrace{\text{sparse cut}}$, approximation

$\hookrightarrow$ S such that $\dfrac{|\partial S|}{\min(|S|, |\bar{S}|)}$ is small

Fact: finding the sparsest cut is NP-hard also - computing $h(G)$.

However, we can find γ_2, and compute an e.vec f for γ_2. We showed that for some $t \in \mathbb{R}$ we then have that $S = \{v \mid f(v) > t\}$ satisfies

$$\frac{|\partial S|}{\min(|S|,|\bar{S}|)} \underset{\substack{\uparrow \\ \text{from hard side proof}}}{\leq} \sqrt{2KR_f} = \sqrt{2K\gamma_2} \underset{\substack{\uparrow \\ \text{easy side of Cheeger}}}{\leq} \boxed{\sqrt{2Kh}}$$

Hoffman's bound

We say $S \subset V$ is independent if $E(S,S) = \emptyset$. The independence number of G is $\alpha_G = \max\{|S| \mid S \text{ indep.}\}$.

The chromatic number of G is the smallest c so that V_G can be colored in c colors, with no neighbors of the same color. Denoted χ_G.

Observe: $\alpha_G \chi_G \geq n$ (in a coloring, the vertices of any color form an indep. set).

<u>Thm</u> (Hoffman's bound):

For k-regular graph $\alpha_G \leq n\left(1 - \frac{k}{\gamma_n}\right)$

Hence, $\chi_G \geq \frac{\gamma_n}{\gamma_n - k}$.

$A \qquad [-k \quad \dots \quad k]$

$\Delta = kI - A \qquad [0 \quad \dots \underset{k}{|} \dots \quad 2k]$

$\left(\gamma_n = 2k \text{ iff } G \text{ is bipartite} \quad \hookrightarrow n\left(1 - \frac{k}{\gamma_n}\right) = \frac{n}{2} \right)$

if $\gamma_n = \left(1 + \frac{1}{m}\right)k$ then $\alpha_G \leq \frac{n}{m+1}$

(ex: $\gamma_n \geq k$ always) $\longrightarrow \chi_G \geq m+1$

<u>Pf</u>: If S is indep., take

$$f = (n-s)\mathbb{1}_S - s\,\mathbb{1}_{\bar{S}} \qquad (s = |S|)$$

$$\gamma_n \geq R_f = \frac{\|\delta f\|^2}{\|f\|^2} = \frac{|\partial S|n}{s(n-s)} \underset{\uparrow}{=} \frac{ksn}{s(n-s)} = \frac{kn}{n-s}$$

S indep.

$\rightarrow n\gamma_n - s\gamma_n \geq kn$

$\rightarrow s \leq n\left(1 - \frac{k}{\gamma_n}\right).$ $\square$

"one-sided expander"

$\overbrace{\gamma_2 \gg 0}$ $\rightarrow$ Cheeger (no sparse cut)

$\gamma_n \ll 2h \rightarrow$ Hoffman (no large indep.)

$\underbrace{0 \ll \gamma_2 < \gamma_n \ll 2k}$ $\rightarrow$ EML (pseudo random)

"two-sided expander"

Existence

Q: Fix $k \geq 3$. Is there some $\varepsilon > 0$ so that $\exists$ inf. family of k-regular graphs with $h(G) \geq \varepsilon$?

Pinsker: Yes. (Also - Kolmogorov + Bardzin)

In fact, most are:

"Thm": generate a random k-reg graph on n vertices. $\exists \varepsilon > 0$ s.t. $\mathrm{Prob}[h \geq \varepsilon] \xrightarrow[n\to\infty]{} 1$

Our model: Choose unif. rand. indep. $\sigma_1, \dots, \sigma_{k/2} \in S_n$, take $V = \{1, \dots, n\}$ connect $i \to \sigma_j(i)$ and also $i \to \sigma_j^{-1}(i)$ (to get an undirected gph)

$$A = \sum_{i=1}^{k/2} P_{\sigma_i} + P_{\sigma_i^{-1}} \quad \left(P_\sigma = \text{permutation matrix of } \sigma\right)$$

(Caveat: only good for even k)

Thm: $\forall k \geq 4 \ \exists \varepsilon > 0$ s.t.

$$\mathrm{Prob}[h(G) \geq \varepsilon] \xrightarrow[n\to\infty]{} 1$$

Pf: Want to show: $P\Big[\underbrace{\min_{1 \le |S| \le \frac{n}{2}} \frac{|\partial S|}{|S|} < \varepsilon}_{\text{union over many events}}\Big] \to 0$

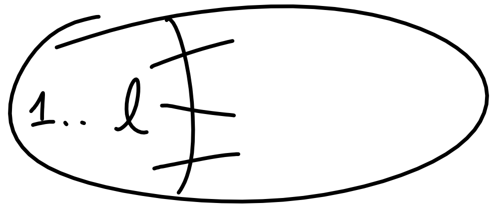
Union bound:

$$P\Big[\bigcup_{1 \le |S| \le n/2} \Big\{\sigma_1, \ldots, \sigma_{k/2} \Big| \frac{|\partial S|}{|S|} < \varepsilon\Big\}\Big]$$

$$\le \sum_{1 \le |S| \le n/2} P\Big[\frac{|\partial S|}{|S|} < \varepsilon\Big] = \sum_{1 \le \ell \le n/2} \binom{n}{\ell} P\Big[\frac{\partial\{1..\ell\}}{\ell} < \varepsilon\Big]$$

When is $|\partial\{1..\ell\}| < \varepsilon \ell$?

$|\partial| < \varepsilon \ell$



If $|\partial\{1..\ell\}| < \varepsilon \ell$ then

$|N_{1..\ell}| \le \ell + \varepsilon \ell$, where

$N_S = \{w \mid \exists s \in S,\ s \sim w\}$ (note we can have $S \cap N_S \neq \emptyset$)

so $\exists T \subseteq V \setminus S$ with $|T| = \varepsilon \ell$

such that $N_{1..\ell} \subseteq \{1..\ell\} \cup T$.

Thus, $P\Big[\frac{|\partial\{1..\ell\}|}{\ell} < \varepsilon\Big] \le P\Big[\bigcup_{\substack{T \subseteq V \setminus 1..\ell \\ |T| = \varepsilon \ell}} \big\{N_{1..\ell} \subseteq \{1..\ell\} \cup T\big\}\Big]$

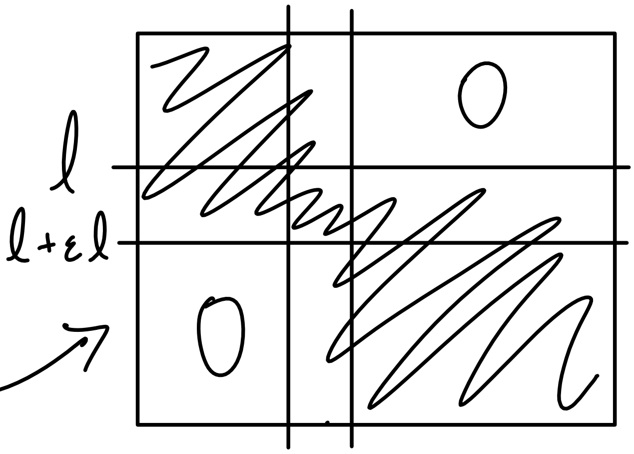
$$\le \sum_T P\big[N_{1..\ell} \subseteq \{1..\ell\} \cup T\big\}\big]$$

$$= \sum_T P\big[N_{1..\ell} \subseteq \{1..\ \ell + \varepsilon \ell\}\big]$$

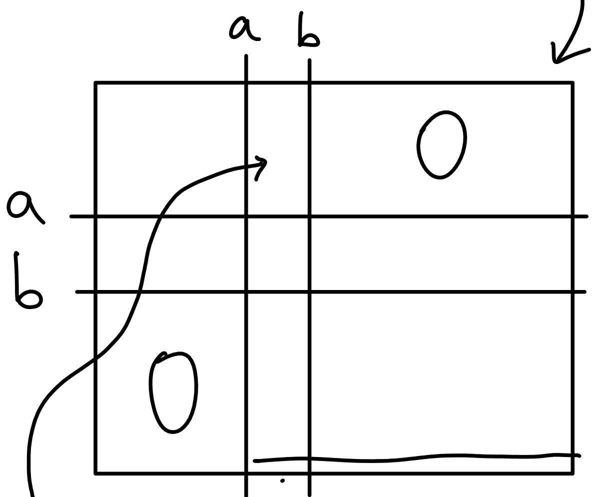
$$= \binom{n-\ell}{\varepsilon\ell} P\left[N_{1..\ell} \subseteq \{1..\ell+\varepsilon\ell\}\right]$$

$N_{1..\ell} \subseteq \{1..\ell+\varepsilon\ell\}$ means that in each $\sigma = \sigma_1, \dots, \sigma_{k/2}$ we had

P_σ



<u>Lemma</u>: for $a<b<n$, there are at most $b!(n-a)!/(b-a)!$ permutations with



<u>Pf</u>: First choose the top a rows: there are $\underbrace{b\cdot(b-1)\cdot\ldots\cdot(b-a+1)}_{(b)_a}$.

Assume we put N ones in

Choose the last $n-b$ rows: $(n-a-N)_{n-b}$ possibilities. This is at most $(n-a)_{n-b}$

Now choose the middle $b-a$ rows - $(b-a)!$ possibilities.

In total we have at most

$$(b)_a \, (n-a)_{n-b} \, (b-a)! = \frac{b! \, (n-a)!}{(b-a)!} \quad \square$$

So, $P\left[N_{1..\ell} \subseteq \{1 .. \ell+\varepsilon\ell\}\right] \leq \left(\frac{(\ell+\varepsilon\ell)! \, (n-\ell)!}{(\varepsilon\ell)! \, n!} \right)^{k/2}$

In total,

$$P\left[h(G) < \varepsilon\right] \leq \sum_{\ell=1}^{n/2} \binom{n}{\ell} \binom{n-\ell}{\varepsilon\ell} \left(\frac{(\ell+\varepsilon\ell)! \, (n-\ell)!}{(\varepsilon\ell)! \, n!} \right)^{k/2}$$

$$(k \geq 4) \qquad \leq \sum_{\ell=1}^{n/2} \binom{n}{\ell} \binom{n-\ell}{\varepsilon\ell} \left(\frac{(\ell+\varepsilon\ell)! \, (n-\ell)!}{(\varepsilon\ell)! \, n!} \right)^{2}$$

we need to show that for some $\varepsilon > 0$, this goes to 0 as $n \to \infty$.

(In fact for $\varepsilon \leq 0.163$ it does, for $\varepsilon = 0.164$ it goes to ∞ as $n \to \infty$)

Remark: using modern methods one can show actually $P\left[h'(G) > k - 2\sqrt{k}\right] \xrightarrow[n\to\infty]{} 1$!

Can you construct G with $h'(G) > k - 2\sqrt{k}$?

"Finding hay in a haystack".

$$\sum_{\ell=1}^{\log n} \binom{n}{\ell}\binom{n-\ell}{\varepsilon\ell}\left(\frac{(\ell+\varepsilon\ell)!\,(n-\ell)!}{(\varepsilon\ell)!\;n!}\right)^2 \le \sum_{\ell=1}^{\log n} n^{\ell}\, n^{\varepsilon\ell}\left((\ell+\varepsilon\ell)^{\ell}\cdot\frac{1}{(n-\ell+1)^{\ell}}\right)^2$$

$$\le \sum_{\ell=1}^{\log n}\left(\frac{n^{(1+\varepsilon)}\,\log^2 n\cdot(1+\varepsilon)^2}{(n-\log n+1)^2}\right)^{\ell} \overset{\text{def}}{=} \sum_{\ell=1}^{\log n} q_n^{\ell}$$

For any $\varepsilon<1$, $q_n \xrightarrow{n\to\infty} 0$ (by calc. I)

$$\Longrightarrow \sum_{\ell=1}^{\log n} q_n^{\ell} \le \sum_{\ell=1}^{\infty} q_n^{\ell} = \frac{q_n}{1-q_n} \xrightarrow{n\to\infty} 0.$$

$$\sum_{\ell=\log n}^{n/2} \binom{n}{\ell}\binom{n-\ell}{\varepsilon\ell}\left(\frac{(\ell+\varepsilon\ell)!\,(n-\ell)!}{(\varepsilon\ell)!\;n!}\right)^2 = \sum_{\log n}^{n/2} \frac{(\ell+\varepsilon\ell)!^2\,(n-\ell)!^2}{\ell!\,(n-\ell(1+\varepsilon))!\,(\varepsilon\ell)!^3\,n!}$$

$$\underset{\text{Cheating!}}{\xrightarrow{\varepsilon\to 0}} \quad \sum_{\ell=\log n}^{n/2}\binom{n}{\ell}^{-1} \le \frac{n}{2}\binom{n}{\log n}^{-1} \le \frac{n}{2}\left(\frac{\log n}{n-\log n+1}\right)^{\log n} \xrightarrow[n\to\infty]{} 0$$

Use Sterling: $n! \approx \left(\frac{n}{e}\right)^n$, or $\boxed{\log n! \approx n\log n - n}$

In log the "$-n$" part cancels out:

$$2(\ell+\varepsilon\ell)+2(n-\ell) = \ell+n-\ell(1+\varepsilon)+3\varepsilon\ell+n$$

So you are left with "$\log n! \approx n \log n$", from there everything is tedious analysis.

Schreier Graphs

If Γ is a group, $S \subseteq \Gamma$, we defined

$$\mathrm{Cay}(\Gamma, S) = \left(\Gamma, \left\{ \overset{g}{\circ} \longrightarrow \overset{gs}{\circ} \;\middle|\; \begin{matrix} g \in \Gamma \\ s \in S \end{matrix} \right\}\right)$$

if $S = S^{-1}$ then this is an undirected graph.

It is always k-regular for $k = |S|$.

so if we want undirecte, k odd, S must have an element with $s^{-1} = s$ (s of order 2).
(or $s = e$)

11/5/20

Recall: if G is a group, a <u>G-set</u> is a set X on which G acts. Namely, there is a mult. rule $\cdot : G \times X \longrightarrow X$ so that: ① $\forall g, g' \in G$, $x \in X$: $g(g'x) = (gg')x$

② $\forall x \in X$: $e \cdot x = x$

example: $GL_n(\mathbb{F})$ acts on $\mathbb{F}^n$

S_n acts on $\{1, \dots, n\}$

every G acts on itself (by $g \cdot g' = gg'$)
D_n acts on the vertices of the n-gon

If G is a group, X a G-set and $S \subseteq G$, we define the Schreier graph:

$$Sch(\underset{\text{"acts"}}{G \circlearrowright} X, S) = \left(X, \left\{(x \to sx) \,\middle|\, \begin{matrix} x \in X \\ s \in S \end{matrix}\right\}\right).$$

Example: the model we gave last week came from taking $G = S_n$, $X = \{1, \ldots, n\}$, and S = random $k/2$ permutations & their inverses.
(and if we want odd k, we can choose a permutation of order 2 - a product of transpositions).

Characters

A character of a finite abelian group G is a hom. $\chi : G \to \mathbb{C}^\times$ (actually $|\chi(g)| = 1$ since G is finite $\to$ $\chi(g)^{|G|} = \chi(g^{|G|}) = \chi(e) = 1$)
The set of all chars is denoted $\hat{G}$,

and its a group (by $(\chi\chi')(g) = \chi(g)\chi'(g)$)

$\hat{G}$ is called the dual group of G.

Point: if $\chi \in \hat{G}$, for any $S \subseteq G$, χ is an eigenvector of $\mathrm{Cay}(G,S)$:

$$(A\chi)(g) = \sum_{s \in S} \chi(sg) = \left(\sum_{s \in S} \chi(s)\right)\chi(g).$$

Fact: $|\hat{G}| = G$ (actually even $\hat{G} \cong G$) and the elts. of $\hat{G}$ are lin. ind. in $\mathbb{C}^G$.

This means we found <u>all</u> e.vecs of A!

Take $G = \mathbb{Z}/n \times \mathbb{Z}/n = \mathbb{Z}_n^2$. For each $\alpha \in \mathbb{Z}_n^2$ define $\chi_\alpha(\beta) = \omega^{\alpha_1\beta_1 + \alpha_2\beta_2} = \omega^{\alpha^t\beta}$

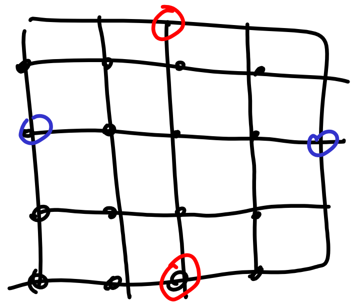
$$(\omega = e^{\frac{2\pi i}{n}})$$

We got n^2 chars., they are orthogonal;

$$\langle \chi_\alpha, \chi_\beta \rangle = \sum_{g \in G} \chi_\alpha(g)\overline{\chi_\beta(g)} = n^2 \delta_{\alpha,\beta}$$

(It follows that they are indep $\rightarrow$

form a basis of $\mathbb{C}^G$).



Take $S = \left\{ \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \right\}$

$\Gamma = \mathrm{Cay}(G, S)$. $A_c = \mathrm{Adj}_\Gamma$

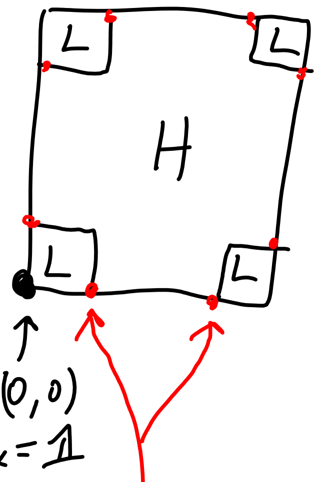
For $\alpha \in \mathbb{Z}_n^2$, χ_α is an e.vec of A_c with e.val $\lambda_\alpha = \sum_{s \in S} \chi_\alpha(s) = \sum_s \omega^{\alpha^t s}$

$$= \omega^{\alpha_1} + \omega^{-\alpha_1} + \omega^{\alpha_2} + \omega^{-\alpha_2} = 2\left(\cos\tfrac{2\pi}{n}\alpha_1 + \cos\tfrac{2\pi}{n}\alpha_2\right)$$

if α_1, α_2 are close to 0 (or n), this is close to $4 = k$.

Divide $\mathbb{Z}_n^2 = L \sqcup H$

$$L = \left\{ \alpha \;\middle|\; \begin{matrix} \min(\alpha_1, n-\alpha_1) < \frac{n}{6} \\ \min(\alpha_2, n-\alpha_2) < \frac{n}{6} \end{matrix} \right\}$$



Writing $\mathbb{C}^G = \underbrace{\hat{L}}_{\text{span}\{\chi_\alpha \mid \alpha \in L\}} \oplus \hat{H}$ ← span$\{\chi_\alpha \mid \alpha \in H\}$

we get for $f \in \hat{H}$:

$$\frac{\langle A_c f, f\rangle}{\langle f, f\rangle} \leq \max_{\alpha \in H} \lambda_\alpha = \boxed{2\left(1 + \cos\tfrac{2\pi}{6}\right)} = 3.$$

Our goal now is to add egdes that will handle $\hat{L}$. 12/5/20

Look at a new graph:

$$\mathrm{Sch}\left(GL_2(\mathbb{Z}/n) \circlearrowright (\mathbb{Z}/n)^2, \left\{\begin{pmatrix}1 & \pm 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ \pm 2 & 1\end{pmatrix}\right\}\right)$$

Call its adj. operator A_s. (both A_c and A_s are 4-regular & act on $\mathbb{C}^{\mathbb{Z}_n^2}$).

