22/10/17

Ramanujan graphs and complexes - Parzan

$G=(V,E)$ undirected a graph (finite) $A: \mathbb{R}^V \longrightarrow \mathbb{R}^V$

$$(Af)(v) = \sum_{w \sim v} f(w)$$

If G is k-regular, then

$A\mathbb{1} = k\mathbb{1} \longrightarrow k$ is an eigenvalue of A.

Exercise: λ is a e.value of $A \longrightarrow |\lambda| \le k$

Defn: A k-reg graph is an ε-expander if

$$\mathrm{Spec}(A) \subseteq \{k\} \cup [-\varepsilon, \varepsilon] \quad \boxed{\text{multiplicity } 1}$$

A (k,l)-regular bipartite graph is an ε-bip expander if

$$\mathrm{Spec}(A) \subseteq \{\pm\sqrt{kl}\} \cup [-\varepsilon, \varepsilon]$$

Example: $K_n^{(\mathrm{parzan})}$ - complete graph $A_{K_n^{\mathrm{parzan}}} = J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ & \ddots & & \\ 1 & \cdots & \cdots & 1 \end{pmatrix}$

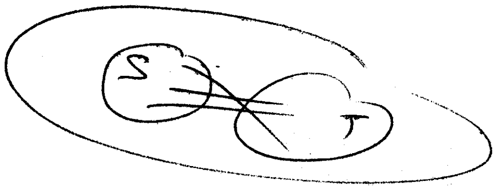
$\mathrm{Spec}(A_{K_n^{\mathrm{par}}}) = \{n, 0, \ldots, 0\} \longrightarrow$ it is 0-expander.

Expander mixing lemma

G k-reg graph, n vertices

$S,T \subseteq V$. What is #edges between S and T ?

Expectancy $k\frac{|S||T|}{n}$



Lemma: If G is an ε-expander

$$\left|\, |E(S,T)| - \frac{k|S||T|}{n} \right| \leq \varepsilon\sqrt{|S||T|}$$

Proof: Take O.N. basis v_i $\quad Av_i = \lambda_i v_i \quad \lambda_1 = k \quad v_1 = \frac{\mathbb{1}}{\sqrt{n}}$

$v_i \perp \mathbb{1} \ \forall i > 0$

For $f \in \mathbb{R}^V$ $\quad f = \sum_{i=1}^{n} f^i v_i$, where $f^i = \langle f, v_i \rangle$

$$\langle A\mathbb{1}_S, \mathbb{1}_T \rangle = \langle v \to \begin{smallmatrix}\#\text{neigh of}\\ v \text{ in } S\end{smallmatrix}, \mathbb{1}_T \rangle = \sum_{v \in T} \begin{smallmatrix}\#\text{neigh of}\\ v \text{ in } S\end{smallmatrix} = |E(S,T)|$$

$$\langle A\mathbb{1}_S, \mathbb{1}_T \rangle = \langle \sum A\mathbb{1}_S^i v_i, \sum \mathbb{1}_T^j v_j \rangle$$

$$= \sum_i \lambda_i \mathbb{1}_S^i \mathbb{1}_T^i = k \underset{\overset{\shortparallel}{\frac{|S|}{\sqrt{n}}}}{\mathbb{1}_S^1} \underset{\overset{\shortparallel}{\frac{|T|}{\sqrt{n}}}}{\mathbb{1}_T^1} + \sum_{i=2}^{n} \lambda_i \mathbb{1}_S^i \mathbb{1}_T^i$$

$$= \frac{k|S||T|}{n}$$

$$\Rightarrow \left| |E(S,T)| - \frac{k|S||T|}{n} \right| \leq \left| \sum_{i=2}^{n} \lambda_i \mathbb{1}_S^i \mathbb{1}_T^i \right|$$

$$\leq \varepsilon \left| \sum_{i=2}^{n} \mathbb{1}_S^i \mathbb{1}_T^i \right| \leq \varepsilon \sum_{i=1}^{n} |\mathbb{1}_S^i \mathbb{1}_T^i| \leq \varepsilon \sqrt{\left(\sum_{i=1}^{n} |\mathbb{1}_S^i|^2\right)\left(\sum_{i=1}^{n} |\mathbb{1}_T^i|^2\right)}$$

$$= \varepsilon \sqrt{|S||T|}$$

HW: Prove EML for (k,l)-bipartite graphs.

Cor: $\varepsilon \neq 0$ unless $G = K_n^{\text{parzan}}$ or $G = \emptyset$

Proof: If $\varepsilon = 0$ then $|E(S,T)| = \frac{k|S||T|}{n} \Rightarrow \forall v, w \in V$

$$\{0,1\} \ni |E(\{v\},\{w\})| = \frac{k}{n} \Rightarrow \begin{matrix} k = 0 \\ \text{or } k = n. \end{matrix}$$

After fixing k - ε is not too small either

For which ε - can you construct k-regular expanders?

Random walk

P_0 prob measure on V (state at time 0)

P_t prob measure of RW after t steps.

$$P_t = \left(\frac{A}{k}\right)^t P_0$$

How fast does $P_t \to \frac{\mathbb{1}}{n}$?

For K_n^{partan} $P_1 = \frac{\mathbb{1}}{n}$

For an ε-expander $P_t p_0 = \sum_i \left(\frac{A}{k}\right)^t p_0^i v_i$

$$= \underbrace{p_0^1 \frac{\mathbb{1}}{\sqrt{n}}}_{\frac{\mathbb{1}}{n}} + \underbrace{\sum_{i=2}^{n} \left(\frac{\lambda_i}{k}\right)^t p_0^i v_i}_{\text{error term}}$$

$$|\cdot| \leq \left(\frac{\varepsilon}{k}\right)^t \sum_{i=2}^{n} |p_0^i|^2 \leq \left(\frac{\varepsilon}{k}\right)^t$$

If G is disconnected $\longleftrightarrow$ k has multiplicity ≥ 2

Stronger bound on ε, after fixing k.

$$tr(A^2) = \sum_{i=1}^{n} \lambda_i^2 \leq k^2 + (n-1)\varepsilon^2$$

$$tr(A^2) = \sum_i A^2_{ii} = \text{[redacted]}\ 2\,\#\text{edges} = kn$$

$$kn \leq k^2 + (n-1)\varepsilon^2 \leq k^2 + n\varepsilon^2$$

$$k - \frac{k^2}{n} \leq \varepsilon^2 \Rightarrow \varepsilon \geq \sqrt{k - \frac{k^2}{n}} \xrightarrow[n\to\infty]{} \sqrt{k}$$

Alon-Boppana thm For fixed k and $n \to \infty$ $\varepsilon \geq 2\sqrt{k-1}$

Formally: for $\forall \varepsilon < 2\sqrt{k-1}$ there are [redacted] no ε-expanders with $n > n_0(\varepsilon, k)$

Proof: $tr(A^{2m}) \leq k^{2m} + (n-1)\varepsilon^{2m}$

$$tr(A^{2m}) = \sum_{i=1}^{n} (A^{2m})_{ii} = \begin{matrix}\text{num of closed paths}\\ \text{of length } 2m\end{matrix} \geq \begin{matrix}\#\text{backtracking}\\ 2m\text{-cycles}\end{matrix}$$

$$= n \begin{matrix}\#\text{closed paths with origin}\\ v_0 \text{ in a } k\text{-regular tree}\end{matrix} \overset{(ex.)}{\geq} n \frac{1}{m+1}\binom{2m}{m} k(k-1)^{m-1}$$

$$\varepsilon \geq \sqrt[2m]{\frac{n}{n-1}\frac{1}{m+1}\binom{2m}{m}k(k-1)^{m-1} - \frac{k^{2m}}{n-1}} \underset{n\to\infty}{\approx} \sqrt[2m]{\frac{4^m}{m+1}\frac{1}{\sqrt{\pi m}}k(k-1)^{m-1}} \text{ [redacted]}$$

$$\xrightarrow[m\to\infty]{} 2\sqrt{k-1}$$

Def: A k-reg graph is Ramanujan if it is a $2\sqrt{k-1}$-expander

"A 'bipartite' k-reg graph is 'bipartite Ramanujan' if it is a $2\sqrt{k-1}$ bipartite expander

We know:

① If $k=p^m+1$, p prime $\exists\ \infty$-many k-reg ~~Ram~~ graphs. & Ramanujan bipartite

other k - unkown $k=7$?