We'll show that A_s contracts $f \in \hat{L}$, $f \perp \mathbb{1} = \chi_0$

$$\left(\frac{\langle A_s f, f\rangle}{\langle f, f\rangle} < \delta \text{ for some } \delta < 4.\right)$$

For a fin. abel. gp. G, we "saw" G has $|G|$ chars., and they are orthogonal $\Rightarrow$ form a basis of $\mathbb{C}^G$. The change of basis from $\{\delta_g\}_{g \in G}$ (the "standard" basis) and $\{\chi\}_{\chi \in \hat{G}}$ is called "Fourier transform".

For $f \in \mathbb{C}^G$, and $\chi \in \hat{G}$, define $\hat{f} \in \mathbb{C}^{\hat{G}}$

$$\boxed{\hat{f}(\chi) = \langle f, \chi\rangle = \sum_{g \in G} f(g)\overline{\chi(g)}}.$$

We will write $\hat{f}(\alpha)$ instead of $\hat{f}(\chi_\alpha)$,

getting: $\boxed{\hat{f}(\alpha) = \sum_{\beta \in G} f(\beta)\overline{\chi_\alpha(\beta)} = \sum_{\beta} \omega^{-\alpha^t \beta} f(\beta)}$.

Since $\{\chi_\alpha\}$ are almost an O.N.B., we get: $\boxed{f = \frac{1}{n^2}\sum_\alpha \hat{f}(\alpha)\chi_\alpha}$

(since $\langle \chi_\alpha, \chi_\beta \rangle = n^2 \cdot \delta_{\alpha,\beta}$). We also get

$\boxed{\langle f, h \rangle = \frac{1}{n^2}\langle \hat{f}, \hat{h} \rangle}$ (Parseval)

Ex: (Fourier inversion): $\hat{\hat{f}}(\alpha) = n^2 f(-\alpha)$.

For $X \in GL_2(\mathbb{Z}_n)$ define $M_X : \mathbb{C}^G \to \mathbb{C}^G$

$(M_X f)(\alpha) = f(X\alpha)$

so $A_S = M_B + M_{B^{-1}} + M_{B^t} + M_{B^{t^{-1}}}$ $\left(B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right)$

We have $\langle A_S f, f \rangle = \frac{1}{n^2}\langle \widehat{A_S f}, \hat{f} \rangle$

$= \frac{1}{n^2}\langle \widehat{(M_B + M_{B^{-1}} + \dots) f}, \hat{f} \rangle$

$= \frac{1}{n^2}\langle \widehat{M_B f} + \widehat{M_{B^{-1}} f} + \dots, \hat{f} \rangle = \frac{1}{n^2}\left[\langle \widehat{M_B f}, \hat{f} \rangle + \dots\right]$

<u>Observe</u>: $\widehat{M_X f}(\alpha) = \sum_{\beta \in G} \omega^{-\alpha^t \beta}(M_X f)(\beta)$

$$= \sum_{\beta} \omega^{-\alpha^t \beta} f(X\beta) \overset{\beta \mapsto X^{-1}\beta}{=} \sum_{\beta} \omega^{-\alpha^t X^{-1}\beta} f(\beta)$$

$$= \sum_{\beta} \omega^{-(X^{t^{-1}}\alpha)^t \beta} f(\beta) = \hat{f}(X^{t^{-1}}\alpha) = (M_{X^{t^{-1}}} \hat{f})(\alpha)$$

i.e.: $\widehat{M_X f} = M_{X^{t^{-1}}} \hat{f}$. We get magic:

$$\boxed{\begin{aligned} \widehat{A_s f} &= \widehat{M_B f} + \widehat{M_{B^{-1}} f} + \widehat{M_{B^t} f} + \widehat{M_{B^{t^{-1}}} f} \\ &= M_{B^{t^{-1}}} \hat{f} + M_{B^t} \hat{f} + M_{B^{-1}} \hat{f} + M_B \hat{f} = A_s \hat{f} \end{aligned}}$$

so $R_f^{A_s} = \dfrac{\langle A_s f, f\rangle}{\langle f, f\rangle} \underset{\uparrow \text{ Parseval}}{=} \dfrac{\langle \widehat{A_s f}, \hat{f}\rangle}{\langle \hat{f}, \hat{f}\rangle} = \dfrac{\langle A_s \hat{f}, \hat{f}\rangle}{\langle \hat{f}, \hat{f}\rangle} = R_{\hat{f}}^{A_s}$.

We want to show $R_f < \delta < 4$ for $f \in \hat{L}$, $f \perp \mathbb{1}$

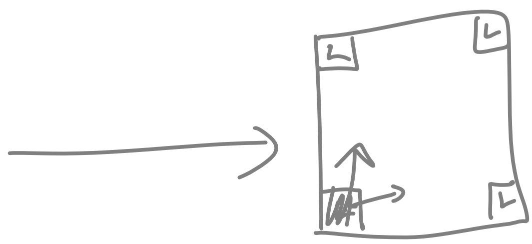
or: $f \in \widehat{L \setminus \{0\}} = \mathrm{Span}\{\chi_\alpha \mid 0 \neq \alpha \in L\}$.

For such f, $\hat{f}$ is supported on $L \setminus \{0\}$.

$\langle A_s \hat{f}, \hat{f} \rangle$ is small

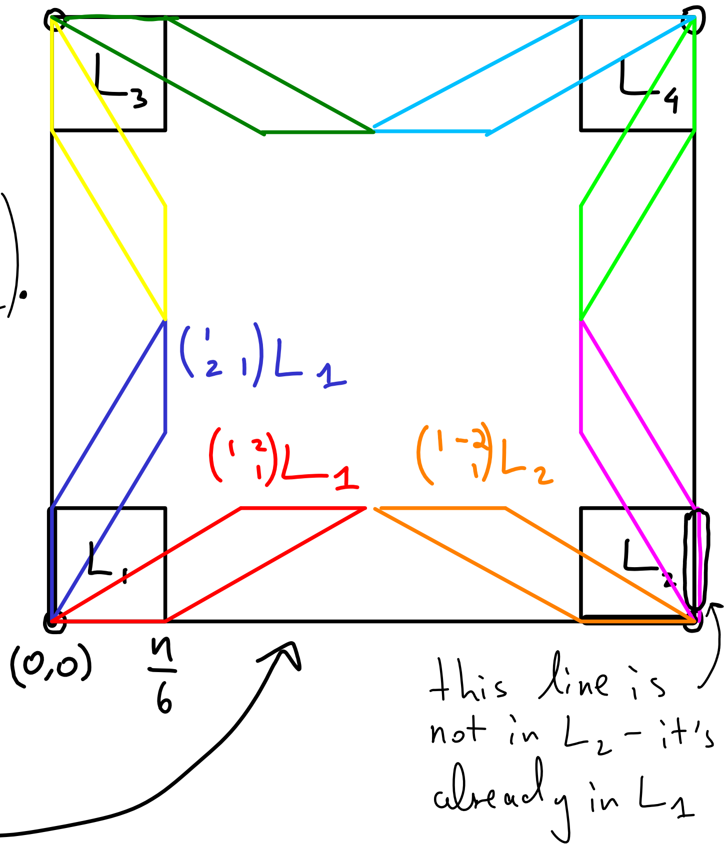
$\Updownarrow$

"A expands $\hat{f}$"



Claim: For any $S \subseteq L \setminus 0$, $|\partial S| \geq |S|$.

Proof: Divide $L \setminus \{0\}$ to 4 disjoint squares $L_1, \dots, L_4$ (assigning their "intersections" arbitrarily). For each L_i pick two appropriate shearings among $B, B^{-1}, B^t, B^{t^{-1}}$ so that in total we get eight disjoint parallelograms.



This means that for any $S \subset L \setminus (0,0)$ we have $|N_S| \geq 2|S|$ $\left(N_S = \operatorname{supp} A_s \mathbb{1}_S\right)$ and therefore, $|\partial S| \geq |N_S| - |S| \geq |S|$ ■

$\boxed{\text{For any } t > 0}$, taking $S = L_t^{|\hat{f}|} = \{\alpha \mid |\hat{f}(\alpha)|^2 > t\}$ $\boxed{\text{we get } \frac{|\partial L_t^{|\hat{f}|}|}{|L_t^{|\hat{f}|}|} > 1}$. However, we proved (page 24) that for any $p : V \to \mathbb{R}$, $\boxed{\exists\, t > 0 \text{ with}}$

$$\boxed{\frac{|\partial L_t^p|}{|L_t^p|} \leq \sqrt{2k R_p^\Delta}}.$$

Thus: $\boxed{1 < 2k R^\Delta_{|\hat{f}|} = 8 R^\Delta_{|\hat{f}|}}$.

Combining everything: For $\mathbb{1} \perp f \in \hat{L}$ with $\|\hat{f}\| = 1$,

$$R_f^{A_s} = R_{\hat f}^{A_s} = \langle A_s \hat f, \hat f\rangle \overset{\text{triangle}}{\leq} \langle A_s |\hat f|, |\hat f|\rangle$$

$$= \langle (4I-\Delta)|\hat f|, |\hat f|\rangle = 4 - \underbrace{\langle \Delta |\hat f|, |\hat f|\rangle}_{R^{\Delta}_{|\hat f|}} \leq 4 - \frac{1}{8}.$$

The Gabber-Galil graph is 8 regular with $A = A_c + A_s$.

For $\mathbb{1} \perp f$, $\|f\| = 1$, write $f = f^L + f^H$ $(f^L \in \hat L, f^H \in \hat H)$

$$\langle A_c f, f\rangle = \langle A_c f^H, f^H\rangle + \langle A_c f^L, f^L\rangle$$

all χ_α are A_c-e.vecs

$$\leq 3\|f^H\|^2 + 4\|f^L\|^2$$

$$= 4 - \|f^H\|^2.$$

Thus if $\|f^H\| \geq \varepsilon$ (determine ε later)

then

$$\langle Af, f\rangle = \langle A_c f, f\rangle + \langle A_s f, f\rangle \boxed{\leq 8 - \varepsilon^2}$$

And, if $\|f^H\| < \varepsilon$ then

$$\langle Af, f\rangle = \langle A_c f, f\rangle + \langle A_s f, f\rangle$$

$$\leq 4 + \langle A_s f^L, f^L\rangle + \langle A_s f^L, f^H\rangle + \langle A_s f^H, f\rangle$$

$$\leq 4 + \left(4 - \frac{1}{8}\right) + 4\varepsilon + 4\varepsilon$$

$= \boxed{8 - \frac{1}{8} + 8\varepsilon}$. We need to choose ε to make both $\boxed{\text{bounds}}$ < 8.

Taking $\varepsilon = \frac{1}{65}$ we get $\lambda_2 \leq \langle Af, f\rangle \leq 7.9998$ $\square$

Gabber-Galil prove $\lambda_2 \leq 5\sqrt{2} \approx 7.07$.

Remark: It follows from Margulis + Kazhdan that this is a family of two-sided expanders, but with no explicit bound.

18/5/20

Thm (Margulis): pick $d \geq 3$, and $S \subseteq SL_d(\mathbb{Z}) = G$ s.t. $\langle S \rangle = G$ $\left(\text{e.g.: } S = \left\{ \begin{pmatrix} 1 & & \pm 1 \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}\right)$

Then $\Gamma_n = \mathrm{Sch}\left(SL_d(\mathbb{Z}) \circlearrowright (\mathbb{Z}/n)^d, S\right)$ is a family of expanders. Doesn't give an explicit $\lambda(\Gamma_n)$.

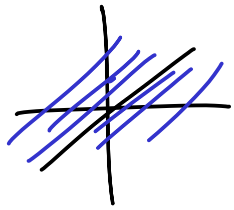
We took $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ - Margulis doesn't apply. In fact, the Gabber-Galil graph is the Schreier graph of $\mathrm{Aff}_2(\mathbb{Z}) = \left\{ v \mapsto Av + b \,\middle|\, \begin{matrix} A \in GL_2(\mathbb{Z}) \\ b \in \mathbb{Z}^2 \end{matrix} \right\}$

$$\mathrm{Sch}\left(\mathrm{Aff}_2(\mathbb{Z}) \circlearrowright \mathbb{Z}_n^2, \left\{ v \mapsto B^{\pm 1} v, v \mapsto B^{t^{\pm 1}} v, v \mapsto v \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v \mapsto v \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}\right)$$

$$\mathrm{Aff}_d(\mathbb{Z}) = GL_d(\mathbb{Z}) \ltimes \mathbb{Z}^d \hookrightarrow GL_{d+1}(\mathbb{Z})$$

$$\mathrm{Aff}_2(\mathbb{Z}) = \left\{ v \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} v + \begin{pmatrix} e \\ f \end{pmatrix} \right\} \longrightarrow \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{Z})$$

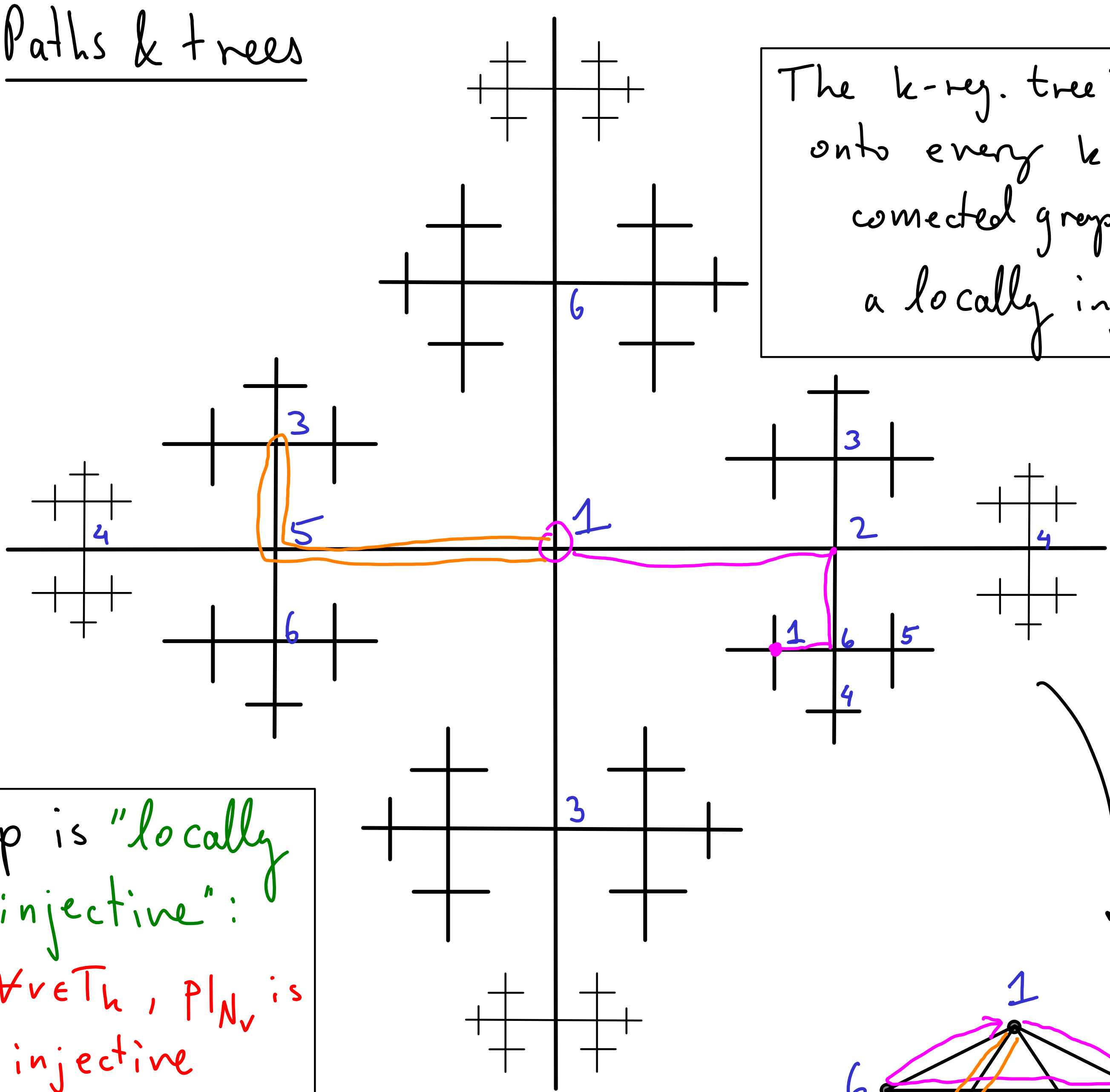
Geometrically, $\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} * \\ * \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ * \\ 1 \end{pmatrix}$



plane at height 1

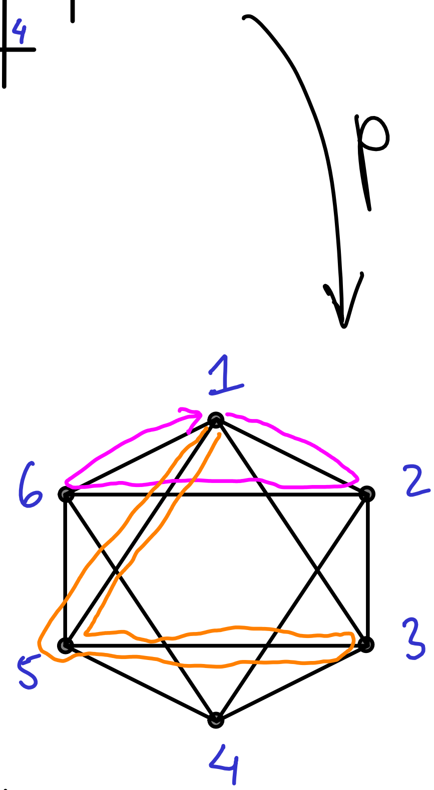
Margulis applies for GL_3...

Paths & trees



The k-reg. tree T_k maps onto every k-reg. connected graph Γ by a locally inj. map

p is "locally injective": $\forall v \in T_k$, $p|_{N_v}$ is injective



(think why connectedness of the graph Γ implies that the map $T_k \to \Gamma$ is onto).

Assume we chose $p : T_k \twoheadrightarrow \Gamma$. Any path

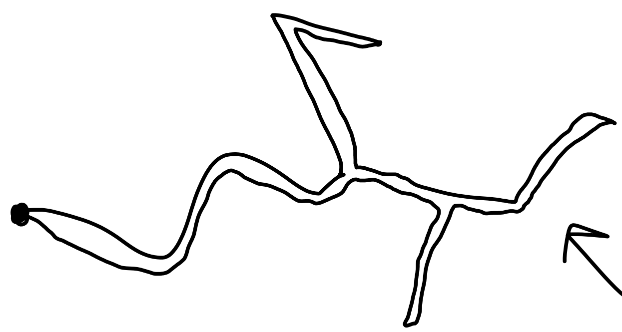
$\gamma = v_1 \to v_2 \to v_3 \to \dots \to v_\ell$ in T_k we can "push" to Γ by p: $p(\gamma) = p(v_1) \to p(v_2) \to \dots \to p(v_\ell)$

AND, any path "downstairs" (in Γ) $\gamma = v_1 \to \dots \to v_\ell$ can be "lifted" to a path $\tilde{\gamma}$ in T_k, with $p(\tilde{\gamma}) = \gamma$.

Furthermore, after choosing the start point of $\tilde{\gamma}$ (any vertex in $p^{-1}(v_1)$), $\tilde{\gamma}$ is completely determined (pigeonhole principle at every step).

A closed path in T_k always descends to a closed path in Γ. When does a closed path γ in Γ lift to a closed path in T_k?

19/5/20



When it is "completely backtracking"

Namely: a path in Γ is called completely backtracking if it lifts to a closed path in T_k.

Why is this useful?

Recall: we showed for a k-regular graph

$$\lambda(G) \geq \sqrt{k} - o(1) \quad (\text{as } n \to \infty).$$

Thm (Alon–Boppana): k-reg. graphs satisfy

$$\lambda(G) \geq 2\sqrt{k-1} - o(1) \quad (\text{as } n \to \infty)$$

In other words, $\forall \varepsilon > 0$, there are only fin. many k-reg graphs with $\lambda(G) \leq 2\sqrt{k-1} - \varepsilon$.

This is optimal: A graph with $\lambda(G) = 2\sqrt{k-1}$ is called a Ramanujan graph, and they indeed exist, with n arbitrary large (at least when $k = p^m + 1$ 80', for general k it is known if we allow $\lambda_n = -k$, and observe $\lambda(G) = \max(\lambda_2, |\lambda_{n-1}|)$ (2015)).

Denote $\rho = 2\sqrt{k-1}$.

We need to show: $\forall \varepsilon > 0$, $\exists n_0 = n_0(\varepsilon, k)$, s.t. $n > n_0$, $\lambda(G) \geq \rho - \varepsilon$.

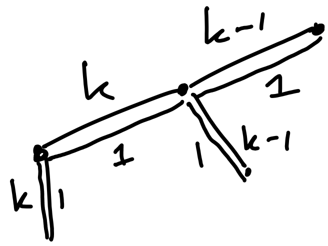
Pf: $\mathrm{tr}(A^{2t}) = \sum_{i=1}^{n} (A^{2t})_{i,i} = \sum_{i=1}^{n} \#\left\{\begin{matrix}\text{closed } 2t\text{-path}\\ \text{from } i \text{ to } i\end{matrix}\right\}$

choose a covering map $p: T_k \to \Gamma$, and $\tilde{i} \in p^{-1}(i)$. Every closed path of length $2t$ around $\tilde{i}$ (in T_k) descends to a closed path around i (in Γ). So, $\mathrm{tr}(A^{2t}) \geq \sum_{i=1}^{n} \#\left\{\begin{matrix}\text{closed } 2t\text{-paths}\\ \text{from } \tilde{i} \text{ to } \tilde{i}\end{matrix}\right\}$

(and once $2t \geq \mathrm{girth}(\Gamma)$, this a strict inequality)

$= n \cdot \#\left\{\begin{matrix}\text{closed } 2t\text{-paths}\\ \text{around any } v_0 \in T_k\end{matrix}\right\} \stackrel{\text{def}}{=} n \cdot B_{2t}^{(k)}$

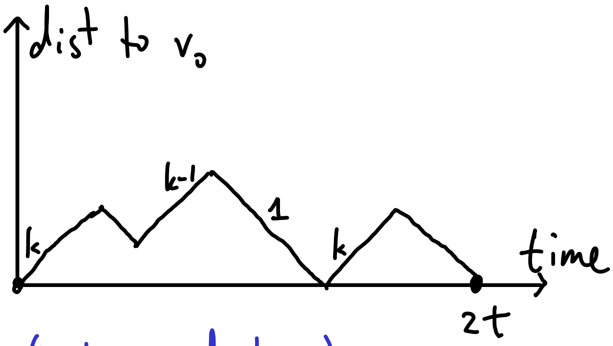
Let's bound $B_{2t}^{(k)}$:



t times 1

t times $k/k-1$

For a closed $2t$-path, we can plot the "dist to v_0" function:



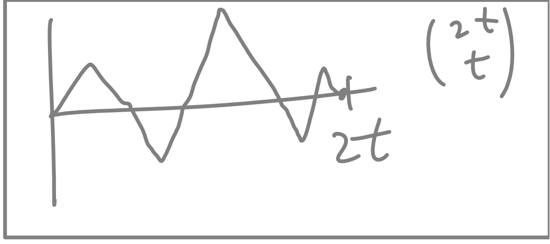
given such a distance profile, there are

$k^{\#\text{times dist}=0} \cdot (k-1)^{t-(\#\text{times dist}=0)}$ of paths around v_0

giving this distance profile. This is $\geq (k-1)^t$.

There are C_t different profiles (C_t - Catalan Number), by definition.

We know $C_t = \frac{1}{t+1}\binom{2t}{t}$ (try to prove this!)



Thus,

$$B_{2t}^{(k)} \geq C_t (k-1)^t \underset{\text{Stirling}}{\sim} \frac{4^t}{t^{3/2}\sqrt{\pi}}(k-1)^t \geq \frac{(4(k-1))^t}{t^2} = \frac{\rho^{2t}}{t^2}$$

so that

$$\frac{n\rho^{2t}}{t^2} \leq nB_{2t}^{(k)} \leq \mathrm{tr}\left(A^{2t}\right) = k^{2t} + \sum_{i=2}^{n} \lambda_i^{2t} \leq k^{2t} + n\lambda^{2t}$$

$\Rightarrow \lambda \geq \sqrt[2t]{\frac{\rho^{2t}}{t^2} - \frac{k^{2t}}{n}}$. Given $\varepsilon > 0$, choose t_0 such that $\sqrt[2t_0]{\rho^{2t_0}/t_0^2 - 1} \geq \rho - \varepsilon$ $\left(\text{using } \sqrt[2t]{\frac{\rho^{2t}}{t^2} - 1} \xrightarrow{t\to\infty} \rho\right)$, and then for $n \geq k^{2t_0}$, we get $\lambda \geq \rho - \varepsilon$ $\square$

LPS Ramanujan graphs (Lubotzky-Phillips-Sarnak)

Pick $p \equiv 1 \pmod 4$ (prime). Find all ways to write $p = a^2 + b^2 + c^2 + d^2$, a odd positive.

e.g.: $5 = 1^2 + (\pm 2)^2 + 0 + 0$
$\quad = 1 + 0 + (\pm 2)^2 + 0$
$\quad = 1 + 0 + 0 + (\pm 2)^2$

Fact (Jacobi): $\exists\ (p+1)$ such decompositions.

(Jacobi: $8(p+1)$ ways to write $p = a^2 + b^2 + c^2 + d^2$)

Pick prime $q \equiv 1 \pmod 4$, $q \neq p$. Then, there is "i" = "$\sqrt{-1}$" $\in \mathbb{F}_q$ (since $\mathbb{F}_q^\times$ is cyclic of size $q-1$, and $4 \mid q-1$, so $\exists$ elt. of order $4 \to \sqrt{-1}$).

LPS graph $(p+1)$-reg:

$$X^{p,q} = \mathrm{Cay}\left(PGL_2(\mathbb{F}_q), \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \middle| \begin{matrix} a^2+b^2+c^2+d^2=p \\ a \text{ odd positive} \end{matrix} \right\}\right)$$

$i = \sqrt{-1} \in \mathbb{F}_q$

is Ramanujan (uses heaviest 20-th century math).

Remark: for $\alpha = a+bi$, $\beta = c+di$ (in $\mathbb{Z}[i]$)

we have $N(\alpha) + N(\beta) = \alpha\bar{\alpha} + \beta\bar{\beta} = a^2+b^2+c^2+d^2 = p$

and $\det\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = N(\alpha) + N(\beta)$.

Open Q.: is there an ∞ family of 7-regular Ram. graphs? (allowink $\lambda_n = -7$, there is - Marcus-Spielman-Srivastava '15, using Bilu-Linial).

Recall: In our k-regular model (takink random uniform ind. $\sigma_1, \dots, \sigma_{k/2} \in S_n$), we showed (Pinsker) that $\exists \varepsilon > 0$ $P[\lambda_2 \leq k - \varepsilon] \xrightarrow{n \to \infty} 1$.

Alon's conjecture: $\forall \varepsilon > 0$, $P[\lambda \leq 2\sqrt{k-1} + \varepsilon] \xrightarrow{n \to \infty} 1$!

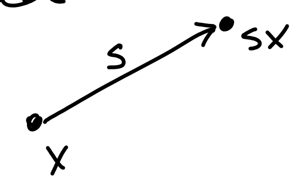
(Random regular graphs are almost Ramanujan)
Proved by Friedman 2004.

25/5/20

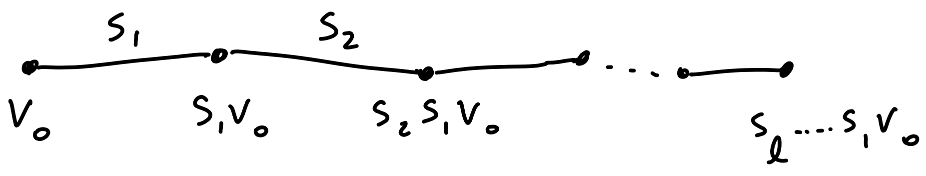
<u>Thm (Broder-Shamir)</u>: In our random k-reg. model, $\forall \varepsilon > 0 : P\left[\lambda \leq \sqrt{2}\, k^{3/4} + \varepsilon\right] \to 1$.

Recall that our model is the Schreier graph of $S_n \curvearrowright \{1, \dots, n\}$ with $k/2$ random generators $\sigma_1, \dots, \sigma_{k/2}$, and their inverses.

For any Schreier graph $Sch(G \curvearrowright X, S) = \Gamma$ the edges in Γ are S-labeled (so our edges are labeled by $\sigma_1^{\pm 1}, \dots, \sigma_{k/2}^{\pm 1}$).



Paths of length ℓ in Γ are determined by
1. a starting vertex
2. a word of length ℓ in S



Instead, we'll look at words in a fixed alphabet $x_1, \dots, x_{k/2}$ (and words = also allowing inverses ~~of~~ letters).

This is the Free group on $k/2$ letters:

$$F_r = \left\{ \begin{matrix} \text{words in} \\ x_1, \dots, x_r \text{ \& inverses} \end{matrix}, \text{concatenation} \right\}$$

$$F_2 : \; x_1 x_2 x_1^{-3} x_2, \; (x_2^{-1} x_1 x_2)(x_2^{-1} x_1^{-1} x_2) = \emptyset$$

So we will identify ℓ-paths in $\Gamma = Sch(G \curvearrowright X, S)$

$$\updownarrow$$

starting vertex + word of len ℓ in $F_{k/2}$

Here $|S| = k$, and we assume $S = S^{-1}$

so we write $S = \{s_1, \dots, s_{k/2}\} \cup \{s_1^{-1}, \dots, s_{k/2}^{-1}\}$

and then the path corresponding to the word $x_1 x_2^{-1} x_3$ is

$$v_0 \longrightarrow s_1 v_0 \longrightarrow s_2^{-1} s_1 v_0 \longrightarrow s_3 s_2^{-1} s_1 v_0$$

Observation: such a path completely backtracking iff the word is actually the trivial element of $F_{k/2}$. $(x_1 x_2 x_2^{-1} x_3 x_3^{-1} x_1^{-1})$

(Hint: $\mathrm{Cay}(F_{k/2}, \{x_1^{\pm 1}, \dots, x_{k/2}^{\pm 1}\})$ is T_k).

Now look at our graph $(\Gamma = \mathrm{Sch}(S_n \curvearrowright [n], \sigma_1^{\pm 1}, \dots, \sigma_{k/2}^{\pm 1}))$

a path $\leftrightarrow$ starting vertex $i \in [n] = \{1, \dots, n\}$

& word $x_{m_\ell}^{\varepsilon_\ell} x_{m_{\ell-1}}^{\varepsilon_{\ell-1}} \cdots x_{m_1}^{\varepsilon_1}$ $\quad \left(\begin{matrix} m_j \in [k/2] \\ \varepsilon_j = \pm 1 \end{matrix} \right)$

This path is closed iff $\sigma_{m_\ell}^{\varepsilon_\ell} \cdots \sigma_{m_1}^{\varepsilon_1}(i) = i$:

$$\underset{i}{\bullet} \longrightarrow \underset{\sigma_{m_1}^{\varepsilon_1}(i)}{\bullet} \longrightarrow \underset{\sigma_{m_2}^{\varepsilon_2}\sigma_{m_1}^{\varepsilon_1}(i)}{\bullet} \longrightarrow \bullet \; \cdots \; \bullet \longrightarrow \underset{i?}{\bullet}$$

For $w = x_{m_\ell}^{\varepsilon_\ell} \cdots x_{m_1}^{\varepsilon_1}$ and $\sigma = (\sigma_1, \dots, \sigma_{k/2})$, we denote $w(\sigma) = \sigma_{m_\ell}^{\varepsilon_\ell} \cdots \sigma_{m_1}^{\varepsilon_1} \in S_n$.

<u>Pf</u>: For even t,

$$\mathrm{tr}(A^t) = \#\text{closed paths of length } t = \sum_{i=1}^n \sum_{w \in \Sigma^t} \delta_{w(\sigma)(i), i}$$

where $\Sigma = \{x_1, \dots, x_{k/2}, x_1^{-1}, \dots, x_{k/2}^{-1}\}$.

Also $\mathrm{tr}(A^t) = k^t + \sum_{i=2}^n \lambda_i^t \geq k^t + \lambda^t$, or

$$\lambda^t \leq \mathrm{tr}(A^t) - k^t = \sum_i \sum_w \delta_{w(\sigma)(i), i} - k^t.$$

$$\Rightarrow \mathbb{E}(\lambda^t) \leq \sum_{i=1}^n \sum_{w \in \Sigma^t} \mathbb{E}(\delta_{w(\sigma)(i), i}) - k^t$$

$$= \sum_{i=1}^{n} \sum_{w \in \Sigma^t} \mathbb{P}[w(\sigma)(i) = i] - k^t$$

$$= \left(n \cdot \sum_{w \in \Sigma^t} \mathbb{P}[w(\sigma)(1) = 1] \right) - k^t$$

$$= n \cdot \sum_{w \in \Sigma^t} \underbrace{\left(\mathbb{P}[w(\sigma)(1) = 1] - \frac{1}{n} \right)}$$

if $w(\sigma)$ is uniform in S_n (e.g.: $w = x_1$)
then this is zero!

26/5/20

Let's define $p_w = \mathbb{P}[w(\sigma)(1) = 1] - \frac{1}{n}$, so

$$\mathbb{E}(X^t) \leq n \sum_{w \in \Sigma^t} p_w .$$

If $w = \emptyset$ (w comletely cancels out), then in every instance of $\sigma_1, \dots, \sigma_{k/2}$, the path w is completely backtracking, so $p_\emptyset = 1 - \frac{1}{n}$.

If $w = x_1$ then $w(\sigma) = \sigma_1 \sim \mathrm{Uniform}(S_n)$ so $p_{x_1} = 0$.

(Thm (Puder-P): $p_w = 0$ $(\forall n)$ iff $\exists \varphi \in \mathrm{Aut}(F_{k/2})$ with $\varphi(w) = x_1$)

Exer.: for $w \in F_{k/2}$ and $\varphi \in \mathrm{Aut}(F_{k/2})$, $p_w = p_{\varphi(w)}$.

Examples: $p_{x_1^m} = \frac{\#\mathrm{div}(m) - 1}{n}$, $p_{x_1 x_2 x_1^{-1} x_2^{-1}} = \frac{1}{n^2 - n}$

For $w \in \Sigma^t$, denote by w' its reduced form.

$w = x_1 x_2 x_2^{-1}$ $w' = x_1$

Main Claim:

types of words after reduction	bound for p_w	bound for the number of such words
$\emptyset$ or conjugate to a power	1	$P(t)\,(2\sqrt{k})^t$
all others	$\frac{P(t)}{n^2}$	k^t

e.g.: $xy^2zy^2zx^{-1}$

$P(t)$ means some pol. in t (can be a different one each time)

The claim gives the proof:

$$\mathbb{E}(\lambda^t) \leq n\sum_w p_w \leq n P(t)(2\sqrt{k})^t + k^t \frac{P(t)}{n}$$

$$= P(t)\sqrt{k}^{\,t}\left(n\,2^t + \frac{\sqrt{k}^{\,t}}{n}\right)$$

We'll take the t that gives the best bound, which happens when $n2^t = \frac{\sqrt{k}^{\,t}}{n} \implies n = \left(\frac{k}{4}\right)^{t/4} \Big|\; t = 4\log_{k/4} n$

this t gives $\mathbb{E}(\lambda^t) \leq P(t)\left(\sqrt{2}\,k^{3/4}\right)^t$. Thus for any

$\varepsilon > 0$, $\mathbb{P}\left[\lambda > \sqrt{2}\,k^{3/4} + \varepsilon\right]\left(\sqrt{2}\,k^{3/4}+\varepsilon\right)^t \leq \mathbb{E}(\lambda^t) \leq P(t)\left(\sqrt{2}\,k^{3/4}\right)^t$

(Markov: for a non-negative RV X, $\mathbb{E}(X^t) \geq a^t\,\mathbb{P}(X \geq a)$)

$$\Rightarrow \mathbb{P}[\lambda > \_\_] \leq P(t)\left(\frac{\sqrt{2}\,k^{3/4}}{\sqrt{2}\,k^{3/4}+\varepsilon}\right)^t \xrightarrow[n\to\infty]{} 0. \quad \square$$

(up to the claim)

$n \to \infty$
$\Downarrow$
$t = 4\log_{k/4} n \to \infty$

Claim 1: The number of trivial/power words is bounded by $p(t)(2\sqrt{k})^t$.

Pf: Write $|w'| = \text{length}(w') = t - 2r$ (there were r cancellations in w).

Write: $w = *** ((()()))** ((())())**$

where $*$ – a letter which survives in w'

$()$ – cancelled letters.

(not unique: $xx^{-1}xx^{-1} = ()() = (()))$

Write $w' = uv^j u^{-1}$ (with no cancelations)

We assumed $j \geq 2$, so that:

$$t = |w| = |w'| + 2r = 2|u| + j|v| + 2r \geq 2(|u| + |v| + r)$$

There are at most $\binom{t}{r}$ choices for the locations of the "(", and then k^r choices for their content.

After this, the loc. and cont. of the ")" is determined. There are $\leq t^2$ choices for $|u|, |v|$ (which determine j), and $\leq k^{|u|}, k^{|v|}$ choices for their content. In total, we have the bound

$$\sum_{r=0}^{t/2} \binom{t}{r} k^r t^2 k^{|u|+|v|} \leq t^2 \sum_{r=0}^{t/2} \binom{t}{r} k^{t/2}$$

$$= t^2 k^{t/2} \sum_{r=0}^{t/2} \binom{t}{r} \leq t^2 (2\sqrt{k})^t \quad \square$$

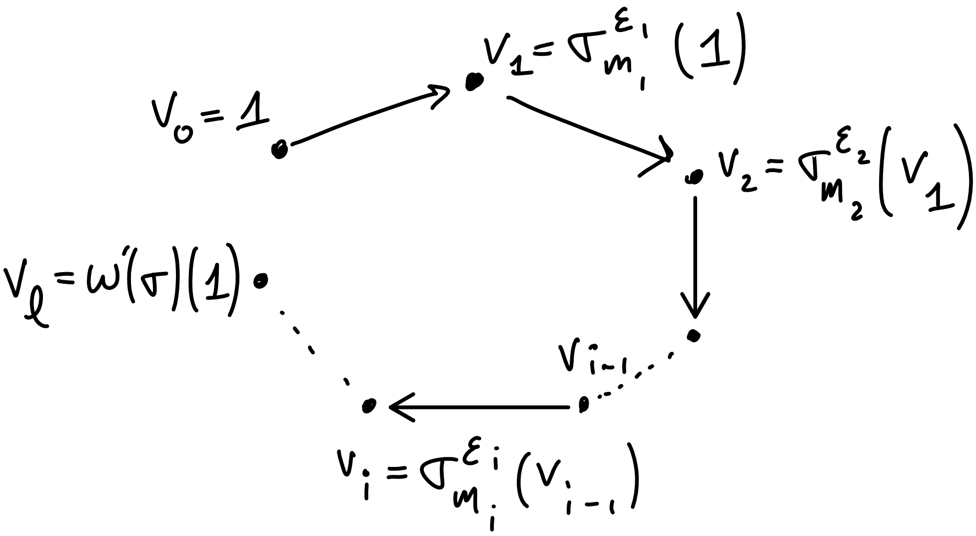
<u>Claim 2</u>: If w is not trivial nor a power then $p_w \le \frac{2t^4+2t}{n^2}$.