explicit num theoretic construction

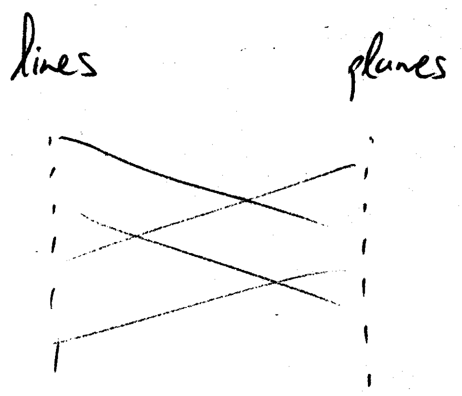
Lubotzky, Phillips, Sarnak, Margulis 1985

② $\forall k\ \exists$ inf many bipartite Ramanujan graphs

Marcus, Spielman, Srivastava 2013

based on Bilu-Linial

Example: $X = \mathbb{P}^2\mathbb{F}_p = \left(V = \text{lines and planes in } \mathbb{F}_p^3,\ E = \text{containment } (l,p) \text{ s.t. } l \subset p \right)$



$|V| = \#\text{lines} + \#\text{plane}$

$= (p^3-1)/(p-1) + (p^3-1)/(p-1) = 2p^2+2p+2$

non zero vertices

non zero scalars giving the same lines

bijection lines planes $V \mapsto V^\perp$

degree If l is a line $l \le \mathbb{P} \le \mathbb{F}_p^3 \leftrightsquigarrow l/l \le ? \le \mathbb{F}_p^3/l$

$0 \le \#\text{lines} \le \mathbb{F}_p^2 \rightsquigarrow \frac{p^2-1}{p-1} = 1+p$

$\mathbb{P}^2\mathbb{F}_p$ is a $(p+1)$-reg bipartite graph on $2(p^2+p+1)$ vertices

$$A_X = \begin{array}{cc} \text{lines} & \text{planes} \\ \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right) \end{array} \qquad A_X^2 = \begin{array}{cc} & l_2 \\ l_1 & \left(\begin{array}{c|c} \boxed{\cdot} & 0 \\ \hline 0 & \end{array}\right) \end{array}$$

$$\begin{array}{c}\#\text{paths from } l_1 \text{ to } l_2 \\ \text{of length } 2\end{array} = \begin{array}{c}\#\text{planes containing} \\ l_1 \text{ and } l_2\end{array} = \begin{cases} p+1 & l_1 = l_2 \\ 1 & l_1 \neq l_2 \end{cases}$$

so $A_X^2 = \left(\begin{array}{c|c} J & 0 \\ \hline 0 & J \end{array}\right) + p\,\mathrm{Id} \Longrightarrow Sp(A_X^2) = \{\underset{\text{multip } 2}{\underset{\uparrow}{(p+1)^2}}, p\}$

$J =$ all 1 matrix

$$\Longrightarrow Spec(A_X) = \{\underset{\substack{\text{multiplicity} \\ 1 \text{ for each}}}{\underset{\uparrow}{\pm(p+1), \pm\sqrt{p}}}\}$$

so $\mathbb{P}^2\mathbb{F}_p$ is a $\sqrt{k-1}$ expander $k=p+1$

twice better than Ramanujan!!

The catch is: $n \approx 2k^2$ (fixed for a fixed degree)

Q: Given a group G is there a generating set S s.t. $Cay(G,S)$ is an expandor ε-expander

Q: Given a family of groups G_n is there a global $\varepsilon > 0$ s.t. all the family is an ε-expanders (with fixed k)

e.g. S_n?

Example: $SL_2(\mathbb{F}_p) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$ is an infinite family of groups (for all p) with 2 generators. Do they form an expander family.

Thm: Abelian group do not give expanders.

Before that: What is the diam. of ε-expander?

Discussion: Take $p_0 = \mathbb{1}_{v_0}$. We saw that $\|p_t - \frac{\mathbb{1}}{n}\|^2 \le \left(\frac{\varepsilon}{k}\right)^{2t}$

On the other hand $\|p_t - \frac{\mathbb{1}}{n}\| \ge \frac{1}{n^2} \underbrace{\#\text{vertices not visited after } t \text{ steps}}_{M_t}$

$M_t \le n^2 \left(\frac{\varepsilon}{k}\right)^{2t}$, Taking $t = \log_{\frac{k}{\varepsilon}} n + 1$

Gives $M_t < 1 \Rightarrow \text{diam} \le \log_{\frac{k}{\varepsilon}}(n)$

For an Abelian group, the growth of random walks is polynomial

$|B_t(e)| = |\{s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} : n_1 + \dots + n_k \le t\}| \le \binom{t+k}{k} \sim t^k$

$s_1 s_2 s_1 s_3 = s_1^2 s_2 s_3$ in an Abelian group

$\Rightarrow |B_{diam}(e)| = n \Rightarrow diam \gtrsim \sqrt[k]{n}$

Rem:
r-Nilpotent groups for fixed r are also not expanders.

Margulis: $\forall S \underset{\text{finite generating}}{\subseteq} SL_3(\mathbb{Z})$ s.t. $Cay(SL_3(\mathbb{F}_p), S \bmod p)$ is a family of expanders.

Gabor-Galil computed the ε for a similar family of graphs for a specific S.

Idea: Kazhdan's property (T)

Ramanujan graphs and complexes

5/11/17

Today: T_k the k-regular tree and in particular $\mathrm{Spec}(A_{T_k})$

Tomorrow: hyperbolic plane and the spec of its adjacency matrix.

For which λ is there $f : V(T_k) \longrightarrow \mathbb{C}$, s.t. $Af = \lambda f$?

Consider spherical functions: fix "center" v_0 and look at functions which are constant on $S_m(v_0) = \{v : \mathrm{dist}(v, v_0) = m\}$, $\forall m \in \mathbb{N}_0$

This is enough: If $Af = \lambda f$, pick v_0 s.t. $f(v_0) \neq 0$ and look on

$$f_{sph}(v) = \frac{1}{\#} \sum_{w \in S_{\mathrm{dist}(v_0,v)}(v_0)} f(w) \equiv \frac{1}{|\{w : \mathrm{dist}(v_0,w) = \mathrm{dist}(v_0,v)\}|} \sum_{\substack{w: \\ \mathrm{dist}(v_0,w) \\ = \mathrm{dist}(v_0,v)}} f(w)$$

$$\left.\begin{array}{l} * \ f_{sph} \text{ is not zero} \\ * \ \|f_{sph}\|_2^2 \leq \|f\|_2^2 \\ * \ Af_{sph} = \lambda f_{sph}. \end{array}\right\} \underline{\text{exercise.}}$$

Assume f is spherical and $Af = \lambda f$. W.L.O.G. $f(0) = 1$

Denote $f(n)$ the value of a vertex at dist n from v_0

$\Rightarrow \lambda = \lambda f(0) = Af(0) = k f(1)$

$\Rightarrow f(1) = \frac{\lambda}{k}$

$\lambda f(1) = (Af)(1) = (k-1) f(2) + f(0) \longrightarrow$ get $f(2)$

recurssion

$f(0) = 1, \quad f(1) = \frac{\lambda}{k}$

$$f(n) = \frac{\lambda f(n-1) - f(n-2)}{k-1}$$

Get: $f(n) = c_1 \left(\frac{2}{\lambda + \sqrt{\lambda^2 - \rho^2}} \right)^n + c_2 \left(\frac{2}{\lambda - \sqrt{\lambda^2 - \rho^2}} \right)^n$, where $\rho \equiv 2\sqrt{k-1}$

[redacted]

There is a spherical function for every λ. Furthermore it is unique ~~assum~~ (we used here the fact that $f(0) = 1$, $f(1) = \frac{\lambda}{k}$)

When is f in L^2? $\Rightarrow$ $\lambda \in \operatorname{Spec}(A_{T_k})$

$$\| f|_{S_n(v_0)} \|_2^2 = |f(n)|^2 \cdot |S_n(v_0)| = k(k-1)^{n-1} |f(n)|^2 \approx (k-1)^n |f(n)|^2$$

$$= \left(c_1 \left(\frac{2\sqrt{k-1}}{\lambda + \sqrt{\lambda^2 - \rho^2}} \right)^n + c_2 \left(\frac{2\sqrt{k-1}}{\lambda - \sqrt{\lambda^2 - \rho^2}} \right)^n \right)^2$$

$$= \left(c_1 \left(\frac{\rho}{\lambda + \sqrt{\lambda^2 - \rho^2}} \right)^n + c_2 \left(\frac{\rho}{\lambda - \sqrt{\lambda^2 - \rho^2}} \right)^n \right)^2$$

Case 1:

$$|\lambda|>\rho \Rightarrow \quad \alpha \equiv \frac{\rho}{\lambda+\sqrt{\lambda^2-\rho^2}} \qquad \beta \equiv \frac{\rho}{\lambda-\sqrt{\lambda^2-\rho^2}}$$

$\alpha\beta = 1 \Rightarrow$ Since both α,β are real for $|\lambda|>\rho$

$\Rightarrow \alpha \geq 1$ or $\beta \geq 1$. Furthermore $|\alpha|\neq|\beta|$, so $\alpha>1$ or $\beta>1$.

$\Longrightarrow$ exp growth $\|f|_{S_n(v_0)}\| \longrightarrow \infty$

Case 2:

$$|\lambda|\leq\rho \qquad \alpha = \frac{\rho}{\lambda+i\sqrt{\rho^2-\lambda^2}} \qquad \beta=\bar{\alpha}$$. Also $C_2=\overline{C_1}$

So $\|f|_{S_n(v_0)}\|_2^2 = \left[2\mathrm{Re}\left(C_1\left(\frac{\rho}{\lambda+i\sqrt{\lambda^2-\rho^2}}\right)^n\right)\right]^2$

Observe: $\left|\frac{\rho}{\lambda+i\sqrt{\rho^2-\lambda^2}}\right| = 1$

$$\Rightarrow \|f|_{S_n(v_0)}\|_2^2 = \left(2\mathrm{Re}\left(C_1\alpha^n\right)\right)^2$$

For infinitely many n's $\|f|_{S_n(v_0)}\|_2^2 \geq \delta > 0$.

$$\Rightarrow \|f\|_2^2 = \sum_{n\geq 0}\|f|_{S_n(v_0)}\|_2^2 = \infty \longrightarrow$$ for inf many n's this is at least δ.

A_{T_k} has no L^2- eigenfunctions

$\mathrm{Spec}(A_{T_k}) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$

Fact. If A is self adjoint $\lambda \in \mathrm{Spec}(A)$

$\Leftrightarrow$ $\exists$ seq $f_n \in L^2$ s.t. $\frac{\|(A-\lambda I)f_n\|}{\|f_n\|} \xrightarrow[n\to\infty]{} 0$. $\circledast$

Fix k, λ as before. Define for $m \in \mathbb{N}$

$$f_m(v) = \begin{cases} f(v) & \mathrm{dist}(v_0, v) \leq m \longleftrightarrow v \in B_m(v_0) \\ 0 & \text{otherwise} \end{cases}$$

for the spherical e.function of λ we found before.

$$(A - \lambda I) f(n) = \begin{cases} 0 & n \leq m-1 \\ & n = m \\ & n = m+1 \\ 0 & n \geq m+2 \end{cases}$$

Tomorrow: We will show that for $|\lambda| \leq \rho$ those are approximated eigenfunctions.

$\swarrow$

a seq of f_n as above $\circledast$

6/11/17

Random graphs and complexes - Lecture 4

Recap: T_k k-reg tree, $A = A_{T_k}$ its adj. matrix

$\forall \lambda \in \mathbb{C}$ $\exists!$ spherical λ-eigenfunction of A s.t. $f_\lambda(\underset{\substack{\uparrow \\ \text{root of symmetry}}}{v_0}) = 1$

We got

$$\underset{\substack{\downarrow \\ \text{on the } n\text{-th level}}}{f(n)} = \text{some explicit formula}$$

$$\circledast - |S_n(v_0)| \cdot |f(n)|^2 = \frac{k}{k-1} \left| c_1 \alpha^n + c_2 \beta^n \right|^2, \text{ where}$$

$$\alpha, \beta = \frac{\rho}{\lambda \pm \sqrt{\lambda^2 - \rho^2}}, \quad \rho = 2\sqrt{k-1}.$$

[Assuming $|\lambda| \neq |\rho|$]

Since $\alpha\beta = 1$, if $|\lambda| > \rho \Rightarrow$ either $|\alpha|$ or $|\beta|$ are strictly bigger than $1 \Rightarrow \| f|_{S_n(v_0)} \|^2 \longrightarrow \infty$

If $|\lambda| < \rho \Rightarrow |\alpha| = |\beta| = 1$, $\alpha = \bar{\beta}$ $\quad c_1 = \bar{c}_2$

$\| f|_{S_n(v_0)} \|^2 = \left(2 \operatorname{Re}(c_1 \alpha^n) \right)^2$ check that $\alpha = \pm 1$,

$\Rightarrow |\operatorname{Re}(c_1 \alpha^n)| > \delta$ for some $\delta > 0$ and infinitly many n's

$\Rightarrow \| f_\lambda \|^2 = \infty$.

$\Rightarrow$ A has no L^2-eigenfunctions.

Exercise: If A is a selfadjoint bounded operator on a Hilbert space

$$\mathrm{Spec}(A) = \left\{\lambda \in \mathbb{C} : \exists\, f_m \in L^2 \text{ s.t. } \frac{\|Af_m - \lambda f_m\|}{\|f_m\|} \to 0\right\}$$

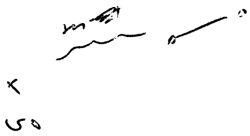
approximated eigenfunctions/ eigenvalues.

Back to T_k and $A = A_{T_k}$

$$f_m^\lambda = \begin{cases} f_\lambda & \text{on } B_m(v_0) \\ 0 & \text{otherwise} \end{cases}$$

$$(A-\lambda I)f_m(n) = \begin{cases} 0 & n \le m-1 \\ 0 & n \ge m+2 \\ f(m) & n = m+1 \\ -(k-1)f(m+1) & n = m \end{cases}$$

$(A-\lambda I)f_m(m+1) = f_\lambda(m)$



$(A-\lambda I)f_m(m) = f(m-1) - \lambda f(m) = f(m-1) - Af(m) = -(k-1)f(m+1)$

$$\frac{\|(A-\lambda I)f_m\|^2}{\|f_m\|^2} = \frac{\left[\|f|_{S(m)}\|^2 + \|f|_{S(m+1)}\|^2\right](k-1)}{\sum_{j=0}^{m}\|f|_{S(j)}\|^2} \le \frac{8(k-1)|C_1|^2}{\text{goes to infinity}} \xrightarrow[n\to\infty]{} 0$$

For $|\lambda| < |\rho|$ $\delta < \|f|_{S(n)}\|^2 = |2\,\mathrm{Re}(C_1\alpha^n)|^2 < 4|C_1|^2$

$\hookrightarrow$ for inf many n.

Conclusion: For $-\rho<\lambda<\rho$ ~~f^{λ}_{n}~~ is approximated eigenvalue

$\Rightarrow \lambda \in \mathrm{Spec}(A)$, i.e. $(-\rho,\rho) \subseteq \mathrm{Spec}(A)$

thm: $\mathrm{Spec}(A)$ is a closed set in $\mathbb{C}$

$\overset{*}{\Rightarrow} \pm\rho \in \mathrm{Spec}(A)$ or equiv. $[-\rho,\rho] \subseteq \mathrm{Spec}(A)$.

A self adjoint $\Rightarrow \mathrm{Spec}(A) \subseteq \mathbb{R}$.

$\boxed{\text{Goal } [-\rho,\rho] \text{ is the spectrum.}}$

For each $|\lambda|>\rho$ we will try to solve the equation

$(A-\lambda I)f = \delta_{v_0}$

Repeating the same argument you get a unique solution g_λ. Show that if $|\lambda|>\rho$, then $g_\lambda \in L^2$.

$\Rightarrow$ This shows that $\delta_v \in \mathrm{Im}(A-\lambda I)$ $\forall$vertex v

$\Rightarrow$ also all linear combinations of δ_v $= \mathrm{Im}(A-\lambda I) \Rightarrow \mathrm{Im}(A-\lambda I) = L^2(V)$

$\Rightarrow A-\lambda I$ is surjective

we already showed that $A-\lambda I$ is injective $\Rightarrow A-\lambda I$ is invertible.