(e.g. $p_{xyx^{-1}y^{-1}} = \frac{1}{n^2-n} \le \frac{1024+8}{n^2}$, $p_{xy^2} = 0$)

<u>Pf</u>: Let w' denote the <u>cyclic</u> reduction of w $(xyx^{-1} \longmapsto y)$. Verify: $p_w = p_{w'}$.

so now we have w' which is cyclically reduced, non-power, of length $0 < \ell \le t$.

Write $w' = X_{m_\ell}^{\varepsilon_\ell} \cdot \ldots \cdot X_{m_1}^{\varepsilon_1}$, and observe the "journey" of 1:



$\text{Prob}(v_\ell = 1)$?

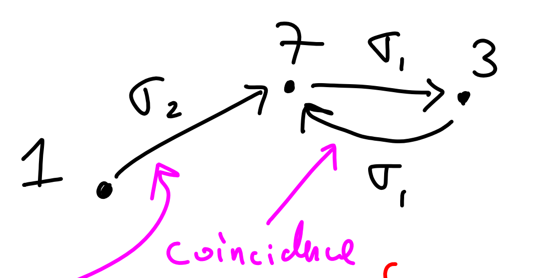
1/6/20

The idea is <u>not</u> to generate $\sigma_1, \ldots, \sigma_{k/2}$ at once, but only as much as we need: first, generate $\sigma_{m_1}^{\varepsilon_1}(1) \in \{1, \ldots, n\}$, then $\sigma_{m_2}^{\varepsilon_2}(\sigma_{m_1}^{\varepsilon_1}(1))$, unless it is already known, e.g.: $m_2 = m_1$, and $\varepsilon_2 = \varepsilon_1$ and $v_1 = 1$ ($w = x_1^2$, $\sigma_1(1) = 1$: $1 \circlearrowleft \sigma_1$)

We say that the i-th step is **forced** if the value of $\sigma_{m_i}^{\varepsilon_i}(v_{i-1})$ is already determined by the choices we made before, and it is **free** otherwise.

$w = \underline{\underline{x_2^{-1}}}\, x_1 x_1 x_2$

forced return



$\sigma_2(1) = 7$

$\sigma_1(7) = 3$

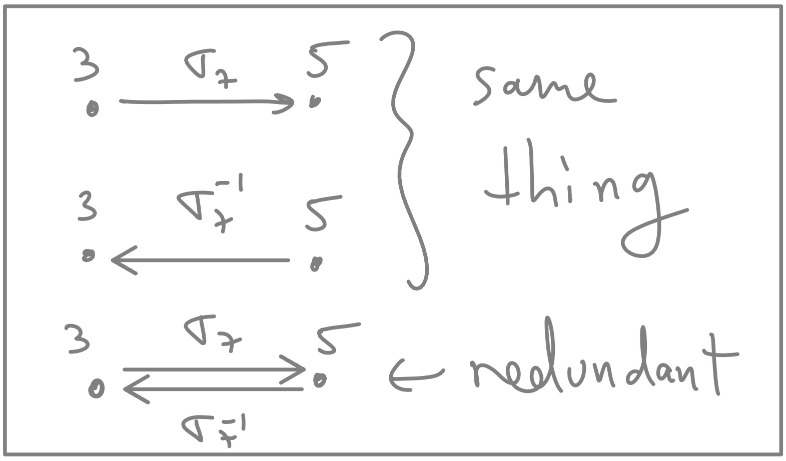
$\sigma_1(3) = 7$

forced: $\sigma_2^{-1}(7) = 1$

So, if we draw the edges of $G = Sch(S_n \curvearrowright \ldots)$ one by one, the i-th step is forced if when we get to $x_{m_i}^{\varepsilon_i}$, the vertex v_{i-1} already has an outgoing edge labeled $\sigma_{m_i}^{\varepsilon_i}$, <u>OR</u> it has an incoming edge labeled $\sigma_{m_i}^{-\varepsilon_i}$.

$\Leftrightarrow$ forced step is one in which we <u>don't</u> draw a new edge

free = one where we draw a new edge.



We say the i-th step is a return if $v_i \in \{v_0, \dots, v_{i-1}\}$.

$$\boxed{\text{A coincidence is a free return.}}$$

Observe: - $v_\ell = 1$ implies there are returns.

- the first return is a coincidence (when w is reduced! $w = x_1^{-1} x_1$ $\quad 1 \xrightarrow{\sigma_1} 7$ first return is forced)

$\Rightarrow$ $v_\ell = 1$ implies there are coincidences, so:

$$P_w + \frac{1}{n} = \mathbb{P}\left[v_\ell = 1\right] \le \mathbb{P}\left[v_\ell = 1, \begin{array}{c}\text{there is only}\\ \text{1 coincidence}\end{array}\right] + \underline{\underline{\mathbb{P}\left[\begin{array}{c}\text{at least 2}\\ \text{coincidences}\end{array}\right]}}.$$

$$\mathbb{P}\left[\begin{array}{c}i\text{-th step}\\ \text{is coin.}\end{array}\right] = \underbrace{\mathbb{P}\left[\begin{array}{c}i\text{-th step}\\ \text{is free}\end{array}\right]}_{\le 1} \underbrace{\mathbb{P}\left[\sigma_{m_i}^{\varepsilon_i}(v_{i-1}) \in \{v_0, \dots, v_{i-1}\} \,\middle|\, \begin{array}{c}i\text{-th}\\ \text{step}\\ \text{free}\end{array}\right]}_{\le \frac{|\{v_0, \dots, v_{i-1}\}|}{n - \begin{array}{c}\#\text{computed}\\ \text{values of } \sigma_{m_i}\\ \text{at time } i\end{array}}}$$

maybe values alredy computed of σ_{m_i} forbid me to land in some of $v_0, \dots, v_{i-1}$.

$\le \frac{i}{n-i} \le \frac{t}{n-t}$. This holds regardless of prev. coincidences, so

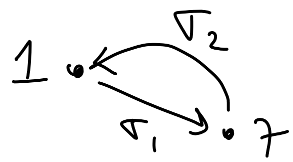
$$\mathbb{P}\left[\begin{array}{c}\text{at least}\\ \text{2 coinc.}\end{array}\right] \le \sum_{0 \le i < j \le t} \mathbb{P}\left[\begin{array}{c}\text{coin. at}\\ i \,\&\, j \text{ step}\end{array}\right] \le t^2 \cdot \left(\frac{t}{n-t}\right)^2 \underset{t \ll n}{\le} \frac{2t^4}{n^2}.$$

Bounding $\mathbb{P}\left[\begin{smallmatrix}\text{only } 1\\ \text{coincidence}\end{smallmatrix} \wedge v_\ell = 1\right]$: 2/6/20

This implies the coincidence was at the last step!

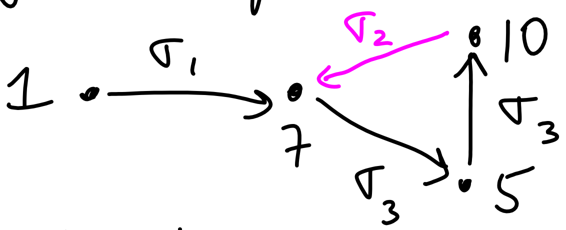
- False for powers: $w = (X_2 X_1)^3$

  can have 1 coinc. earlier:



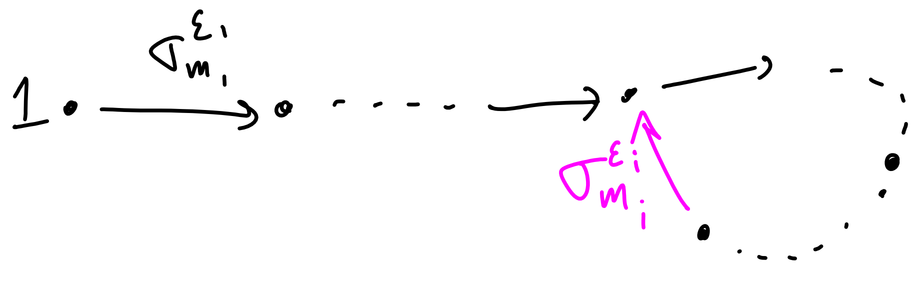
- and for non-cyclically reduced:

$w = X_1^{-1} X_2 X_3^2 X_1$



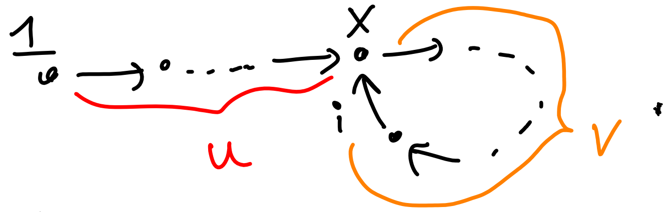
coinc. at $\ell-1$ step

If w is cyc. reduced, non-power, and there is one coin., at time $i < \ell$, then



all next steps are either forced, or go to new vertices. If even one of them goes to a new vertex we can't have $v_\ell = 1$ (without more coincidences)!

Thus, they are all forced, so

we travel along 

Also, we cannot change direction (w is reduced). We can only decide each time we arrive at x whether to turn right or left. After we took left we get to 1 and that's it (has to be step ℓ). If we turned right j times, this means that $w = u^{-1} v^{j} u$.

But we assumed w is non power $\Rightarrow$ $j = 1$

cyc. reduced $\Rightarrow$ $u = \emptyset$.

Namely, the single coinc. has to be at v_ℓ, and in particular $v_0, \dots, v_{\ell-1}$ are different.

$$\mathbb{P}\left[\begin{matrix} v_\ell = 1 \\ \text{one coin.} \end{matrix} \;\middle|\; \begin{matrix} w \text{ non} \\ \text{power} \end{matrix}\right] \le \mathbb{P}\left[\sigma_{m_\ell}^{\varepsilon_\ell}(v_{\ell-1}) = 1 \;\middle|\; \dots\right]$$

$$= \frac{1}{n - \substack{\#\text{generated} \\ \text{vals. of } \sigma_{m_\ell}}} \le \frac{1}{n-\ell} \le \frac{1}{n-t} = \frac{1}{n} + \frac{t}{n(n-t)} \underset{t \ll n}{\le} \frac{1}{n} + \frac{2t}{n^2}$$

all in all, we got w non-power $\Rightarrow$ $p_w \le \dfrac{2t^2 + 2t}{n^2}$.

$\boxed{\text{Broder Shamir}}$

Brodr-Shamir: $\sqrt{2}k^{3/4}+\varepsilon$

Puder '15: by these ideas, get to $2\sqrt{k-1}+0.84$

Friedman: $2\sqrt{k-1}+\varepsilon$

Separating $F_{k/2}$ to more types:

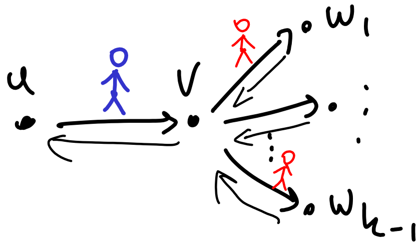
	trivial	powers	3rd	4th	...	$\left(\frac{k}{2}+1\right)^{th}$
e.g.	$\emptyset$	$(x_1x_2)^m$	$x_1x_2x_1^{-1}x_2^{-1}$			x_1
p_w	≈ 1	$\approx \frac{m-1}{n}$	$O\left(\frac{1}{n^2}\right)$	$O\left(\frac{1}{n^3}\right)$	...	0

However, $\mathbb{E}(\lambda^t)$ is sensitie to rare events (e.g. $\lambda=k$ - disconnected), causing Alon's conjecture not to be provable like this (words in $F_{k/2}$, $\mathbb{E}(\lambda^t)$).

NonBackTracking Walks

Random process on $E^{\pm}$ = all directed edges of the (undirected) graph G.

Moving from $u \longrightarrow v$ to a random edge $v \longrightarrow w$ with $w \neq u$.



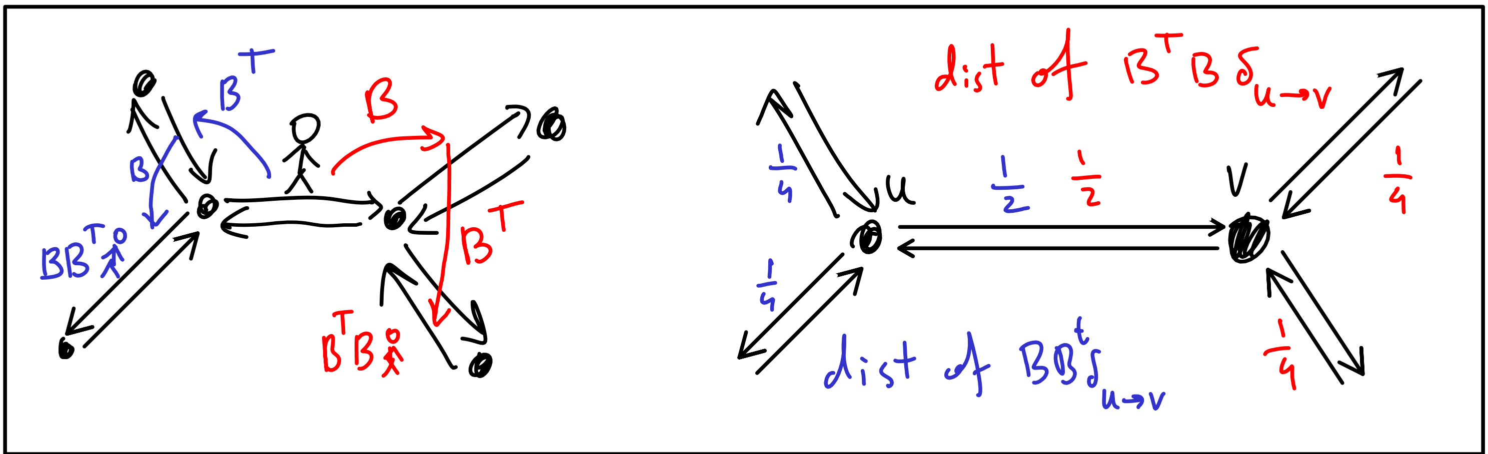
Denote $B:\mathbb{C}^{E^{\pm}} \longrightarrow \mathbb{C}^{E^{\pm}}$

$$(Bf)(u\to v) = \sum_{u\neq w\sim v} f(v\to w)$$

(so $\frac{1}{k-1}B^T$ evolves the dist. of NBRW).

Assume from <u>now</u> that G is <u>$(k+1)$-regular</u> and connected. $B\mathbb{1}=k\mathbb{1}$

B is not self-adjoint $(B\neq B^T)$ and not even normal: $BB^T\neq B^TB$:



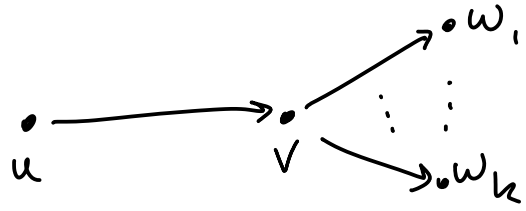
$\Rightarrow$ B is <u>not</u> unitarily diagonalizable.

(no ONB of B-efuncs).

Q: is B diagonalizable?

Let's try to find efuncs for B from efuncs for A. Given $f: V\to\mathbb{R}$ with $Af=\lambda f$, we look for $F: E^{\pm}\to\mathbb{C}$ with $BF=\theta F$ for some $\theta\in\mathbb{C}$, by

$$F(u\to v) = f(u) + \alpha f(v)$$



$$\theta f(u) + \theta\alpha f(v) = \theta F(uv) \overset{?}{=} BF(uv) = \sum_{u\neq w\sim v} F(vw)$$

$$= \sum_{u\neq w\sim v}\Big(f(v) + \alpha f(w)\Big) = k f(v) + \Big(\alpha \sum_{w\sim v} f(w)\Big) - \alpha f(u)$$

$$= k f(v) + \alpha\lambda f(v) - \alpha f(u) = (k+\alpha\lambda) f(v) - \alpha f(u).$$

so we want $\begin{cases}\alpha = -\theta \\ \theta\alpha = k + \alpha\lambda\end{cases} \Longrightarrow -\theta^2 = k - \theta\lambda$

$$\Rightarrow \theta^2 - \theta\lambda + k = 0$$

So $\boxed{\theta^{\pm} = \dfrac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2}}$ (and $\alpha^{\pm} = -\theta^{\pm}$)

We got that $\boxed{f^{\pm}(u\to v) = f(u) - \theta^{\pm} f(v)}$

$(Af = \lambda f)$ satisfies $\boxed{Bf^{\pm} = \theta^{\pm} f^{\pm}}$.

ex.: $\lambda = k+1$, $f = \mathbb{1}$

$$\theta^+ = \frac{k+1+(k-1)}{2} = k \qquad f^+(uv) = f(u) - k f(v) = (1-k)$$

$$f^+ = (1-k)\mathbb{1} \sim \mathbb{1}$$

$$\theta^- = \frac{k+1-(k-1)}{2} = 1 \qquad f^-(uv) = f(u) - f(v) = 0$$

$f^- \equiv 0$ not e.vector!

Reminder: we defined $(Bf)(u\to v) = \sum_{\substack{w\sim v \\ w\neq u}} f(v\to w)$

For $f: V\to\mathbb{R}$ with $Af=\lambda f$ we took

$f_\alpha(u\to v) = f(u)+\alpha f(v)$ and asked if

for some $\alpha,\theta\in\mathbb{C}$, $Bf_\alpha = \theta f_\alpha$. We

got $(B-\theta I)f_\alpha(uv) = -(\alpha+\theta)f(u)+(k+\alpha\lambda-\alpha\theta)f(v)$

so we took $\alpha=-\theta$, and then

$$(B-\theta)f_{-\theta}(uv) = (\theta^2-\lambda\theta+k)f(v)$$

so we took $\theta^{\pm} = \dfrac{\lambda\pm\sqrt{\lambda^2-4k}}{2}$, and we

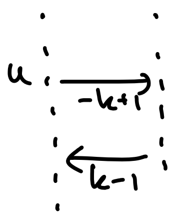
call $f_{\theta^+}=f^+$, $f_{\theta^-}=f^-$, so $Bf^{\pm}=\theta^{\pm}f^{\pm}$.

E.g.: $\underbrace{\lambda=-k-1}$ $\longrightarrow$ $\theta^+=-1$

$f=\mathbb{1}_L-\mathbb{1}_R \longrightarrow f^+(uv) = f(u)-\theta^+f(v) = 0$

And $\theta^-=-k$, $f^-(uv)=f(u)+kf(v)$

so $f^- = C\cdot(\mathbb{1}_{L\to R}-\mathbb{1}_{R\to L})$



- $\lambda=0 \longrightarrow \theta^{\pm}=\pm i\sqrt{k}$
- $\lambda=2\sqrt{k}$ (Worst Ramanujan e.val) $\longrightarrow \theta^+=\theta^-=\sqrt{k}$.

  so $f^+=f^-$.