(open mapping thm)

The case $\lambda=\rho$ $\quad f_\rho(n) = \frac{n(k-2)+k}{k(k-1)^{n/2}} \equiv \Xi$ Harish Chander Ξ function of T_k.

$\Xi(n)>0$

and it majorizes all f_λ for $\lambda\in[-\rho,\rho]$ $\left(\begin{array}{c}|f_\lambda(n)| \le \Xi(n) \\ \text{assuming } f_\lambda(0)=1\end{array}\right)$

Defn: $f\in\mathbb{C}^V$ is tempered if $f\in L^{2+\varepsilon}$ $\forall \varepsilon>0$.

Claim: For $\lambda\in[-\rho,\rho]$, f_λ is tempered, $\boxed{\text{assuming } k\geq 3}$.

Proof: $k\geq3$ Enough to show that Ξ is tempered.

$$\forall p>2 \quad \|\Xi\|_p^p = \sum_{n=0}^{\infty} \underbrace{k(k-1)^{n-1}}_{\left[\substack{\text{replace}\\ \text{by } 1 \text{ for}\\ n=0}\right]} \left(\frac{n(k-2)+k}{k(k-1)^{n/2}}\right)^p \underset{k\geq3}{\leq} \sum_{n=0}^{\infty} \frac{c(k)n}{\left((k-1)^{\frac{p}{2}-1}\right)^n} < \infty \qquad ▨$$

Graph Laplacian $\quad \Delta = I - \frac{1}{k}A$

$A\mathbb{1} = k\mathbb{1} \Rightarrow \Delta\mathbb{1}=0 \qquad \lambda\in \mathrm{spec}(A) \Rightarrow 1-\frac{\lambda}{k}\in \mathrm{spec}(\Delta)$

Δ determins how non constant function behave.

$f: \underset{\substack{\downarrow\\ \text{Euclidean}\\ \text{space}}}{\mathbb{E}^2} \to \mathbb{C} \qquad (Af)(p) = \text{"}\sum_{p'\sim p} f(p')\text{"}$

$$\Delta f(p) = f(p) - \text{"}\underset{p'\sim p}{\mathrm{avg}}(f(p'))\text{"}$$

$$\Delta f(p) = \lim_{r\to 0}\left(f(p) - \frac{1}{\mathrm{Vol}(S_r(p))}\int_{S_r(p)} f(t)\,dt \right)$$

$$= \lim_{r\to 0} f(p) - \frac{1}{2\pi r}\int_0^{2\pi} f(p_x + r\cos\theta,\ p_y + r\sin\theta)\, r\, d\theta$$

"Taylor"
$$= \lim_{r\to 0} f(p) - \frac{1}{2\pi}\int_0^{2\pi} f(p) + r\cos\theta\, f_x(p) + r\sin\theta\, f_y(p)$$

$$+ \frac{r^2\cos^2\theta}{2} f_{xx}(p) + \frac{r^2\sin^2\theta}{2} f_{yy}(p) + r^2\cos\theta\sin\theta\, f_{xy}(p)\ d\theta$$

$$+ \text{little order}$$

$$= \frac{r^2}{4\pi}\int_0^{2\pi}\left(\overset{\cos^2\theta}{f_{xx}(p)} + \overset{\sin^2\theta}{f_{yy}(p)}\right) d\theta = \frac{r^2}{4}\left(f_{xx}(p) + f_{yy}(p)\right) \xrightarrow[r\to 0]{} 0$$

We need to rescale by $\frac{1}{r^2}$ to get

$$\Delta f(p) \equiv \lim_{r\to 0}\frac{1}{r^2}\left(f(p) - \frac{1}{\mathrm{Vol}(S_r(p))}\int_{S_r(p)} f(t)\,dt \right) = -\frac{f_{xx}(p) + f_{yy}(p)}{4}$$

Finally:

$$\Delta_{\mathbb{E}^2} f(p) = -f_{xx}(p) - f_{yy}(p).$$

$$\Delta = \underset{\text{div}}{\nabla^*} \circ \underset{\text{grad}}{\nabla} \to \text{self adjoint.}$$

$\mathbb{E}^2 \Longleftrightarrow \mathbb{H}^2$

$\mathbb{Z}^2 \longleftrightarrow T_k$

$$\mathrm{Spec}(A|_{\mathbb{Z}^2}) = [-4,4] \longleftrightarrow \mathrm{Spec}(A_{T_k}) = [-\rho, \rho]$$

$A\mathbb{1} = 4\mathbb{1}$

$\mathbb{1}$ is not in L^2

but $\mathbb{1}|_{B_n(0)}$ are approximated eigenfunctions

$\mathbb{1}$ is not tempered.

All eigenfunctions are not in L^2 but are tempered.

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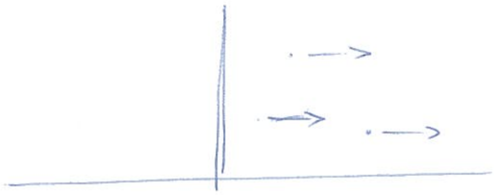
Ramanujan graphs and complexes - Lecture 5

$\mathbb{H} = \{x+iy \in \mathbb{C} : y>0\}$

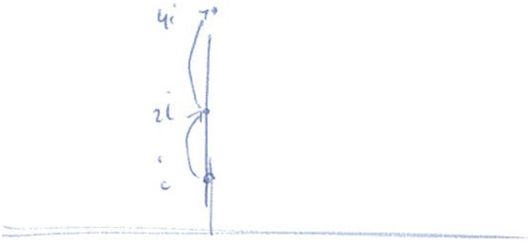
Prescribe geometry on $\mathbb{H}$ by providing the symmetry group (orientation preserving) $PSL_2(\mathbb{R})$.

$PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by $z \longmapsto \frac{az+b}{cz+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$

Möbius transformation.



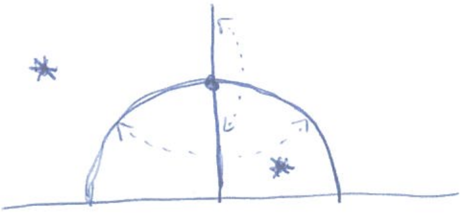
$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $z \longmapsto z+x$ $x \in \mathbb{R}$



$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$ $z \longmapsto az$ $a \in \mathbb{R}_{>0}$

$\Longrightarrow$ dist(iy_1, iy_2) only depend on $\frac{y_1}{y_2}$

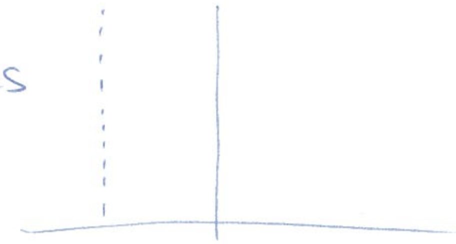
In fact dist$(iy_1, iy_2) = |\ln(\frac{y_1}{y_2})|$



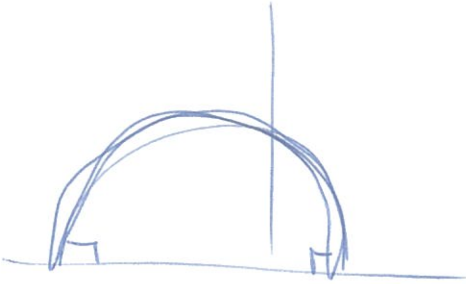
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $z \longmapsto -\frac{1}{z}$ inversion

dis$(\frac{i}{n}, i)$ = dist(ni, i)

Geodesics



or

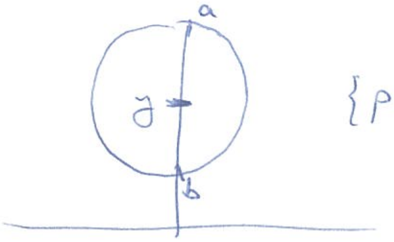


length	area	Laplacian	
$ds^2_{\text{Euclidean}} = dx^2+dy^2$	$da^2= dxdy$	$\Delta_{Euc} = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$	Euclidean
$ds^2_{\text{hyperbolic}} = \frac{dx^2+dy^2}{y^2}$	$da = \frac{dxdy}{y^2}$	$\Delta_{hyp} = y^2 \Delta_{Euc}$	Hyperbolic

Euclidean and hyperbolic metrics on $\mathbb{H} \subseteq \mathbb{C} \cong \mathbb{R}^2$ are conformal, namely angles are preserved (Since for a given point ds_i^2 are the same up to a scalar)

Circles are circles.

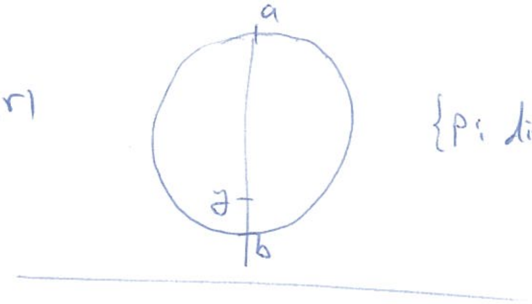
Euc $S(y;r)$



$\{p : dist_E(y,p)=r\} \Rightarrow a-y=y-b=r$

$\Rightarrow a=y+r \quad b=y-r$

Hyp $S(y,r)$



$\{p : dist_H(y,p)=r\}$

$\log\left(\frac{a}{y}\right) = \log\left(\frac{y}{b}\right) = r$

$a = ye^r \quad b = ye^{-r}$

The Euclidean center of this ball is at $y\cosh(r) = \frac{a+b}{2}$

Euclidean center of $S_r^{Hyp}(x+iy)$ is $x+y\cosh(r)$

Euclidean radius of $S_r^{Hyp}(x+iy)$ is $y\sinh(r)$

$$k_t(z) = \frac{\cos t\, z + \sin t}{-\sin t\, z + \cos t}$$

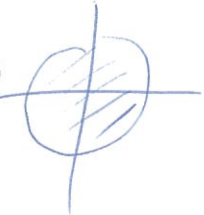
Since $G = PSL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$, and $K = SO(2)$ is the stabilizer of i we get bijections $G/K \xrightarrow{1:1} \mathbb{H}$ $gK = gi$

K maximal compact subgroup of G.

If G is a Lie group and K is a max compact sgroup. G/K has a geometry structure with $G \subseteq \text{Isom}$. G/K is then called the symmetric space for G.

$\mathbb{H}$ is the symmetric space for $PSL_2(\mathbb{R})$

$\mathbb{E}$ " " " " " $\text{Iso}^+(\mathbb{E}^2)$

Disc model for $\mathbb{H}$: $B_1(0) \subseteq \mathbb{C}$  denoted $\mathbb{D}$

The map $z \mapsto \frac{z-i}{z+i}$ takes you from the $\mathbb{H}$ model to the $\mathbb{D}$ model.

Tomorrow
$$\begin{cases} \Delta f = \lambda f \\ f(i) = 1 \\ f(k_t z) = f(z) \quad \forall t \in \mathbb{R} \end{cases}$$

③

$$(\Delta f)(x,y) = \lim_{r\to 0} f(x,y) - \frac{1}{\mathrm{Vol}(S_r^{\mathbb{H}}(x,y))}\int_0^{2\pi} f(x+\sinh(r)\cos t,\ y\cosh(r)+y\sinh(r)\sin(t))\,ds$$

$ds = ds(t) =$ arclength at the point $(x+\sinh(r)\cos t,\ y\cosh(r)+y\sinh(r)\sin t)$

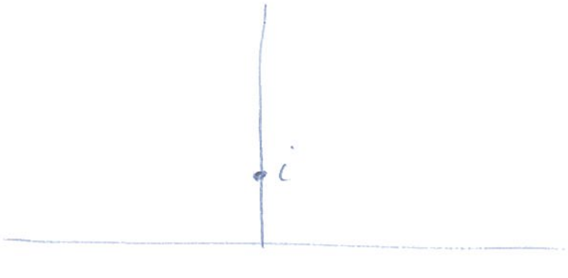
Taylor expansion gives $-\frac{1}{4}y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$

Furthermore $\mathrm{Spec}(\Delta)\subseteq[0,\infty)$ since $\Delta = \mathrm{div}\circ\mathrm{grad}$ so $\Delta\geq 0$.

~~Spherical~~

Spectrum of Δ on $\mathbb{H}$

Spherical functions around i:



$f(z)$ only depends on $\mathrm{dist}(z,i)$

Stabilizer of i in $PSL_2(\mathbb{R}) = \left\{\frac{ai+b}{ci+d}=i\right\} = SO(2)$ rotations in $\mathbb{E}^2$.

rotations in $\mathbb{H}$ around i and rotation in $\mathbb{E}$ are the same.

$\mathrm{Stab}_{\mathrm{Iso}^+(\mathbb{E}_2)}(0) = SO(2)$.

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Lecture 6

Study spectrum of the Δ.

Start with spherical eigenfunctions.

$f : \mathbb{H} \to \mathbb{C}$ $f(k_t z) = f(z)$, $k_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ $\forall t \in \mathbb{R}$

and $\Delta f = \lambda f$, $f(i) = 1$.

This is a boundary value problem of a PDE on $\mathbb{H}$ $\xrightarrow{\text{symmetry}}$ An ODE with boundary values on $[1,\infty)$

Rem: Spherical $\to$ $\nabla f(i) = 0$ an additional boundary condition.

Polar coordinates

$(t,r) \mapsto k_t(e^r i)$ $\Delta = \frac{1}{\sinh(r)} \frac{\partial}{\partial r}\left(\sinh(r) \frac{\partial}{\partial r}\right) + \frac{1}{\sin(r)^2} \frac{\partial^2}{\partial t^2}$

$\Rightarrow$ For f spherical

$$\begin{cases} \Delta f = \frac{1}{\sinh(r)} \frac{\partial}{\partial r}\left(\sinh(r) \frac{\partial f}{\partial r}\right) = \lambda f \\ f(1) = 1 \quad f'(1) = 0 \end{cases}$$

Change of variable : $x = \cosh(r)$

$$\begin{cases} (1-x^2) f''(x) - 2x f'(x) + \lambda f(x) = 0 \\ f(1) = 1 \\ f'(1) = 0 \end{cases}$$

The solution is called Legendre's P_α function.

Another way: Guess solutions.

$\forall m \in \mathbb{C}$ $\quad y^m$ is an e.f. of Δ

$$\Delta y^m = -y^2 \left(m(m-1) y^{m-2} \right) = ~~m~~ m(1-m) y^m$$

these are not in L^p because the integration over x is giving ∞.

We can sphericlize: $\displaystyle (y^m)_{sph} (k_t e^r i) = \int_0^{2\pi} \left(\mathrm{Im} (k_\theta e^r i) \right)^m d\theta$

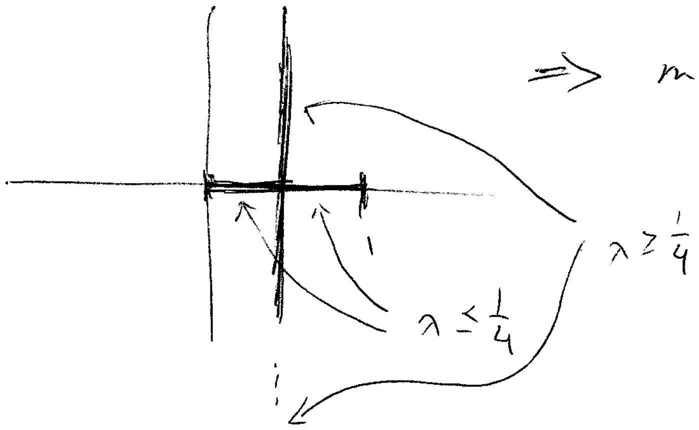
$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{2e^r}{1+e^{2r} - (e^{2r}-1) \cos(2\theta)} \right)^m d\theta = \text{elliptic integral of the second kind}$$

When is $(y^m)_{sph}$ in $L^2(\mathbb{H})$ or is tempered?

Δ is self adjoint and positive semidefinite. Every e.v. and $\lambda \in \mathrm{Spec}(\Delta|_{L^2(\mathbb{H})})$ is ≥ 0.

$\Rightarrow$ If $(y^m)_{sph} \in L^2$ or $\cap L^{2+\varepsilon}$ then $m(1-m) \geq 0$ $\quad m \in \mathbb{C}$

$\Rightarrow m \in [0,1]$ or $m \in \frac{1}{2} + i\mathbb{R}$.



Starting from $\lambda = m(1-m)$ $\quad m = \frac{1 \pm \sqrt{1-4\lambda}}{2}$

Check: for $\lambda = \frac{1}{4}$ $m = \frac{1}{2}$ also $\sqrt{y}\log y$ is an eigenfunction.