$\to$ For $\lambda=\pm(k+1), \pm 2\sqrt{k}$ we get only one e.func of $\mathcal{B}$.

Claim: for $\lambda \neq \pm(k+1), \pm 2\sqrt{k}$, f^+, f^- are lin. independent.

Pf: dfn $f^h(uv) = f(u)$, $f^t(uv) = f(v)$, so

$$\begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} 1 & -\theta^+ \\ 1 & -\theta^- \end{pmatrix} \begin{pmatrix} f^h \\ f^t \end{pmatrix}. \quad \det(\ ) = \theta^+ - \theta^- = \sqrt{\lambda^2 - 4k} \neq 0$$

for $\lambda \neq \pm 2\sqrt{k}$

so if f^+, f^- are lin. dep., so are f^h, f^t.
Thus $f(u) = c \cdot f(v)$ for all $u \sim v$, which implies $f = \mathbb{1}$ or $\mathbb{1}_R - \mathbb{1}_L$ $\left(f(u) = cf(v) = c^2 f(u) \right)$ $\square$
$\Rightarrow c = \pm 1$

Observe: $\theta^+ \cdot \theta^- \overset{*}{=} k$, $\theta^+ + \theta^- = \lambda$ (Vieta)

$$\boxed{\begin{aligned} &\textbf{Claim: } \lambda \in [-2\sqrt{k}, 2\sqrt{k}] \text{ (Ramanujan)} \\ &\iff |\theta^\pm| = \sqrt{k} \overset{*}{\iff} |\theta^\pm| \leq \sqrt{k}. \end{aligned}}$$

$\frac{1}{2}$ **Pf:** if $\lambda \in [\dots]$ then $\sqrt{\lambda^2 - 4k} < 0$ so

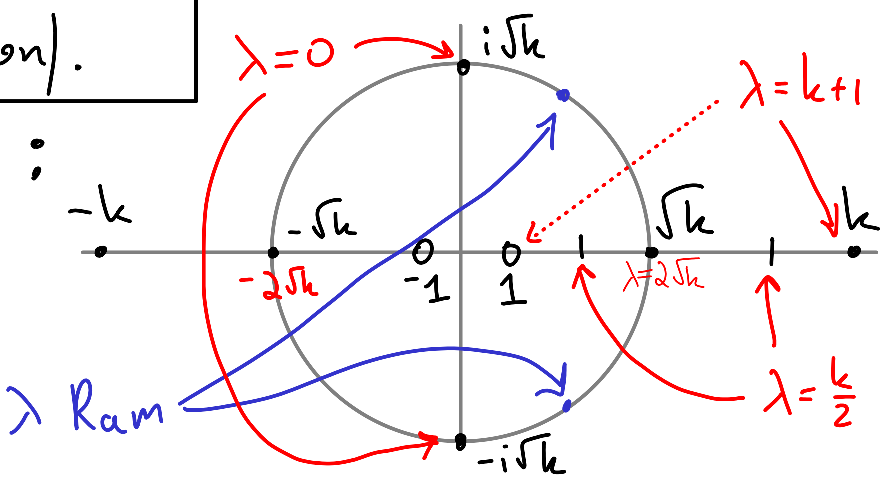
$$\left| \frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2} \right|^2 = \left| \frac{\lambda \pm i\sqrt{4k - \lambda^2}}{2} \right|^2 \underset{\text{Pyt}}{=} \frac{\lambda^2 + (4k - \lambda^2)}{4} = k$$

(finish other direction).

λ to θ schematics:

Red - λ

Black - θ



9/6/20

Proving that the e.vecs we got as $f^{\pm}$ are indep:

Let $Af = \lambda f \longrightarrow Bf^{\pm} = \theta^{\pm} f^{\pm}$

$Ag = \mu g \longrightarrow Bg^{\pm} = \eta^{\pm} g^{\pm}$

$$\langle f^{\pm}, g^{\pm} \rangle = \sum_{u \sim v}^{(2E)} f^{\pm}(uv) \overline{g^{\pm}(uv)}$$

$$= \sum_{u \sim v} \left(f(u) - \theta^{\pm} f(v)\right) \left(g(u) - \overline{\eta^{\pm}} g(v)\right) = C \cdot \langle f, g \rangle$$

$$\boxed{\begin{aligned} &\sum_{u \sim v} f(u) g(u) = \sum_{u} f(u) g(u) \sum_{v \sim u} 1 = (k+1) \langle f, g \rangle \\ &\sum_{u \sim v} \theta^{\pm} \overline{\eta^{\pm}} f(v) g(v) = (k+1) \theta^{\pm} \overline{\eta^{\pm}} \langle f, g \rangle \\ &\sum_{u \sim v} f(u) \overline{\eta^{\pm}} g(v) = \overline{\eta^{\pm}} \langle f, Ag \rangle = \mu \overline{\eta^{\pm}} \langle f, g \rangle \ \ldots \end{aligned}}$$

so if $f \perp g$ then $f^{\pm} \perp g^{\pm}$. Thus, choosing an ONB of evecs for A, we get pairs of e.vecs of B which are orth. between different pairs. In particular $\rightarrow$ independent.

B acts on a $2|E| = (k+1)n$ dim. space, we found $\sim 2n$ e. funcs.

Now, focus on $\lambda = 2\sqrt{k}$ ($\lambda = -2\sqrt{k}$ is similar)

$\theta^+ = \theta^- = \sqrt{k}$ so $f^+ = f^- = uv \mapsto f(u) - \sqrt{k}\, f(v)$.

Recall $f_\alpha(uv) = f(u) + \alpha f(v)$ gave

$$(B-\theta) f_\alpha(uv) = -(\alpha+\theta) f(u) + (k + \alpha\lambda - \alpha\theta) f(v)$$

$$\overset{\substack{\lambda = 2\sqrt{k} \\ \theta = \sqrt{k}}}{\Longrightarrow} (B - \sqrt{k}) f_\alpha(uv) = -(\alpha + \sqrt{k}) f(u) + (k + \alpha\sqrt{k}) f(v)$$

$$= -(\alpha + \sqrt{k})\big(f(u) - \sqrt{k}\, f(v)\big) = -(\alpha + \sqrt{k}) f^+(uv)$$

so $(B - \sqrt{k}) f_\alpha = -(\alpha + \sqrt{k}) f^+$

$\Rightarrow (B - \sqrt{k})^2 f_\alpha = 0$ (for any α)

Take $\alpha = -1-\sqrt{k}$: $\boxed{f' = f_{-1-\sqrt{k}}}$. We get

$Bf' = \sqrt{k} f' + f^+ \Rightarrow [B|_{\langle f^+, f' \rangle}] = \begin{pmatrix} \sqrt{k} & 1 \\ 0 & \sqrt{k} \end{pmatrix}$

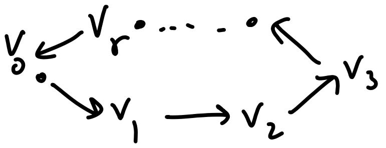
This is a non diagonal Jordan Block, so

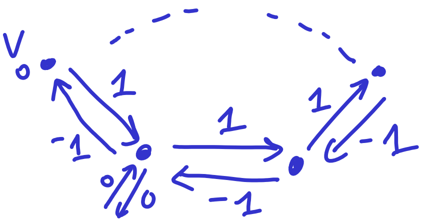
in particular $\pm 2\sqrt{k} \in \mathrm{Spec}\, A \Rightarrow B$ is not diagonalizable

But, in algebraic multiplicities, we

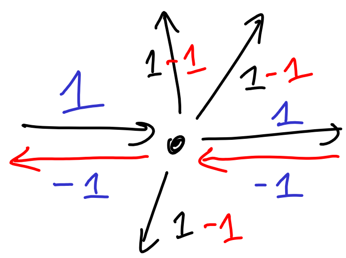
found $\begin{matrix} 2n-1 \\ 2n-2 \end{matrix}$ e.vals $\begin{matrix} G \text{ is not bip.} \\ G \text{ is bip} \end{matrix}$.

Where do the other $2|E|-2n+1+\delta_{bip}$ evals come from?

Let γ:  be a cycle in G.

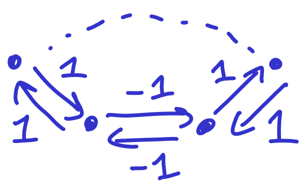
take p_γ:  .

$\boxed{Bp_\gamma = p_\gamma:}$



so from every cycle in G we can create an e.func with eval 1.

Choose a spanning tree T. For every $e \in E \setminus T$ $\exists$ unique cycle γ_e in $T \cup \{e\}$. The functions $\{p_{\gamma_e} \mid e \in E \setminus T\}$ are indep., so we got $|E| - |E_T| = |E| - n + 1$ e.vals $\left(p_{\gamma_{vu}} = -p_{\gamma_{uv}}\right)$

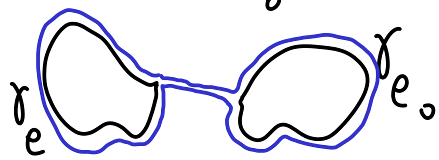
If $\text{len}(\gamma)$ is even, define n_γ: 

Now $\boxed{Bn_\gamma = -n_\gamma}$.

If G is bipartite, all cycles are even, so $\{n_{\gamma_e} \mid e \in E \setminus T\}$ give $|E|-n+1$ times eval -1.

So we found all of Spec B!

If G is not bipartite, there is at least one odd cycle, γ_{e_0}. Then, for every $e_0 \neq e \in E \setminus T$ either γ_e is even, or $\gamma_e + \gamma_{e_0}$ is even ($\gamma_e + \gamma_{e_0}$ means concatenating them)



(connect them arbitrarily in T)

and

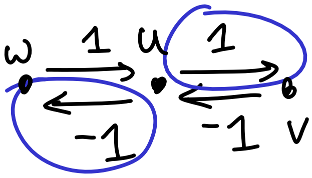
$$\left\{ n_{\gamma_e} \;\middle|\; \begin{matrix} e \in E\setminus T \\ \gamma_e \text{ even} \end{matrix} \right\} \cup \left\{ n_{\gamma_e + \gamma_{e_0}} \;\middle|\; \begin{matrix} e_0 \neq e \in E\setminus T \\ \gamma_e \text{ odd} \end{matrix} \right\}$$

gives us $|E| - n$ indep. -1 e.vals., completing again $\mathrm{Spec}(\mathcal{B})$ to $2|E|$.

We know $f^{\pm} \notin \langle p_\gamma, n_\gamma \rangle$ (different evals), but actually they are orthogonal ($f^{\pm}, f' \perp p_\gamma, n_\gamma$) since for any $f: V \to \mathbb{R}$, $f^h, f^t \perp p_\gamma, n_\gamma$:

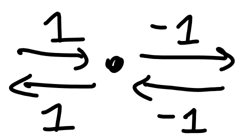
$$\langle f^h, p_\gamma \rangle = \sum_{u \sim v} f^h(uv)\, p_\gamma(uv) = \sum_{u\sim v} f(u)\, p_\gamma(uv) = 0$$

$f(u)p_\gamma(uv) + f(u)p_\gamma(uw) = 0$:



at every $u \in V_\gamma$ p_γ has outgoing $1, -1$

similarly for n_γ:



.

Thus, $\boxed{f^{\pm}, f' \in \langle f^h, f^t \rangle \perp \left\langle p_\gamma, n_\gamma \;\middle|\; \begin{matrix}\text{all} \\ \text{cycles } \gamma\end{matrix} \right\rangle}$

We know $B^TB \neq BB^T \Rightarrow B$ is not unitarily diag.ble
It turns out that the only obstruction is that
f', f^+, f^- for the same $f: V \to \mathbb{R}$ are not orthogonal
To see this it is only left to show that the e.vecs coming from p_γ, n_γ are orth.
But $p_\gamma \not\perp p_{\gamma'}$ when γ, γ' overlap!

However, for any γ, $B^T p_\gamma = p_\gamma$ $\quad \underset{-1}{\overset{1}{\rightleftarrows}} \cdot \underset{-1}{\overset{1}{\rightleftarrows}} \cdot \underset{-1}{\overset{1}{\rightleftarrows}}$
and $B^T n_\gamma = -n_\gamma$. Thus, $B|_{\langle p_\gamma, n_\gamma \mid \gamma \rangle}$ is self-adj.
(as $B|_{\langle p_\gamma, n_\gamma \rangle} = B^T|_{\langle p_\gamma, n_\gamma \rangle}$). So, it <u>has</u> an ONB of eigenvectors.

In total

$$B \underset{\text{equiv.}}{\overset{\text{unitarily}}{\sim}} \begin{pmatrix} k \\ & \boxed{B_{\lambda_2}} \\ & & \boxed{B_{\lambda_3}} \\ & & & \ddots \\ & & & & \boxed{B_{\lambda_s}} \\ & & & & & (-k) \\ & & & & & & \underbrace{1 \ \ddots \ 1}_{\varepsilon - n + 1} \\ & & & & & & & \underbrace{-1 \ \ddots \ -1}_{\varepsilon - n + \delta_{bip}} \end{pmatrix}$$

and (Ihara-Bass-Hashimoto):

$$\boxed{\mathrm{Spec}(B) = \left\{ \frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2} \;\middle|\; \lambda \in \mathrm{Spec}\, A \right\} \cup \{\underbrace{\pm 1, \ldots, \pm 1}_{E-n}\}}$$

(with alg. multip.)

Check!

15/6/20

__Corollary__: G is Ramanujan iff

$$\mathrm{Spec}\, B_G \subseteq \{ z \in \mathbb{C} \mid |z| \leq \sqrt{k} \} \cup \{\pm k\}$$

(Ramanujan: $A_G \subseteq \{ z \in \mathbb{R} \mid |z| \leq 2\sqrt{k} \} \cup \{\pm(k+1)\}$)

__Spectral analysis of non-normal ops.__

Recall: if T is normal,

$$\|T\| := \max_{\|v\|=1} \|Tv\| = \max\{|\lambda| \mid \lambda \in \mathrm{Spec}\, T\}$$

(we used this for $T = A_G|_{Z(G) = \mathbb{1}^\perp}$)

__Def__: for an op T on an inn. prod. space, its singular values are $\mathrm{Sing}\, T = \{\sqrt{\lambda} \mid \lambda \in \mathrm{Spec}(T^*T)\}$

(Denoting $\mathrm{Spec}\, T^*T = \lambda_1, \ldots, \lambda_n$, $\sigma_i := \sqrt{\lambda_i}$)

Note: T^*T is always self-adjoint with non-neg. spectrum (we showed this when we looked at $\Delta = \delta^* \delta$)

__Claim__: $\|T\| = \max(\mathrm{Sing}(T))$

Pf: for any $\|v\| = 1$, $\|Tv\|^2 = \langle Tv, Tv \rangle$

$$= \langle T^* T v, v \rangle \underset{\text{Rayl.}}{\leq} \max\{\lambda \mid \lambda \in \operatorname{Spec} T^* T\} = \max_{\sigma \text{ sing.}} \{\sigma^2\}$$

$\Rightarrow \|Tv\| \leq \max\{\sigma\}$.

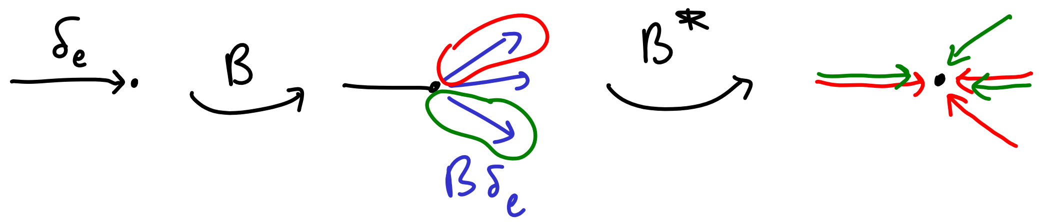
OTOH, if $\lambda \in \operatorname{Spec} T^*T$, and $T^*Tv = \lambda v$ with $\|v\| = 1$, then

$$\|Tv\|^2 = \langle Tv, Tv \rangle = \langle T^* T v, v \rangle = \lambda \Rightarrow \|Tv\| \geq \sqrt{\lambda} \quad \square$$

Sad fact: $\operatorname{Sing}(B_G) = \{k, k, k, k, \dots, k, \text{ and smaller ones}\}$

$B^* B \mathbb{1} = k^2 \mathbb{1}$

Why?



so $\{e \mid \operatorname{tail}(e) = v\}$ is a disconnected component of the graph whose adj op. is B^*B (which is k^2-regular).

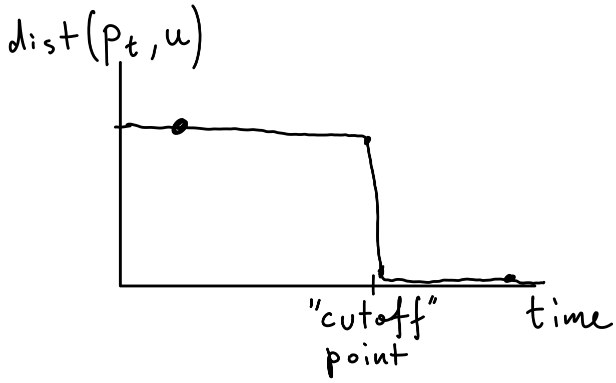
In other words, $B^* B \mathbb{1}_{\{e \mid \operatorname{tail}(e) = v\}} = k^2 \mathbb{1}_{\{e \mid \operatorname{tail}(e) = v\}}$.

$k + k(k-1)$

So we get $\sqrt{k^2} = k$ n times, and in particular $\max \operatorname{Sing}(B|_{\mathbb{1}^\perp}) = \max \operatorname{Sing}(B) = k$.

Diaconis/Aldous cutoff phenomenon

For some natural Markov chains, the distance of the dist. from the stationary one looks like this:



Riffle-Shuffle: Verts = 52! ordering of deck of cards.
"7 shuffles are enough"

2004': Peres conjectured that all transitive expanders have this property.

At the time, no family of bounded degree expanders was known to have this property!

2016': Ramanujan graphs have it (Lubetzky-Peres)

16+22/6/20

The ε-mixing time of a reg. graph G (w.r.t. some dist. on $\mathbb{R}^V$) is $t_\varepsilon^G = \min\{t \mid \text{dist}(p_t^{v_0}, u) < \varepsilon \ \forall v_0 \in V\}$.

We kept looking at $\text{dist}(f,g) = \|f-g\|_2$. In prob. theory, the most useful dist is total-variation dist:

if p, p' prob. measures on X $(p: X \to [0,1], \ \sum p = 1)$

$$\text{dist}_{TV}(p,p') = \max_{A \subseteq X} |p(A) - p'(A)| \underset{\text{exer!}}{=} \frac{1}{2}\|p-p'\|_1 .$$

A family of graphs $\{G_n\}$ has the cutoff phenom.

if $\forall \varepsilon > 0$, $\displaystyle\lim_{n\to\infty} \frac{t_\varepsilon^{G_n}}{t_{1-\varepsilon}^{G_n}} \longrightarrow 1$ (w.r.t. dist_{TV}).

So from now on we focus on $\text{dist}(f,g) = \frac{1}{2}\|f-g\|_1$

Motivating example (why L_1-cutoff occurs)

Say we have a family of k-reg. graphs $\{G_n\}$, with $|V(G_n)| = n$, and $\lambda(G_n) \leq \sqrt{k}$ $\forall n$ (which is impossible by Alon-Boppana). Then we have cutoff: after fixing $\varepsilon > 0$,

$$\| p_t^{v_0} - u \|_1 \leq \sqrt{n} \| p_t^{v_0} - u \|_2 \leq \sqrt{n} \left(\frac{\lambda(G_n)}{k} \right)^t \leq \sqrt{\frac{n}{k^t}} \leq 2\varepsilon.$$

if we take $t = \log_k n + \log_k \frac{1}{4\varepsilon^2}$

OTOH, $\| p_t^{v_0} - u \|_1 = \sum_v | p_t^{v_0}(v) - \frac{1}{n} |$. We split this:

$$\left.\begin{aligned} \sum_{v \notin S^t(v_0)} | p_t^{v_0}(v) - \tfrac{1}{n} | &= (n - |S^t(v_0)|) \tfrac{1}{n} = 1 - \frac{|S^t(v_0)|}{n} \geq 1 - \frac{k^t}{n} \\ \sum_{v \in S^t(v_0)} | p_t^{v_0}(v) - \tfrac{1}{n} | &= \sum_{v \in S^t(v_0)} p_t^{v_0}(v) - \tfrac{1}{n} = 1 - \frac{|S^t(v_0)|}{n} \geq 1 - \frac{k^t}{n} \end{aligned}\right\} \geq 1 - \varepsilon$$

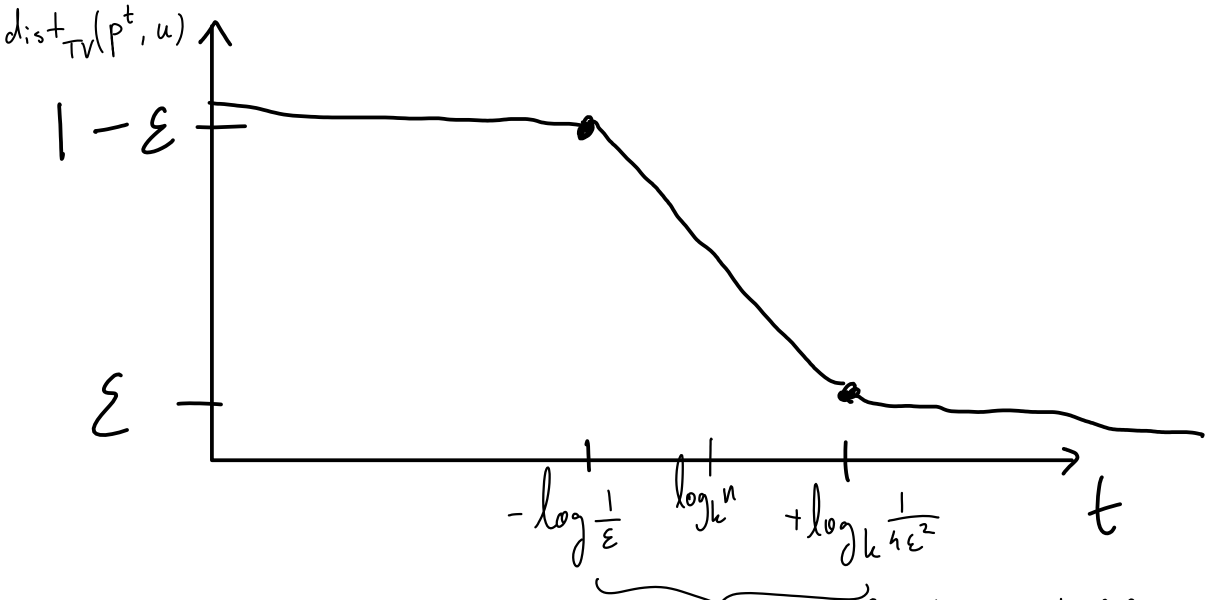
$$p_t^{v_0}(v) = \frac{\#\{t\text{-paths } v_0 \rightsquigarrow v\}}{k^t} \geq \frac{1}{k^t} > \frac{1}{n}$$

$$\boxed{\text{taking } t = \log_k n - \log_k \tfrac{1}{\varepsilon}}$$

Thus, $\| p^t - u \|_{TV} = \frac{1}{2} \| p^t - u \|_1 \geq 1 - \varepsilon$.

Remark: in the lower bound we didn't look at the spectrum at all, so it holds for all k-reg. graphs, and also for digraphs!

in total:



the size of the "cutoff window" doesn't grow with n!

In particular:

$$1 \leq \frac{t_{\varepsilon}^{G_n}}{t_{1-\varepsilon}^{G_n}} \leq \frac{\log_k n + \log_k \frac{1}{4\varepsilon^2}}{\log_k n - \log \frac{1}{\varepsilon}} \xrightarrow{n\to\infty} 1$$

Furthermore, the cutoff happened around $t = \log_k n$, which is optimal for k-reg. graphs. So, we say we have optimal cutoff.

<u>Conj (Peres, '04)</u>: every family of transitive expanders exhibits cutoff.

<u>Thm (Lubetzky-Peres '16)</u>: Ram. graphs have cutoff.

<u>Note</u>: if G_n is a fam. of $(k+1)$-reg. Ram graphs, then $D_n = \{E^{\rightleftarrows}(G_n), \{uv \to vw \mid w \neq u\}\}$ is a family of (in other words $A_{D_n} = B_{G_n}$) directed k-reg. graphs with $\lambda(D_n) \leq \sqrt{k}$, like we assumed in the Motivating Example.

However, for a non-normal graph D we only have $\|p_t^{v_0} - u\|_2 \le \left(\frac{\sigma(D)}{k}\right)^t$ where

$\sigma(D) = \max \mathrm{Sing}(A_D|_{\mathbb{1}^\perp})$, and we saw that $\sigma(A_{D_n}) = \sigma(B_{G_n}) = k$ for any $(k+1)$ reg G...

Deus ex machina: we saw $\mathbb{C}^{E^\pm}$ decomposes into 1-dim & 2-dim orthogonal subspaces which are stable under B_G (G is $(k+1)$-reg Ram. graph) (if it was only 1-dim $\Leftrightarrow B_G$ is normal).

Lets look at one of these 2-dim subspaces:

For $\lambda \in \mathrm{Spec}\, A$, take $W_\lambda = \langle f^+, f^- \rangle$ (or $\langle f^+, f' \rangle$ in the special cases $\lambda = \pm 2\sqrt{k}$), and observe

$$B_\lambda = [B|_{W_\lambda}]_{\{f^+, f^-\} \text{ or } f'}$$

Assume from now G is not bipartite, so

$B = \mathrm{diag}(k, B_{\lambda_2}, B_{\lambda_3}, \dots, 1, \dots, 1, -1, \dots, -1)$, hence

$B^t = \mathrm{diag}(k^t, B_{\lambda_2}^t, B_{\lambda_3}^t, \dots, 1, \dots, 1, -1^t, \dots, -1^t)$

and in particular $\|B^t|_{\mathbb{1}^\perp}\|_2 \le \max_{2 \le j < n} \|B_{\lambda_j}^t\|_2$ (Pythagoras).

Gram-Schmidt: for any $A \in GL_n(\mathbb{C})$, there is a (unique) decomp. $A = QR$ with R upp-triang, Q unitary (and $r_{ii} > 0 \ \forall i$). (exer: figure this out)

Schur Decomposition: every $A \in M_n(\mathbb{C})$ decomposes into $A = QRQ^* = QRQ^{-1}$ with R upp-triang. and Q unitary.

In other words, any oper. T on fin. dim. inner-prod. space V has an O.N.B of V with respect to which it is upper triangular (with Spec T on the diagonal, since $\det(A - xI) = \prod(A_{ii} - x)$).

Pf: $[T]_{(v_1,\dots,v_n)}$ is triang. iff $Tv_j \in \langle v_1,\dots,v_j \rangle \ \forall j$ (and $v_j \notin \langle v_1,\dots,v_{j-1}\rangle \to$ basis!). Find $Tv_1 \in \langle v_1 \rangle$ (and $v_1 \neq 0$) (F.T.o.L.A). Now, observe that T is well defined on $V/\langle v_1\rangle$: $T(v + \langle v_1 \rangle) = T\bar{v} = \overline{Tv} = (Tv + \langle v_1\rangle)$, since $T(v + \alpha v_1 + \langle v_1 \rangle) = Tv + \alpha T v_1 + \langle v_1 \rangle = Tv + \langle v_1 \rangle$.

Now, find e.vec of $T \circlearrowright V/\langle v_1 \rangle$: $T\bar{v}_2 = \lambda \bar{v}_2$ $\left(\bar{v}_2 \neq 0 \Leftrightarrow v_2 \notin \langle v_1 \rangle\right)$

This means that $Tv_2 \in \lambda v_2 + \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle$. Now T is well def on $V/\langle v_1, v_2\rangle$, find $T\bar{v}_3 = \lambda \bar{v}_3$, ...

We found basis $v_1, \dots, v_n$ in which $[T]$ is triang. But it is still not an ONB. However,

performing Gram-Schmidt: $\begin{cases} v_1 \leftarrow v_1/\|v_1\| \\ v_2 \leftarrow v_2 - \langle v_2, v_1\rangle v_1 \\ v_2 \leftarrow v_2/\|v_2\| \\ v_3 \leftarrow v_3 - \langle v_3, v_1\rangle v_1 - \langle v_3, v_2\rangle v_2 \\ \vdots \end{cases}$
does not change the property that $Tv_j \in \langle v_1, \dots, v_j\rangle$.

$\square$

Thus, we have $[B_\lambda] = [B|_{\langle f^+, f^-\rangle}] = \begin{pmatrix} \theta^+ & b' \\ 0 & \theta^- \end{pmatrix}$
for some $b' \in \mathbb{C}$, and $|\theta^+| = |\theta^-| = \sqrt{k}$ (for Ramanujan!)
with respect to some O.N.B. of $\langle f^+, f^-\rangle$.

This is crucial: $\left\| \begin{pmatrix} 1 & 1000 \\ 0 & 2 \end{pmatrix} \right\|_2 \sim 1000$

equivalent matrices $\Big\downarrow$ (same operator!)

$\left\| \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\|_2 = 2.$

$\|A\|_2 \leq \left\| \left(|a_{ij}| \right)_{ij} \right\|_2$ in general (triangle ineq.),
so we can look at $|B_\lambda| = \begin{pmatrix} \sqrt{k} & b \\ 0 & \sqrt{k} \end{pmatrix}$. Naively,
we have $\|B_\lambda^t\|_2 \leq \|B_\lambda\|_2^t = k^t$ which doesn't give us anything.

$\uparrow$ max $\mathrm{Sing}(B_\lambda) = k$, since we know k has multiplicity n in $\mathrm{Sing}(B_G)$

However, since we made B_λ triangular we can actually compute its powers:

$$\|B_\lambda^t\|_2 \leq \| |B_\lambda|^t \|_2 = \left\| \sqrt{k}^t \begin{pmatrix} 1 & \frac{b}{\sqrt{k}} \\ & 1 \end{pmatrix}^t \right\|_2 = \sqrt{k}^t \left\| \begin{pmatrix} 1 & \frac{b}{\sqrt{k}} \\ & 1 \end{pmatrix}^t \right\|_2$$

$$= \sqrt{k}^t \left\| \begin{pmatrix} 1 & \frac{tb}{\sqrt{k}} \\ & 1 \end{pmatrix} \right\|_2 \leq \sqrt{k}^t \left(1 + \frac{tb}{\sqrt{k}}\right) \leq 2tb\sqrt{k}^{t-1}$$

$\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}$

and $b \leq k$ since $\|B_\lambda\|_2 \leq \|B\|_2 \leq k$ (k-reg digraph), so any entry of $[B_\lambda]$ is also bounded by k. Thus,

$$\boxed{\|B_\lambda^t\|_2 \leq \sqrt{k}^t (1 + t\sqrt{k}) \leq 2t\sqrt{k}^{t+1}}.$$

23/6/20

Thm: (Lubetzky–Peres '16) NBRW on $(k+1)$-reg. Ram. graphs has optimal cutoff (optimal: at $t = \log_k n$), with cutoff window of size $O(\log\log n)$.

(This is enough for cutoff: $\frac{\log n + c_1 \log\log n}{\log n - c_2 \log\log n} \xrightarrow{n\to\infty} 1$)

Pf: We already know

$$\mathrm{dist}_{TV}\left(P^{v_0}_{\log_k n - \log_k \frac{1}{\varepsilon}}, u\right) \geq 1 - \varepsilon \quad \left(\text{for any } k\text{-reg. walk}\right).$$

On the other side, take $t = \log_k n + 3\log_k \log_k n$:

$$\mathrm{dist}_{TV}(P_t^{v_0}, u) = \frac{1}{2}\|P_t^{v_0} - u\|_1 \leq \frac{\sqrt{n(k+1)}}{2} \|P_t^{v_0} - u\|_2 = \frac{\sqrt{N}}{2} \left\| \left(\frac{B^T}{k}\right)^t (P_0^{v_0} - u) \right\|_2$$

$$\leq \frac{\sqrt{N}}{2k^t} \|B|_{\mathbb{1}^\perp}\|_2 \leq t\sqrt{k \cdot \frac{n}{k^t}(k+1)}$$

$N = n(k+1) = |E^\pm|$

$$= (\log n + 3\log\log n)\sqrt{k\,\frac{(k+1)\,\cancel{n}}{\cancel{n}\cdot(\log n)^3}} \xrightarrow{n\to\infty} 0 .$$

Thus, $\forall \varepsilon > 0$, for n large enough we get

$$\mathrm{dist}_{TV}\left(p_{v_0}^{\log n + 3\log\log n}, u\right) < \varepsilon$$

so $t_\varepsilon^G \le \log_k n + 3\log_k \log_k n$.

In total, in $\left[\log n - \log\frac{1}{\varepsilon}, \log n + 3\log\log n\right]$ the TV distance drops from $1-\varepsilon$ to ε. $\square$

<u>Thm (Lubetzky-Peres)</u>: SRW on $(k+1)$-reg. Ram. graphs has cutoff at $t = \frac{k+1}{k-1}\log_k n$, with window of size $O(\sqrt{\log_k n})$.

Recall there is a cover map (locally inj.) $p : T_{k+1} \twoheadrightarrow G$, and after fixing $\tilde{v}_0$ $(p(\tilde{v}_0) = v_0)$, paths starting at v_0 are in 1-1 correspondence with paths (in T_{k+1}) starting at $\tilde{v}_0$. (given by p)

In particular, $p\left(B_r^{T_{k+1}}(\tilde{v}_0)\right) = B_r^G(v_0)$ $\left(B_r(v) = \{w \mid \mathrm{dist}(v,w) \le r\}\right)$,

so $|B_r^G(v_0)| \le |B_r^{T_{k+1}}(\tilde{v}_0)| = |B_r^{T_{k+1}}|$ (same for any vertex)

exer: $|B_r^{T_{k+1}}| \le r k^r$ (for $r \ge 2$)

Claim: Fix $\varepsilon > 0$. For $r_0 = \log_k n - 2\log_k \log_k n$,

$|B_{r_0}(v_0)| < \varepsilon n$ for n large enough.

Pf: $\frac{1}{n}|B_{r_0}(v_0)| \le \frac{1}{n}|B_{r_0}^{T_{k+1}}| \le \frac{1}{n} r_0 k^{r_0}$

$$= \frac{1}{n}(\log_k n - 2\log_k \log_k n)\left(\frac{n}{(\log_k n)^2}\right) \xrightarrow{n\to\infty} 0. \qquad \square$$

In NBRW, after t steps we are at $S_t^T(\tilde{v}_0)$.

In SRW, we are at $B_t^T(\tilde{v}_0)$, but this is not good enough: most of B_t is at S_t, but the walk only gets to S_t with prob. $\left(\frac{k}{k+1}\right)^{t-1} \xrightarrow{t\to\infty} 0$.

Intuitively, I'm expecting to be around the $\frac{k-1}{k+1}\cdot t$ sphere: except when at $\tilde{v}_0$, we advance away from $\tilde{v}_0$ with prob. $\frac{k}{k+1}$, and toward $\tilde{v}_0$ with prob. $\frac{1}{k+1}$.

So, the expectancy of change in dist. to $\tilde{v}_0$ is around

$$\frac{k}{k+1}\cdot 1 + \frac{1}{k+1}(-1) = \frac{k-1}{k+1}.$$ Formally:

Define Distribution $\mathcal{D}: \begin{cases} 1 & k/k+1 \\ -1 & 1/k+1 \end{cases}$

$\varepsilon := \mathbb{E}(\mathcal{D}) = \frac{k-1}{k+1}$ $\qquad \sigma := \sigma(\mathcal{D}) = \sqrt{\mathbb{E}(\mathcal{D}^2) - \varepsilon^2} = \frac{2\sqrt{k}}{k+1}$.

We consider SRW $v_0, v_1, v_2, \ldots$ on G, and lift it to SRW $\tilde{v}_0, \tilde{v}_1, \ldots$ on T_{k+1} ($v_0, \tilde{v}_0$ are fixed).

$d_t := \mathrm{dist}_{T_{k+1}}(\tilde{v}_t, \tilde{v}_0)$, and $X_t := d_t - d_{t-1}$ $\left[\text{so } d_t = \sum_{j=1}^{t} X_j\right]$

When $d_{t-1} \neq 0$ $\left(\Leftrightarrow \tilde{v}_{t-1} \neq \tilde{v}_0\right)$, $X_t \sim \mathcal{D}$ and are indep.
when $\tilde{v}_{t-1} = \tilde{v}_0$, $X_t \equiv 1$.

Define $Y_t = \begin{cases} X_t & \tilde{v}_{t-1} \neq \tilde{v}_0 \\ \text{new var.} \sim \mathcal{D} & \text{otherwise} \end{cases}$, and $d'_t = \sum_{j=1}^{t} Y_j$.

Note $d'_t \leq d_t$. Also, Y_t are i.i.d so

$$\tilde{d}'_t := \frac{d'_t - t\varepsilon}{\sqrt{t}\,\sigma} \xrightarrow{\text{dist.}} \mathcal{N}(0,1) \quad \text{by C.L.T.}$$

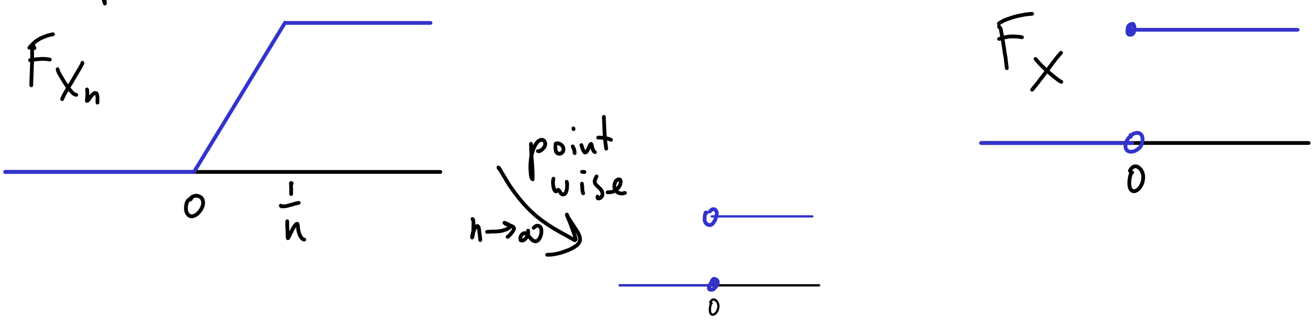
(<u>Remark</u>: d_t and d'_t behave as $\left(\frac{1}{k+1}, \frac{k}{k+1}\right)$-biased random walks on $\mathbb{N}$, and on $\mathbb{Z}$, respectively)

<u>Claim</u>: $\dfrac{d_t - t\varepsilon}{\sqrt{t}\,\sigma} \xrightarrow{\text{dist}} \mathcal{N}(0,1)$.