Fact: For $m \in \frac{1}{2} + i\mathbb{R} \Leftrightarrow \lambda \geq \frac{1}{4}$ $\quad (y^m)_{sph} \in L^{2+\varepsilon}$ $\forall \varepsilon$

$m \in [0,1] \setminus \{\frac{1}{2}\} \Leftrightarrow \lambda < \frac{1}{4}$ $\quad$ No!

$\Rightarrow \mathrm{Spec}(\Delta|_{\mathbb{H}}) = [\frac{1}{4}, \infty)$.

At the critical point

We call $(\sqrt{y})_{sph}$ the Harish-Chandra Ξ-function. It dominates all $(y^m)_{sph}$ for $m \in \frac{1}{2} + i\mathbb{R}$ $(\lambda \geq \frac{1}{4})$.

$f_\lambda = (y^m)_{sph} = P_{m-1}$ Legendre Poly.

In particular $\Xi(k_t e^r i) = P_{-\frac{1}{2}}(\cosh(r))$.

Using Taylor we get $\Xi(k_t e^r i) = \frac{\sqrt{2}\,(3\log(2) + \log(\cosh(r)))}{\pi\sqrt{\cosh(r)}} + O((\cosh r)^{-3/2})$

as $r \to \infty$

$$\sim \frac{r}{e^{r/2}}$$

$$\|\Xi\|_p^p = C\int_0^\infty \text{length}(S_{e^r}(i)) \cdot \left(\frac{r}{e^{r/2}}\right)^p dr \approx C\int_0^\infty e^r \left(\frac{r}{e^{r/2}}\right)^p dr \underset{\substack{\Updownarrow \\ \boxed{p>2}}}{<} \infty$$

Summary

	T_k	$\mathbb{H}$
Spectrum	$[-2\sqrt{k-1}, 2\sqrt{k-1}]$	$[\frac{1}{4}, \infty)$
Ramanujan	$Spec \subseteq \{\pm k\} \cup Spec(T_k)$	?

Every k-reg graph has universal cover T_k so $\Gamma \backslash T_k = X$

for $\Gamma \leq Aut(T_k)$.

Hyperbolic surface $\Gamma \backslash \mathbb{H}$ $\Gamma \leq Iso(\mathbb{H}) = PSL_2(\mathbb{R})$ Γ-discrete

We can study $\Delta_{Hyp}^{on}(\Gamma \backslash \mathbb{H})$

$\Gamma\backslash\mathbb{H}$ is called Ramanujan surfaces

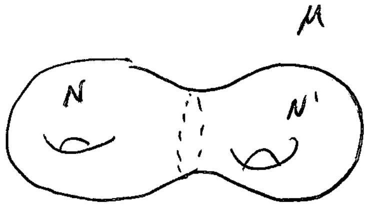
$$\mathrm{Spec} \subseteq \underset{\text{constant func.}}{\{0\}} \cup \left[\tfrac{1}{4}, \infty\right)$$

Cheeger - Buser - Mazya inequality

If X is a ~~hyperbolic surface~~ complete Riemannian manifold of finite volume and λ 2nd e.v. of Δ, then

$$\frac{h^2}{4} \leq \lambda \leq \underset{\text{curvature}}{C}h + 10h^2$$

$$h(M) = \min_{M = N \amalg N'} \frac{\overset{\partial N}{\overset{\shortparallel}{\mathrm{Vol}(N \cap N')}}}{\mathrm{Vol}(N) \underset{\text{min}}{\wedge} \mathrm{Vol}(N')}$$



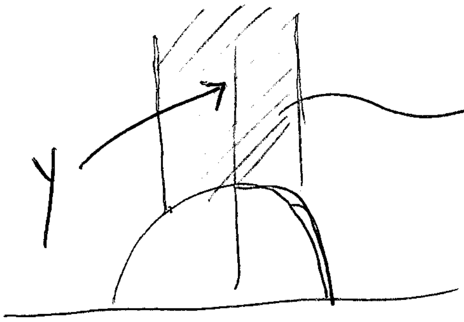
Conj (Selberg): For arithmetic Γ $\Gamma\backslash\mathbb{H}$ is Ramanujan.

e.g. $\Gamma = PSL_2(\mathbb{Z})$ or $\Gamma(N) = \{A \in PSL_2(\mathbb{Z}) : A \equiv I \bmod N\}$.

Thm (Selberg): $\lambda(\Gamma\backslash\mathbb{H}) \geq \frac{3}{16}$.

$\Gamma = \Gamma(N)$ for some N (maybe also true for general arithmetic Γ)

$X(N) = \Gamma(N) \backslash \mathbb{H}$ $\quad$ $X(1) = PSL_2(\mathbb{Z}) \backslash \mathbb{H}$

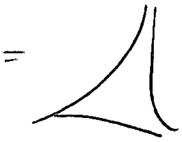


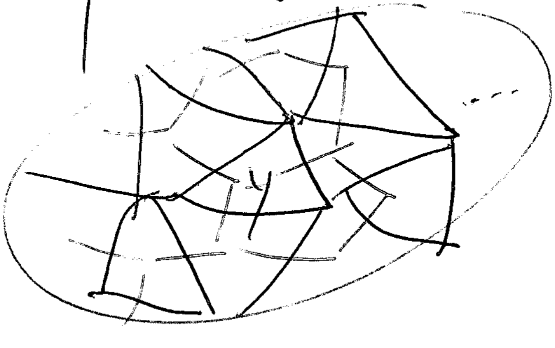
fundamental domain for $\Gamma(1) \rightsquigarrow \lambda \approx 80$

Since $\Gamma(N) \trianglelefteq \Gamma(1)$ a fund. domain for $\Gamma(N)$ can be obtained by gluing $\Gamma(1)$-translations of Y.

Selberg $\rightarrow h > \varepsilon > 0$.

$X(17)$ is composed of many copies of Y triangulation

$Y =$ 



Y is quite connected due to Buser's inequality.

The dual graph is a 3-regular graph which is a good expander

$$\begin{tikzcd}
PSL_2(\mathbb{R}) \arrow[r] \arrow[d] & \mathbb{H} \\
PSL_2(\mathbb{Q}_p) \arrow[r] & T_k
\end{tikzcd}$$

Ihara / Serre / Tits / Bruhats

Ramanujan graphs and complexes 17/11/17

Lecture 7

Goal: For $k=p^r+1$, T_k has an "arithmetic structure".

$$X_p^d = PGL_d(\mathbb{Z}[\tfrac{1}{p}]) / PGL_d(\mathbb{Z}),$$

① $PGL_d(\mathbb{Z})$ inv. $d\times d$ matrices with entries in $\mathbb{Z}$ with determinant ± 1 $/ \{\pm I\}$

Rem:
In general $GL_d(R) = \{A \in M_{d\times d}(\underset{\to \text{ a ring of coefficients}}{R}) : \det(A) \in \overset{\text{inv in } R}{R^*}\} \equiv \{A \in M_{d\times d}(R) : A^{-1} \in M_{d\times d}(R)\}$

②

$$\left(\mathbb{Z}[\tfrac{1}{p}] = \left\{ \frac{a}{p^m} \in \mathbb{Q} : \begin{matrix} a \in \mathbb{Z} \\ m \in \mathbb{Z} \end{matrix} \right\} \text{ - the minimal ring containing } \mathbb{Z} \text{ and } \frac{1}{p} \right)$$

$$PGL_d(\mathbb{Z}[\tfrac{1}{p}]) = \left\{ A \in M_{d\times d}(\mathbb{Z}[\tfrac{1}{p}]) : \det A \in \mathbb{Z}[\tfrac{1}{p}]^* = \{\pm p^m : m \in \mathbb{Z}\} \right\} \Big/ \{\pm p^m\}$$

Example: $d=2$

$$PGL_2(\mathbb{Z}) \ni \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$PGL_2(\mathbb{Z}[\tfrac{1}{p}]) \ni \begin{pmatrix} p & 3 \\ 0 & 1 \end{pmatrix}$$

Clearing denominators

$PGL_d(\mathbb{Z}[\frac{1}{p}]) = \{A \in M_d(\mathbb{Z}) : \det = \pm p^n\} / \{\pm p^m\}$ we can always multiply by p^n and get a matrix in $M_d(\mathbb{Z})$

Take $A \in PGL_d(\mathbb{Z}[\frac{1}{p}])$. there is a $\overset{\boxed{\text{unique}}}{\vee}$ scaling $\overset{\text{by a power of } p}{\vee}$ (rep. of) A with entries in $\mathbb{Z}$ and coprime entries $\gcd\{a_{ij} : 1 \le i,j \le d\} = 1$,

[Why does $p^m A$ has integral coprime entries for some m.
After clearing denominators $\det(\underbrace{p^m A}_{\in \mathbb{Z}}) = p^{dm} \det(A) = p^{dm+k} \longrightarrow$
No prime different than p divides all entries of $p^m A$.