$\boxed{29/6/20}$

<u>Reminder</u>: a seq. of R.V. X_n converges in dist. to a R.V. X (or to the dist. of X)
if $F_{X_n}(t) \xrightarrow[n\to\infty]{} F_X(t)$ at every $t \in \mathbb{R}$ at which F_X is cont.
where $F_Y(t) = \mathbb{P}[Y \leq t]$ (the C.D.F.)

<u>Example</u>: $X_n \sim U([0, \frac{1}{n}])$ $\qquad X \equiv 0$



Claim: $\frac{d_t - t\varepsilon}{\sqrt{t}\,\sigma} \overset{dist}{\Longrightarrow} \mathcal{N}(0,1)$

Pf: With prob. 1, the walk $\{\tilde{V}_t\}_{t=0}^{\infty}$ only returns to the origin a finite # of times:

$\mathbb{P}[\exists C$: there are at most C returns$] = 1$ (exer!)

wrong!

(or : $\mathbb{P}\left[\begin{smallmatrix}\text{visit } \tilde{v}_0 \\ \infty \text{ many times}\end{smallmatrix}\right] = 0$)

Thus $\mathbb{P}[\exists C : |d_t - d_t'| \underset{(\forall t)}{\leq} C] = 1$. Therefore,

$\forall \alpha \quad \mathbb{P}\left[\frac{d_t - d_t'}{\sqrt{t}} \leq \alpha\right] \xrightarrow{t\to\infty} 1$, or in other words $\frac{d_t - d_t'}{\sqrt{t}} \overset{dist.}{\Longrightarrow} \underset{(\text{Dirac})}{\delta_0}$ (constant 0).

Thus, $\frac{d_t - t\varepsilon}{\sqrt{t}\,\sigma} = \underbrace{\frac{d_t - d_t'}{\sqrt{t}\,\sigma}}_{\Downarrow\, \delta_0} + \underbrace{\frac{d_t' - t\varepsilon}{\sqrt{t}\,\sigma}}_{\Downarrow\, \mathcal{N}(0,1)} \overset{dist.}{\Longrightarrow} \mathcal{N}(0,1)$ (complete the details) $\square$

Claim: Fix $\varepsilon > 0$. There exist $C = C_{k,\varepsilon}$ s.t. at

$$t_0 = \underbrace{\frac{k+1}{k-1}}_{\frac{1}{\varepsilon}} \log_k n - C\sqrt{\log_k n}$$

we have $\mathbb{P}[d_{t_0} > r_0] \leq 2\varepsilon$ for n large enough.

Proof: Take $t = t_0$. $\mathbb{P}[d_t > r_0] = \mathbb{P}\left[\frac{d_t - t\varepsilon}{\sqrt{t}\,\sigma} > \frac{r_0 - t\varepsilon}{\sqrt{t}\,\sigma}\right]$

$$\frac{r_0 - t\varepsilon}{\sqrt{t}\,\sigma} = \frac{\cancel{\log n} - 2\log\log n - (\cancel{\log n} - \varepsilon C\sqrt{\log n})}{\sqrt{t}\,\sigma}$$

for n large enough $\geq \dfrac{\varepsilon(c-1)\sqrt{\log n}}{\sqrt{t}\,\sigma} \geq \dfrac{(c-1)\sqrt{\log n}}{\sqrt{\frac{1}{\varepsilon}\log n}\;\sigma} = \dfrac{(c-1)\,\varepsilon^{3/2}}{\sigma}$

30/6/20

so $\mathbb{P}[d_t > r_0] \leq \mathbb{P}\left[\dfrac{d_t - t\varepsilon}{\sqrt{t}\sigma} > \dfrac{(c-1)\varepsilon^{3/2}}{\sqrt{t}\sigma}\right]$

$$\leq \mathbb{P}\left[\mathcal{N}(0,1) > \frac{(c-1)\varepsilon^{3/2}}{\sigma}\right] + \varepsilon \leq 2\varepsilon.$$

by choosing c approp. $\square$

<u>Claim</u>: At time t_0, $\mathrm{dist}(p^{t_0}, u) > 1 - 3\varepsilon$.

<u>Pf</u>: $\mathrm{dist}_{TV}(p^{t_0}, u) \geq u(V \setminus B_{r_0}(v_0)) - p^{t_0}(V \setminus B_{r_0}(v_0))$

$$= \frac{n - |B_{r_0}(v_0)|}{n} - \mathbb{P}[v_{t_0} \notin B_{r_0}(v_0)]$$

if $\tilde{v}_t \in B_r$ then $v_t \in B_r$

$$\geq 1 - \varepsilon - \mathbb{P}[\tilde{v}_{t_0} \notin B_{r_0}(\tilde{v}_0)]$$

$$= 1 - \varepsilon - \mathbb{P}[d_{t_0} > r_0] \geq 1 - 3\varepsilon. \quad \square$$

<u>Upper bound on mixing time</u>

(Note: until now we didn't assume Raman.)

Recall that at $r_1 = \log_k n + 3\log_k \log_k n$

we had NBRW came ε-close to u, by

$$\sqrt{n(k+1)}\,\left\|\left(\frac{B}{k}\right)^{r_1}\Big|_{\mathbb{1}^\perp}\right\|_2 \xrightarrow{n\to\infty} 0.$$

I want to think of NBRW as a process on V:

define $H: \mathbb{R}^{E^{\pm}} \to \mathbb{R}^V$ $\quad (Hf)(v) = \sum_{v \to w} f(v \to w)$

$(S = H^T)$ $\quad S: \mathbb{R}^V \to \mathbb{R}^{E^{\pm}}$ $\quad (Sf)(v \to w) = \frac{1}{k+1} f(v)$

Obs.: the dist. p_{NB}^r of NBRW on V after r steps starting at v_0 is $p_{NB}^r = H\left(\frac{B^T}{k}\right)^r \underbrace{S\,\mathbb{1}_{v_0}}_{\frac{1}{k+1}\mathbb{1}_{\{v_0 \to *\}}}$.

The thing is we still have

$$\frac{\sqrt{n}}{2}\left\| H\left(\frac{B^T}{k}\right)^{r_1} S\left(\mathbb{1}_{v_0} - u\right)\right\|_2 \leq \frac{\|H\|_2\|S\|_2\sqrt{n}}{2}\left\|\left(\frac{B^T}{k}\right)^{r_1}\Big|_{\mathbb{1}^\perp}\right\| \xrightarrow{n\to\infty} 0.$$

so for n large enough $\boxed{\mathrm{dist}_{TV}(p_{NB}^{r_1}, u) < \varepsilon}$.

Idea: "SRW of t steps should behave like NBRW of $r = \frac{k-1}{k+1}t$ steps"

Formally: define $t_1 = \frac{k+1}{k-1}\log_k n + C\sqrt{\log_k n}$

(same C as in t_0). Convince yourself that $\mathbb{P}[d_{t_1} < r_1] < 2\varepsilon$ for n large enough.

Now, $p^t(v) = \sum_{\substack{r=0 \\ r \equiv t \ (\mathrm{mod}\ 2)}}^{t} \mathbb{P}[v_t = v \mid d_t = r]\,\mathbb{P}[d_t = r]$

define $p_r^t(v) = \mathbb{P}[v_t = v \mid d_t = r]$ so $p^t = \sum p_r^t\,\mathbb{P}[d_t = r]$

So $\|p^t - u\|_{TV} = \left\| \sum_{r=0}^{t} \mathbb{P}[d_t = r](p_r^t - u) \right\|_{TV}$

$$\leq \sum_{r=0}^{t} \mathbb{P}[d_t = r] \, \|p_r^t - u\|_{TV}$$

$$\leq \mathbb{P}[d_t < r_1] + \sum_{r=r_1}^{t} \mathbb{P}[d_t = r] \, \|p_r^t - u\|_{TV}$$

$t = t_1$

$$\leq 2\varepsilon + \max_{r_1 \leq r \leq t_1} \|p_r^t - u\|_{TV} .$$

So it is left to show that for any $r_1 \leq r \leq t_1$, $\|p_r^t - u\|_{TV} < \varepsilon$, in order to obtain that

$\mathrm{dist}(p^{t_1}, u) < 3\varepsilon$ for n large enough.

so in $\left[\frac{k+1}{k-1} \log_k n \pm C_\varepsilon \sqrt{\log_k n}\right]$ we got from TV dist. $1 - 3\varepsilon$ to 3ε.

Back to p_r^t: I claim that

$$p_r^t(v) = \frac{\#\{\tilde{v} \in S_r(\tilde{v}_0) \mid p(\tilde{v}) = v\}}{|S_r(\tilde{v}_0)|}$$

Why? SRW on T_{k+1}, starting from $\tilde{v}_0$, gives the same prob. to the entire r-sphere around $\tilde{v}_0$, so $p_r^t(v) = \mathbb{P}[v_t = v \mid d_t = r]$

$$= \mathbb{P}[\tilde{v}_t \in p^{-1}(v) \mid d_t = r] = \frac{|p^{-1}(v) \cap S_r(\tilde{v}_0)|}{|S_r(\tilde{v}_0)|}$$

(assuming $\begin{matrix} r \le t \\ r \equiv t \pmod 2 \end{matrix}$).

In particular, p_r^t doesn't depend on t!

<u>Punchline:</u> $p_r^t = p_{NB}^r$ because pulling both of them to T_{k+1} gives $\underset{\text{(uniform)}}{\mathcal{U}}(S_r(\tilde{v}_0))$!

So finally,

$$\mathrm{dist}_{TV}(p_r^t, u) = \mathrm{dist}_{TV}(p_{NB}^r, u) \overset{r \ge r_1}{\le} d(p_{NB}^{r_1}, u) < \varepsilon$$

for $r \geq r_1 = \log n + 3 \log \log n$.

— Fin —

Expander graphs (80571) – Exercise 1

Unless stated otherwise, $G = (V, E)$ is a k-regular graph with $n < \infty$ vertices.

(1) (a) Prove that if $Af = kf$ then f is *locally constant*, namely: $f(v) = f(w)$ whenever $v \sim w$.

  (b) Prove that if $\lambda_2 = k$ then G is disconnected.

  (c) Optional: Prove that the multiplicity (number of appearances) of k in $Spec(A)$ equals the number of connected components of $G^{(\dagger)}$.

(2) (a) Prove that if G is connected and $-k \in Spec(A)$ then G is bipartite.

  (b) Optional: Find what is the multiplicity of $-k$ in $Spec(A)$.

(3) (Infinite graphs!) Let $G = T_k$ be the k-regular tree$^{(\ddagger)}$.

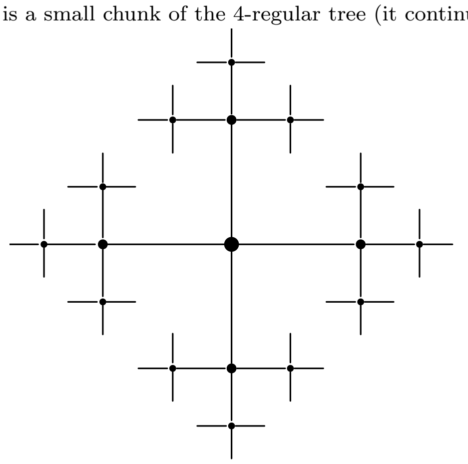
  (a) For $k = 2$ (where you can take $V(T_2) = \mathbb{Z}$), show there exists a **non-constant** function with $Af = 2f$. (compare this with question 1(a) and 1(b)).

  (b) Show that for **every** $\lambda \in \mathbb{R}$ there exists a non-zero function on the vertices $f\colon V(T_k) \to \mathbb{R}$ with $Af = \lambda f$.

(4) Assume G is a bipartite ε-expander. Show that SRW on G with a lazy first step satisfies $\|P_t - \mathbf{u}\| \le \left(\frac{\varepsilon}{k}\right)^{t-1}$.

$^{(\dagger)}$You can check the definition in `https://en.wikipedia.org/wiki/Component_(graph_theory)` - in general wikipedia is a good place to look for definitions you forgot, but you can always email me.

$^{(\ddagger)}$A tree is a connected graph with no cycles. There is a unique k-regular tree (think about it), and it is infinite. Do not be confused with computer science "k-regular tree" which can have leaves at the "bottom", and a "root" vertex with different degree. Here is a small chunk of the 4-regular tree (it continues in all directions):



Expander graphs (80571) – Exercise 2

(1) Let G be a finite group and $S \subseteq G$ a symmetric set ($S^{-1} = S$).

   (a) Show that $\mathrm{Cay}(G,S)$ is connected iff $\langle S \rangle = G$.

   (b) Assuming $\langle S \rangle = G$, show that $\mathrm{Cay}(G,S)$ is bipartite iff there exists a (normal) subgroup $H \leq G$ of index two, such that $S \subseteq G \backslash H$.

(2) For each $\Gamma_n = \mathrm{Cay}\left(D_n, \left\{\sigma, \sigma^{-1}, \tau\right\}\right)$,[(†)] construct a (non-constant) eigenfunction with eigenvalue $\lambda \overset{n\to\infty}{\longrightarrow} 3$, showing that Γ_n do not form a family of expanders (or bipartite expanders).

(3) Assume G is a finite **non-regular** graph, with no vertices of degree zero. Define $D\colon \mathbb{R}^V \to \mathbb{R}^V$ by $(Df)(v) = \deg(v) f(v)$, and $M = D^{-1}A$. Put on $\mathbb{R}^V$ the inner product

$$\langle f, g \rangle = \sum_{v \in V} \deg(v)\, f(v)\, g(v)\,.$$

   (a) Show that M is self-adjoint (w.r.t. the inner product above!), and deduce that its spectrum is real and that $\mathbb{R}^V$ has an O.N.B. of M-eigenvectors.

   (b) Show that if P_t is the distribution of SRW at time t, then $P_{t+1} = M^T P_t$.

   (c) Recall that M and M^T have the same eigenvalues (for general M). For our M, show that $\mathrm{Spec}(M) \subseteq [-1,1]$ and find an eigenvector with eigenvalue 1 for M, and for M^T (these are not the same eigenvectors!)

   (d) Show that 1 appears once in $\mathrm{Spec}(M)$ iff G is connected, and that if G is connected then $-1 \in \mathrm{Spec}(M)$ iff G is bipartite. What is the limit distribution of SRW on G, when G is connected and non-bipartite?
   Hint: show that M^t is self-adjoint w.r.t. the inner product $\langle f, g \rangle = \sum_{v \in V} \frac{f(v)g(v)}{\deg(v)}$.

[(†)] The Dihedral group D_n is the symmetry group of a regular n-gon; σ denotes rotation by $\frac{2\pi}{n}$, and τ a reflection.

Expander graphs (80571) – Exercise 3

April 21, 2020

As always G is k-regular with n vertices.

(1) (a) Prove the slightly stronger version of the Expander Mixing Lemma: for $S, T \subseteq V$ in a ε-expander,

$$\left| |E(S,T)| - \frac{k|S||T|}{n} \right| \leq \varepsilon \sqrt{|S|\left(1 - \frac{|S|}{n}\right)|T|\left(1 - \frac{|T|}{n}\right)}$$

(b) Taking $T = \overline{S}$, conclude that for an ε-expander G

$$h'(G) = \min_{\varnothing \neq S \subsetneq V} \frac{|\partial S| n}{|S| \left|\overline{S}\right|} \geq k - \varepsilon.$$

(2) Let G be a bipartite ε-expander, with $V = R \sqcup L$.

(a) Prove that for $S \subseteq R$ and $T \subseteq L$,

$$\left| |E(S,T)| - \frac{2k|S||T|}{n} \right| \leq \varepsilon \sqrt{|S||T|}$$

(b) Optional: find an improvement as in question 1.

Expander graphs (80571) – Exercise 4

May 29, 2020

In all questions $G = (V, E)$ has n vertices and no loops/multiple-edges.
It does not have to be regular.
Eigenvalues: λ_i - Adjacency, μ_i - Markov ($D^{-1}A$), γ_i - Laplacian ($\Delta = D - A$).

(1) Define the *Edge Laplacian* $\Delta_e \stackrel{def}{=} \delta\delta^* : \Omega(G) \to \Omega(G)$ (recall that $\Delta = \delta^*\delta$).

(a) Show that $f \in \ker \Delta_e$ iff f obeys "Kirchhoff's law", namely the total incoming/outgoing flow at every vertex is zero.

(b) As discussed in class, choosing a direction for each edge in G gives a basis for $\Omega(G)$. Show that the various possible choices all end up in the same matrix representation for Δ (but not so for δ, δ^*, Δ_e).

(c) Show that $\mathrm{Spec}(\Delta_e) = \mathrm{Spec}(\Delta) \sqcup \{\underbrace{0, \ldots, 0}_{|E|-|V| \text{ times}}\}$.

(2) While different choices of edge directions give different matrices for Δ_e, the absolute values of the entries remain the same (check this). Recall that for a $m \times n$ matrix A

$$\|A\|_1 := \max_{v \neq 0} \frac{\|Av\|_1}{\|v\|_1} = \max_j \left\{ \sum_{i=1}^{m} |a_{ij}| \right\}, \qquad \|A\|_\infty := \max_{v \neq 0} \frac{\|Av\|_\infty}{\|v\|_\infty} = \max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\},$$

and show that

$$\|\Delta_e\|_1 = \|\Delta_e\|_\infty = \max_{v \sim w} \{\deg(v) + \deg(w)\}.$$

Using the inequality $\|A\| \leq \sqrt{\|A\|_1 \|A\|_\infty}$ (which you are welcome to prove if you never did), conclude that

$$\gamma_n \leq \max_{v \sim w} \{\deg(v) + \deg(w)\}.$$

Optional: prove this is an equality iff G is bipartite with each side having a constant degree.

(3) Prove as much as you can from the following, or find other bounds:

$$\operatorname*{avg}_{v \in V} \deg(v) \leq \lambda_1 \qquad\qquad \lambda_1 \leq \max_{v \in V} \deg(v)$$

$$\sqrt{\max_{v \in V} \deg(v)} \leq \lambda_1$$

$$\max_{v \in V} \deg(v) + 1 \leq \gamma_n \qquad\qquad \gamma_n \leq n$$

$$\lambda_2 < 0 \Rightarrow G = K_n$$

Hints (in white color):
Optional: if G has no triangles (loops of length three) then $\lambda_n \leq -\sqrt{\max_{v \in V} \deg(v)}$.

Expander graphs (80571) – Exercise 5

June 1, 2020

(1) For a word $w \in \mathbf{F}_{k/2}$ we denoted $p_w = \mathbb{P}\left[w(\sigma)(1) = 1\right] - \frac{1}{n}$, where $\sigma = \left(\sigma_1, \ldots, \sigma_{k/2}\right)$ are random uniform independent permutations in S_n.

(a) Show that $p_w = \mathbb{P}\left[w(\sigma)(j) = j\right] - \frac{1}{n}$ for any $1 \le j \le n$.

(b) Show that $p_w = p_{w'}$ when w' is the cyclic reduction of w.

(c) Show that $p_{x_1^m} = \frac{d(m)-1}{n}$ (for $m \le n$) where $d(m)$ is the number of divisors of m.

(d) Show that $p_w = p_{\varphi(w)}$ for any $\varphi \in \operatorname{Aut}\left(\mathbf{F}_{k/2}\right)$ (this gives another proof for (b)).