$\Rightarrow$ We can identify $GL_d(\mathbb{Z}[\frac{1}{p}])/_{p^m}$ with $\left\{ \begin{array}{l} \text{integral } \overset{\text{coprime entries}}{\text{primitive}} \\ \text{matrices with det} \\ \text{a power of } p \end{array} \right\}$

A lattice in $\mathbb{Z}^d$ is the $\mathbb{Z}$-span of d indep $\overset{\text{over } \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{R}}{\vee}$ vectors.

$= A \cdot \mathbb{Z}^d$ for some $A \in M_d(\mathbb{Z})$, $\det(A) \neq 0$

When is $A\mathbb{Z}^d = B\mathbb{Z}^d$? Iff $B = AC$, where $C \in GL_d(\mathbb{Z})$

(If $A = BC$, then $A\mathbb{Z}^d = BC\mathbb{Z}^d = B\mathbb{Z}^d$
If $A\mathbb{Z}^d = B\mathbb{Z}^d \Rightarrow$ every vector in $A\mathbb{Z}^d$ is spanned by vectors of $B \Rightarrow \exists C$ s.t. $B\overset{\in M_d(\mathbb{Z})}{C} = A$
$\exists C' \in M_d(\mathbb{Z})$ $AC' = B$ $\Rightarrow$ $C'C = I$ $\Rightarrow$ $C \in GL_d(\mathbb{Z})$)

So $\left\{ \begin{matrix} \text{integral} \\ \text{lattices} \end{matrix} \right\} \longleftrightarrow \{ A \in M_d(\mathbb{Z}) : \det A \neq 0 \} / GL_d(\mathbb{Z})$

$\left\{ \begin{matrix} \text{primitive} \\ p\text{-lattices} \end{matrix} \right\} \longleftrightarrow \left\{ A \in M_d(\mathbb{Z}) : \begin{matrix} \det A = \pm p^m \\ A \text{ primitive} \end{matrix} \right\} / GL_d(\mathbb{Z})$

- Primitive lattice: $L = A\mathbb{Z}^d$ such that $\frac{L}{m}$ is not a lattice for $m > 1$ $\left(\frac{L}{m} \not\subseteq \mathbb{Z}^d \text{ any more} \right)$

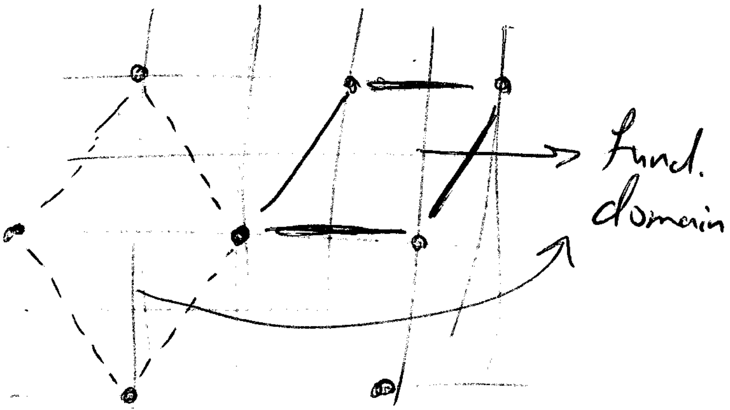
$L = A\mathbb{Z}^d$ is primitive $\underset{\substack{+ \\ \text{well} \\ \text{defined}}}{\Longleftrightarrow}$ A is primitive.

well defined $\longrightarrow$ indep of the choice of A generating the lattice.

- p-lattice: Lattice with covolume a power of p

Co-Vol(L) = Vol$(\mathbb{R}^d / L)$ = Vol of fundamental domain

$= |\det(A)|$
for A s.t. $L = A\mathbb{Z}^d$.



$$\left\{\begin{matrix}\text{primitive}\\ p\text{-lattices}\end{matrix}\right\} \longleftrightarrow \left\{A \in M_d(\mathbb{Z}) : \begin{matrix}\det A = \pm p^n\\ \text{primitive}\end{matrix}\right\} \Big/ GL_d(\mathbb{Z}) \longleftrightarrow PGL_d(\mathbb{Z}[\tfrac{1}{p}]) \Big/ PGL_d(\mathbb{Z})$$

explanation

$$PGL_d(\mathbb{Z}[\tfrac{1}{p}]) = \left\{\begin{matrix}\text{integral}\\ \text{primitive}\\ p\text{-matrices}\end{matrix}\right\} \Big/ \pm 1$$

$$\Rightarrow PGL_d(\mathbb{Z}) = GL_d(\mathbb{Z}) \big/ \pm 1$$

$$PGL_d(\mathbb{Z}[\tfrac{1}{p}]) \Big/ PGL_d(\mathbb{Z}) = \left\{\begin{matrix}\text{inte. prim}\\ p\text{-matrices}\end{matrix}\right\} \Big/ \{\pm 1\} \Bigg/ GL_d(\mathbb{Z}) \big/ \{\pm 1\}$$

$$= \left\{\begin{matrix}\text{inte. prim.}\\ p\text{-matrices}\end{matrix}\right\} \Big/ GL_d(\mathbb{Z}) = \left\{\begin{matrix}\text{primitive}\\ p\text{-lattices}\end{matrix}\right\}$$

Lecture 8

Integral lattice in $\mathbb{Z}^d$ - $\mathbb{Z}$ span of d linearly indep. vectors of $\mathbb{Z}^d$.

= Subgroup of $\mathbb{Z}^d$ not contained in any (proper) subspace of $\mathbb{R}^d$.

= $A\mathbb{Z}^d$ for some $A \in M_d(\mathbb{Z})$ with $\det A \neq 0$

When is $A\mathbb{Z}^d = B\mathbb{Z}^d$? Iff $A = BC$ for some $C \in GL_d(\mathbb{Z})$.

Fix a prime p.

Bijections

$$A\mathbb{Z}^d \xleftrightarrow{\text{bij}} \{A \in M_d(\mathbb{Z}) : \det A \neq 0\} / GL_d(\mathbb{Z})$$

$\downarrow$ primitive

Primitive p-lattice

$$A\mathbb{Z}^d \xleftrightarrow{\text{bij}} \{A \in M_d(\mathbb{Z}) : |\det A| = p^m \text{ primitive}\} / GL_d(\mathbb{Z})$$

$$\xleftrightarrow{\text{bij}} GL_d(\mathbb{Z}[\tfrac{1}{p}]) / GL_d(\mathbb{Z})$$

lattices		matrices $M_d(\mathbb{Z})$
$A\mathbb{Z}^d$	$\leftarrow$	A
primitive lattice L $\frac{1}{m}L \not\subseteq \mathbb{Z}^d$ $\forall m > 1$	$\longleftrightarrow$	primitive matrix A $\gcd(A_{ij}) = 1$
p-lattice $\mathrm{covol}(L) = p^m$ for some $m \in \mathbb{N}$	$\longleftrightarrow$	$\lvert\det A\rvert = p^m$

We are interested in $X_p^d = \{$all primitive p-lattices in $\mathbb{Z}^d\}$

Claim: $X_p^d \cong PGL_d(\mathbb{Z}[\frac{1}{p}]) / PGL_d(\mathbb{Z})$

Each A in $PGL_d(\mathbb{Z}[\frac{1}{p}])$ has a unique scaling by some p^m which is integral and primitive.

<u>Corollary</u>: $PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts transitively on X_p^d.

<u>Goal</u>: Find rep. for the cosets.

Say A is a primitive, integral p-matrix $|\det(\cdot)| = p^m$ for $m \in \mathbb{N}$

<u>Take $d=2$</u>

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ elementry op. over $\mathbb{Z}$ do not change the column span $\Leftrightarrow$ the lattice $\longleftrightarrow$ left $GL_d(\mathbb{Z})$ coset .