(2) The De Bruijn graph $G_{k,s}$ is the k-regular **directed graph** with

$$\begin{aligned} V_{G_{k,s}} &= [k]^s \qquad ([k] = \{1, \ldots, k\}) \\ E_{G_{k,s}} &= \{(a_1, \ldots, a_s) \to (a_2, \ldots, a_s, t) \,|\, a_i, t \in [k]\}. \end{aligned}$$

Show that $\operatorname{Spec} A_{G_{k,s}} = \{k, 0, \ldots, 0\}$, and deduce that no analogue of the Alon-Boppana theorem holds for directed graphs.

(3) (Alon-Boppana) We'll show that the second Laplacian eigenvalue γ_2 of (k-regular) G satisfies

$$\gamma_2 < k - 2\sqrt{k-1} + \frac{2\sqrt{k-1} - 1}{\left\lfloor \frac{\operatorname{diam}(G)}{2} \right\rfloor - 1}. \tag{1}$$

(a) Show that $\operatorname{diam}(G) \to \infty$ as $n = |V| \to \infty$, and deduce that (1) implies the Alon-Boppana theorem as stated in class$^{(\dagger)}$.

(b) Fix $v \in V$, and denote

$$\begin{aligned} S_j &= S_j(v) = \{w \in V \,|\, \operatorname{dist}(v, w) = j\} & s_j &= |S_j| \\ E_j &= E(S_j, S_{j+1}) & e_j &= |E(S_j, S_{j+1})|. \end{aligned}$$

Fix $b \in \mathbb{N}$, denote $\rho = \sqrt{k-1}$, and define $f^v \colon V \to \mathbb{R}$ by

$$f^v(w) = \begin{cases} 1 & w = v \\ \rho^{1-j} & w \in S_j,\ 1 \le j \le b \\ 0 & w \in S_j,\ b < j. \end{cases}$$

Show that (note both sums **include** $j = b$):

$$\begin{aligned} \|f^v\|^2 &= 1 + \sum\nolimits_{j=1}^{b} s_j \rho^{2-2j} \\ \|\delta f^v\|^2 &= (\rho - 1)^2 \sum\nolimits_{j=1}^{\boldsymbol{b}} e_j \rho^{-2j} + (2\rho - 1) e_b \rho^{-2b}. \end{aligned}$$

(c) For $j \ge 1$ show that $e_j \le \rho^2 s_j$ and $s_{j+1} \le \rho^2 s_j$, and deduce also that $s_b \le \frac{1}{b} \sum_{j=1}^{b} \rho^{2(b-j)} s_j$.

(d) Combine (b) and (c) to show that

$$\|\delta f^v\|^2 < \left((\rho - 1)^2 + \frac{2\rho - 1}{b}\right) \|f^v\|^2 = \left(k - 2\sqrt{k-1} + \frac{2\sqrt{k-1} - 1}{b}\right) \|f^v\|^2.$$

(e) Now, let $v, w \in V$ be vertices with $\operatorname{dist}(v, w) = \operatorname{diam}(G)$, and fix $b = \left\lfloor \frac{\operatorname{diam}(G)}{2} \right\rfloor - 1$. Show that $f^v \perp f^w$ and $\delta f^v \perp \delta f^w$. Take $f = f^v + \alpha f^w$ with α for which $f \perp \mathbb{1}$, and use it to deduce (1).

$^{(\dagger)}$In fact it is stronger, as it addresses λ_2 and not only $\lambda = \max(\lambda_2, |\lambda_n|)$.

Expander graphs (80571) – Exercise 6

September 8, 2020

(1) For a $(k+1)$-regular Ramanujan graph, let $Af = \lambda f$ with $\|f\| = 1$ and $\lambda \notin \left\{\pm 2\sqrt{k}, \pm (k+1)\right\}$. We showed that in some ONB $\mathscr{B}$

$$[B_\lambda]_{\mathscr{B}} = \left[B|_{\langle f^+, f^- \rangle}\right]_{\mathscr{B}} = \begin{pmatrix} \vartheta^+ & b \\ 0 & \vartheta^- \end{pmatrix},$$

with $|\vartheta^+| = |\vartheta^-| = \sqrt{k}$ and $|b| \leq k$.

(a) Show that $\left\|f^h\right\|^2 = \left\|f^t\right\|^2 = k+1$, and $\left\langle f^h, f^t \right\rangle = \lambda$.

(b) Define $f^\perp = f^h - \frac{\vartheta^-}{k} f^t$, and show that $f^+ \perp f^\perp$,(†) and that $[B_\lambda]_{\{f^+, f^\perp\}} = \begin{bmatrix} \vartheta^+ & \frac{(1-k)}{k}\vartheta^- \\ 0 & \vartheta^- \end{bmatrix}$.(‡)

(c) Show that $\left\|f^+\right\|^2 = (k+1)^2 - \lambda^2$ and $\left\|f^\perp\right\|^2 = \frac{(k+1)^2 - \lambda^2}{k}$, and compute $[B_\lambda]_{\mathscr{B}}$ completely. Verify you got $|b| = k-1$, showing that our naive bound $|b| \leq k$ was almost optimal.

(d) Optional: Verify from your computation that $\|B_\lambda\|_2 = k$ (we already know this, since we saw that k appears n times in $\operatorname{Sing}(B)$, and $(n-1)$ of these have to come from the B_λ-blocks).

(e) Optional: Compute $\|B^t\|_2$ precisely (by computing $\|B_\lambda^t\|_2$), using some software (or if you are brave, by hand).

(2) (a) We proved that for a k-regular graph/digraph, $\|p_t^{v_0} - \mathbf{u}\|_1 \geq 2 - 2\varepsilon$ when $t = \log_k n - \log_k(1/\varepsilon)$. Prove that $\operatorname{dist}_{TV}(p_t^{v_0}, \mathbf{u}) \geq 1 - \varepsilon$ directly from the definition of dist_{TV}.

(b) Prove that $\operatorname{dist}_{TV}(p, p') = \frac{1}{2}\|p - p'\|_1$ (for distributions p, p').

(3) Let X_t be a positively-biased walk on $\mathbb{N}$, namely, $X_t = 1$ when $X_{t-1} = 0$, and when $X_{t-1} \neq 0$,

$$X_t \sim \begin{cases} X_{t-1} + 1 & \text{with probability } p \\ X_{t-1} - 1 & \text{with probability } 1 - p \end{cases}$$

with $p > \frac{1}{2}$.

(a) Show that with probability one, the walk returns to 0 only a finite number of times.

(b) Deduce that with probability one SRW on the k-regular tree ($k \geq 3$) returns to the starting point only a finite number of times.

(c) Optional: for $n \geq 1$, compute the probability that $X_{t+1}, X_{t+2}, \ldots \geq n$ given that $X_t = n$.

(†) Where did $-\frac{\vartheta^-}{k}$ come from? I solved $\left\langle f^+, f^h + x f^t \right\rangle = 0$ for you as it is a bit tedious, but you can do it yourself if you want to see all the details.

(‡) Tip: we computed $Bf_\alpha = -\alpha f^h + (k + \alpha\lambda) f^t$ in class, and $f^\perp = f_{-\frac{\vartheta^-}{k}}$.

Expander graphs (80571) – Final Assigment

September 8, 2020

Two cycles (=closed paths) in a graph are called equivalent if one is obtained from the other by a cyclic rotation, or in other words, by starting from a different point in it; we denote by $[\gamma]$ the equivalence class of γ. A cycle is called cyclically nonbacktracking (CNB) if it is nonbacktracking and so are also the cycles equivalent to it. A cycle is called primitive if it is CNB, and it is not obtained by repeating some cycle twice or more times. Finally, a prime in a graph G is an equivalence class of primitive cycles. The *Ihara zeta function* of a finite connected graph G is defined by

$$\zeta_G(u) = \prod_{[\gamma]} \frac{1}{1 - u^{\mathrm{len}(\gamma)}},$$

where the product is over all primes in G. It is not a priori clear that the product converges for any $u \in \mathbb{C}$, but Ihara showed (and you will too) that ζ_G is actually given by a rational function, for u small enough.

(1) (a) **(5)** Show that for $n \geq 3$ the n-cycle graph C_n has two primes, and Compute ζ_{C_n}.

(b) **(5)** The cycle rank of a connected graph is $r(G) = |E| - |V| + 1$. Show that any (connected) graph with $r(G) \geq 2$ has infinitely many primes (hint: think first on a ∞ shaped graph).

(c) **(5)** Recall Riemann's zeta function,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

and sketch the argument why unique factorization in $\mathbb{N}$ implies that these two expression are indeed equal (for $s > 1$). Allow yourself to ignore all issues of convergence.

(d) **(5)** Show that in graphs a "product" (namely, concatenation) of primes **can** be a prime, so in particular, there is no unique factorization. On the other hand, show that every CNB cycle can be uniquely written as a power of a primitive cycle.

(e) **(10)** Let $N_G(n)$ be the number of CNB cycles of length n in G. Show that for $|u| < 1$,

$$\ln \zeta_G(u) = \sum_{n=1}^{\infty} \frac{N_G(n)}{n} \cdot u^n.$$

You may be sloppy about convergence issues, as before. Hint: use Taylor for $\ln(1 - x)$.

(f) **(10)** Show that $N_G(n) = \mathrm{tr}(B^n)$ (where $B = B_G$ is the nonbacktracking walk operator on directed edges), and use this to deduce Hashimoto's theorem (1989):

$$\zeta_G(u) = \prod_{\vartheta \in \mathrm{Spec}(B)} \frac{1}{1 - \vartheta u} = \frac{1}{\det(I - uB)}$$

for u small enough (every ϑ repeats in $\mathrm{Spec}(B)$ according to its algebraic multiplicity).[†]

[†]Note that we now discovered that $\zeta_G(u)$ coincides with a rational function on an open set, so it can be analytically continued to a meromorphic function.

(g) **(5)** Assuming that G is $(k+1)$-regular, deduce Ihara's theorem (1966):

$$\zeta_G(u) = \frac{1}{(1-u^2)^{r(G)-1}\det(I - uA + u^2 kI)}.$$

(h) **(5)** Deduce Sunada's observation (1986) that Ramanujan graphs satisfy a sort of "Riemann hypothesis": assuming G is $(k+1)$-regular, it is Ramanujan iff every pole of $\zeta_G(u)$ with $0 < \Re(-\log_k u) < 1$ satisfies $\Re(-\log_k u) = \frac{1}{2}$.

(2) For $K \gneq k \geq 3$, let $G = (L \sqcup R, E)$ be a connected bipartite $(K+1, k+1)$-biregular graph.[(†)] Namely, $\deg\big|_L \equiv K+1$ and $\deg\big|_R \equiv k+1$. Denote $n = |L|$ (so that $|R| = \frac{K+1}{k+1}n$).

(a) **(5)** Show that $\mathfrak{p} := \sqrt{(K+1)(k+1)}$ and $-\mathfrak{p}$ are eigenvalues of $A := A_G$, and that every $\lambda \in \operatorname{Spec}(A)$ satisfies $|\lambda| \leq \mathfrak{p}$.[(‡)]

(b) **(5)** Show that the multiplicity of $0 \in \operatorname{Spec} A_G$ is at least $\frac{K-k}{k+1}n$.

(c) **(15)** Assume $Af = \lambda f$ with $f \neq 0$ and $\underline{0 < \lambda < \mathfrak{p}}$.[(§)] Construct eigenfunctions $f^{\pm\pm}$ of $B = B_G$ with eigenvalues

$$\vartheta^{\pm\pm} = \pm\sqrt{\frac{1}{2}\left(\lambda^2 - K - k \pm \sqrt{\lambda^4 - 2(K+k)\lambda^2 + (K-k)^2}\right)}$$

(or simpler, $\vartheta^{\pm\pm}$ are the roots of $z^4 + (K + k - \lambda^2) z^2 + Kk$).

(d) **(5)** Check what happens if you take $\lambda = \mathfrak{p}$ in the last question (How many eigenfunctions do you get? With which ϑ?)

(e) **(10)** For $0 \leq \lambda \leq \mathfrak{p}$, show that $|\vartheta^{\pm\pm}| \leq \sqrt[4]{Kk}$ iff $\lambda \in \left[\sqrt{K} - \sqrt{k}, \sqrt{K} + \sqrt{k}\right]$. (Do you see the significance of $\sqrt[4]{Kk}$?)

Remark: This connects to the regular case as follows: the adjacency spectrum of the $(k+1)$-regular tree is $\operatorname{Spec}(A_{T_{k+1}}) = \left[-2\sqrt{k}, 2\sqrt{k}\right]$ (the "Ramanujan region"), and that of the $(K+1, k+1)$-biregular tree is

$$\operatorname{Spec}\left(A_{T_{K+1,k+1}}\right) = \left[-\sqrt{K} - \sqrt{k}, -\sqrt{K} + \sqrt{k}\right] \cup \{0\} \cup \left[\sqrt{K} - \sqrt{k}, \sqrt{K} + \sqrt{k}\right].$$

(f) **(10)** Show that for $\lambda \neq \sqrt{K} \pm \sqrt{k}$ the four eigenfunctions $f^{\pm\pm}$ are independent.

(g) **(25)** Study what happens for $\lambda = 0$, and show that

$$\operatorname{Spec}(B) \subseteq \left\{ z \,\middle|\, |z| \leq \sqrt[4]{Kk} \right\} \cup \left\{ \pm\sqrt{Kk} \right\}$$

iff $\operatorname{Spec}(A) \subseteq \operatorname{Spec}\left(A_{T_{K+1,k+1}}\right) \cup \{\pm\mathfrak{p}\}$ **and** the multiplicity of $0 \in \operatorname{Spec} A_G$ is exactly $\frac{K-k}{k+1}n$.[(¶)]

(†) Recommendation: reread what we did in class for bipartite graphs at page 10 of the course notes.

(‡) This is called the "Perron-Frobenius" eigenvalue of a graph, if you want to read more about it.

(§) If you followed my recommendation, you know that there is also f' with $Af' = -\lambda f'$.

(¶) You may assume that $\sqrt{K} \pm \sqrt{k} \notin \operatorname{Spec}(A)$ - these are a headache.

A (Linear) Error-Correcting Code is a subspace $\mathcal{C}$ of $\mathbb{F}_2^n$. The *distance* of $\mathcal{C}$ is

$$\mathrm{dist}_{\mathcal{C}} := \min\left\{\mathrm{dist}_{\mathrm{Ham}}\left(v, v'\right) \,\middle|\, \substack{v,v' \in \mathcal{C} \\ v \neq v'}\right\} = \min\left\{\left\|v - v'\right\|_{\mathrm{Ham}} \,\middle|\, \substack{v,v' \in \mathcal{C} \\ v \neq v'}\right\} = \min\left\{\|v\|_{\mathrm{Ham}} \,|\, 0 \neq v \in \mathcal{C}\right\}$$

(where Ham stands for Hamming distance). The idea is that if $v \in \mathcal{C}$ was transmitted via a noisy medium, as long as less than $\mathrm{dist}_{\mathcal{C}}$ bits were changed, the distorted message is **not** in $\mathcal{C}$, and thus the receiver can deduce that an error occurred[†]. Some common terminology:

- n is called the *block length* of the code
- $\dim \mathcal{C}$ is the *message length* (think why)
- $r_{\mathcal{C}} = \frac{\dim \mathcal{C}}{n}$ is the *rate*
- $\delta_{\mathcal{C}} = \frac{\mathrm{dist}_{\mathcal{C}}}{n}$ is the *relative distance*

A family of codes is called *good* if the rates, and the relative distances of the codes in the family are bounded away from zero[‡].

(3) Fix a code $\mathcal{B}$ (the "Base code") of block length k. Given a k-regular graph G on ***m* vertices**, for each vertex v choose an arbitrary order $e_1^v, \ldots, e_k^v$ on the edges touching v. Define a code with block length $n = km/2$, by identifying $\mathbb{F}_2^n = \mathbb{F}_2^{E(G)}$ and taking

$$\mathcal{C} = \mathcal{C}_G = \left\{f \in \mathbb{F}_2^{E(G)} \,\middle|\, \forall v \in V(G) : \left(f\left(e_1^v\right), \ldots, f\left(e_k^v\right)\right) \in \mathcal{B}\right\}.$$

Namely, f is in $\mathcal{C}$ if every vertex "sees" a word in $\mathcal{B}$. The idea (due to Sipser and Spielman), is that taking a family of very good expanders $\{G\}$ one gets a good family of codes $\{\mathcal{C}_G\}$ (which are also LDPC, if you want to read about it).

(a) **(5)** Show that $r_{\mathcal{C}} \geq 2r_{\mathcal{B}} - 1$, so that if $r_{\mathcal{B}} > \frac{1}{2}$ then $r_{\mathcal{C}}$ are bounded away from zero[§].

(b) **(10)** Assume that $G = (L \sqcup R, E)$ is a bipartite ε-expander, and let $0 \neq f \in \mathcal{C}$. Define $S = \left\{v \in L \,\middle|\, \left(f\left(e_1^v\right), \ldots, f\left(e_k^v\right)\right) \neq \vec{0}\right\}$, and similarly $T = \{v \in R \,|\, \ldots\}$. Show that

$$\delta_{\mathcal{B}} k \sqrt{|S||T|} \leq \|f\|_{\mathrm{Ham}} \leq \frac{2k|S||T|}{m} + \varepsilon\sqrt{|S||T|}.$$

(c) **(5)** Deduce that for $0 \neq f \in \mathcal{C}$

$$\|f\|_{Ham} \geq \delta_{\mathcal{B}}\left(\delta_{\mathcal{B}} - \frac{\varepsilon}{k}\right)\frac{mk}{2},$$

and thus $\delta_{\mathcal{C}} \geq \delta_{\mathcal{B}}\left(\delta_{\mathcal{B}} - \frac{\varepsilon}{k}\right)$.

(d) **(10)** Fix $m \geq 1$, and let $\mathbb{F}$ be a finite field of size 2^m. Recall that the elements of $\mathbb{F}$ correspond to m-tuples of bits: $\mathbb{F} \leftrightarrow \mathbb{F}_2^m$, and that this is also an isomorphism of the additive structure[¶]. We shall thus identify $\mathbb{F}^{2^m}$ (2^m-tuples of elements from $\mathbb{F}$) with $(\mathbb{F}_2)^{m2^m}$. Fix $1 \leq t \leq 2^m$, and define the (Reed-Solomon) code

$$\mathcal{R} = \mathcal{R}_{m,t} = \left\{\left(p\left(0\right), p\left(1\right), \ldots, p\left(2^m - 1\right)\right) \,\middle|\, \begin{matrix} p \in \mathbb{F}[X] \\ \deg p < t \end{matrix}\right\} \leq \mathbb{F}^{2^m} = \mathbb{F}_2^{m2^m}.$$

(so $\mathcal{R}$ has block length $m2^m$). Show that $\mathcal{R}$ has rate $\frac{t}{2^m}$ and distance $\geq 2^m - t + 1$.

(e) **(5)** Find the smallest m for which the code $\mathcal{R}$ (with an appropriate t) can be used as a base code, so that the Sipser-Spielman construction with Ramanujan graphs $\{G\}$ will ensure a good family of codes $\{\mathcal{C}_G\}$.

[†] He can also find the correct message, as long as less than $\frac{\mathrm{dist}_{\mathcal{C}}}{2}$ bits were changed - think why.

[‡] It is common to talk about a "good code", but like in "expander graph", the definition requires a family.

[§] For example, the oldest code in the book, $\mathcal{H}$=Hamming(7,4) (see Wikipedia) has $r_{\mathcal{H}} = \frac{4}{7}$.

[¶] Multiplication in $\mathbb{F}$ is done modulo a fixed irreducible polynomial of degree m, but we don't need this.