[switch columns
multi a column by ± 1
add colum to another]

Using euclid's alg for gcd we obtain $\begin{pmatrix} x & y \\ 0 & \gcd(c,d) \end{pmatrix}$

$\det(\cdot) = x \cdot \gcd(c,d) = \pm p^m \Rightarrow x$ and $\gcd(c,d)$ are powers of $\pm p$

$\Rightarrow$ by multiplying column by ± 1 we can assume both are positive

$A\mathbb{Z}^d = \begin{pmatrix} p^m & z \\ 0 & p^n \end{pmatrix}\mathbb{Z}^d$ by adding / substracting column 1 from 2 we can assume that $0 \leq z \leq p^m - 1$. Finally,

Claim: $X_2^p \longleftrightarrow \left\{ \begin{pmatrix} p^n & a \\ 0 & p^m \end{pmatrix} : \begin{array}{l} 0 \le a < p^m \quad m,n \ge 0 \\ \text{either } n=0 \text{ or } m=0 \text{ or } n,m>0 \text{ and } p \nmid a \end{array} \right\}$

Ex: Prove $\longleftarrow$. Each such matrix gives a diff lattice.

For general d we can do the same

$$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \mathbb{Z}^d \longleftrightarrow \begin{pmatrix} p^m & a & b \\ 0 & p^n & c \\ 0 & 0 & p^l \end{pmatrix} \quad \boxed{\begin{array}{l} 0 \le a,b \le p^m-1 \\ 0 \le c \le p^n-1 \\ \gcd(p^m,p^n,p^l,a,b,c)=1 \end{array}}$$

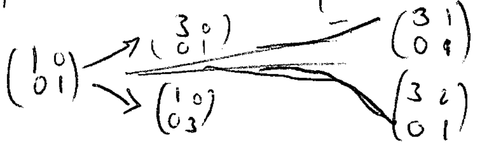
$$X_d^p \longleftrightarrow \left\{ \begin{pmatrix} p^{n_1} & & a_{ij} \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} : \begin{array}{l} 0 \le a_{ij} \le p^{n_i}-1 \qquad 1 \le i \le d \\ \gcd(p^{n_i}, a_{ij}) = 1 \end{array} \right\}$$

X_d^p are the vertices of the Affine Bruhat-Tits building of type $\tilde{A}_{d-1}$ over $\mathbb{Z}[\frac{1}{p}]$.

Claim: X_2^p is a $(p+1)$-reg tree.

Edges (First attempt): We say that (L_1, L_2) is an oriented edge if $L_2 \overset{p}{<} L_1$ or $pL_2 \overset{p}{<} L_1$, where $X \overset{p}{<} Y$ means $X<Y$ and $[Y:X]=p$.

e.g. $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \overset{3}{\le} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$A\mathbb{Z}^2 \overset{3}{\le} \mathbb{Z}^2 \Rightarrow \det A = 3$

$$\begin{tikzcd}
\begin{pmatrix}1&0\\0&1\end{pmatrix} \arrow[r, "e_\infty"] \arrow[dr, "e_0"] \arrow[ddr, "e_1"] \arrow[dddr, "e_2"] & \begin{pmatrix}1&0\\0&3\end{pmatrix} \\
& \begin{pmatrix}3&0\\0&1\end{pmatrix} \\
& \begin{pmatrix}3&1\\0&1\end{pmatrix} \\
& \begin{pmatrix}3&2\\0&1\end{pmatrix}
\end{tikzcd}$$

$d=3$

$$\begin{tikzcd}
\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \arrow[r, "e_{\infty\infty}"] \arrow[dr, "e_{\infty a}"'] \arrow[ddr, "e_{bc}"'] & \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \\
& \begin{pmatrix}1&&\\&p&a\\&&1\end{pmatrix} \quad 0\leq a\leq p-1 \\
& \begin{pmatrix}p&b&c\\&1&0\\&&1\end{pmatrix} \quad b,c=0,\dots p-1
\end{tikzcd}$$

in X_d^p there are $p^{d-1}+p^{d-2}+\dots+p+1$ edges leaving $\mathbb{Z}^d$

There is no $\mathbb{Z}^2 \overset{p}{<} L$, so when is $p\mathbb{Z}^2 \overset{p}{<} L$ $\quad \begin{pmatrix}p&0\\0&p\end{pmatrix} \overset{p}{<} L$

$\Rightarrow$ covol $L = p$

$$\begin{tikzcd}
\begin{pmatrix}p&0\\0&1\end{pmatrix} \arrow[ddr] & \\
\begin{pmatrix}p&1\\0&1\end{pmatrix} \arrow[dr] & \\
\vdots & \begin{pmatrix}1&0\\0&1\end{pmatrix} \\
\begin{pmatrix}p&p-1\\0&1\end{pmatrix} \arrow[ur] & \\
\begin{pmatrix}1&0\\0&p\end{pmatrix} \arrow[uur] &
\end{tikzcd}$$

The bijection between directed edges in $d=2$ does not exists in $d\geq 3$.

<u>Claim</u>: $\forall A \in X_p^d$ A has out deg $\frac{p^d-1}{p-1}$ with the following endpoints $A\cdot A_1, \dots, A\cdot A_{\frac{p^d-1}{p-1}}$, where $(A_i)_{i=1}^{\frac{p^d-1}{p-1}}$ are the out neighbors of $\mathbb{Z}^d$.

When thinking about X_p^d as $PGL_d(\mathbb{Z}[\frac{1}{p}])$

e.g. $\begin{pmatrix}1&\\&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&\\&3\end{pmatrix}$

primitive lattice generated

$$\begin{pmatrix}3&\\&1\end{pmatrix} \longrightarrow \begin{pmatrix}3&\\&1\end{pmatrix}\begin{pmatrix}1&\\&3\end{pmatrix} = \begin{pmatrix}3&\\&3\end{pmatrix} \overset{\downarrow}{=} \begin{pmatrix}1&\\&1\end{pmatrix}$$

<u>Claim</u>: $\Gamma \equiv PGLd(\mathbb{Z}[\frac{1}{p}])$ acts on the graph thus obtained. Furthermore, if we have started with the edge $\begin{pmatrix}1&\\&1\end{pmatrix} \to \begin{pmatrix}p&\\&1\end{pmatrix}$ and required $PGLd(\mathbb{Z}[\frac{1}{p}])$ action the graph we construct, we will be forced to have all these edges. (Since Γ has the elements $\begin{pmatrix}1&1\\&1\end{pmatrix}\begin{pmatrix}1&2\\&1\end{pmatrix}\cdots\begin{pmatrix}1&p-1\\&1\end{pmatrix}\begin{pmatrix}&1\\1&\end{pmatrix}$, those are row operation taking $\begin{pmatrix}1&\\&1\end{pmatrix}$ to itself and $\begin{pmatrix}p&\\&1\end{pmatrix}$ to $\begin{pmatrix}p&1\\&1\end{pmatrix}\cdots\begin{pmatrix}p&p-1\\&1\end{pmatrix}\begin{pmatrix}1&\\&p\end{pmatrix}$)

$$\begin{pmatrix}1&j\\0&1\end{pmatrix}\begin{pmatrix}1&\\&1\end{pmatrix} = \begin{pmatrix}1&j\\0&1\end{pmatrix} \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad A\equiv B \text{ if } A\in B\cdot GLd(\mathbb{Z}).$$

Ramanujan graphs and complexes ① 26/11/17

Lecture

Building of PGL$_d$

vertices

① $PGL_d\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) / PGL_d(\mathbb{Z})$ ~~[illegible]~~

$\quad\quad \shortparallel$

② Primitive p-lattices

$\quad\quad \shortparallel$

③ $X_p^d = \left\{ \begin{pmatrix} p^{n_1} & & a_{ij} \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} : \begin{array}{c} 0 \le a_{ij} < p^{n_i} \\ \text{a primitive matrix} \end{array} \right\}$

Three ways to think about the vertices

edges

② $L_1 \rightarrow L_2$ if $L_2 \overset{p}{\le} L_1$ or $pL_2 \overset{p}{\le} L_1$

③ Define $\frac{p^d-1}{p-1}$ matrices $N_j = \begin{pmatrix} 1 & & & & & 0 & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & p & * & * \cdots * & \\ 0 & & & & 1 & & \\ & & & & & \ddots & 0 \\ & & & & & & 1 \end{pmatrix} \rightarrow 0 \le * \le p-1 \qquad 1 \le j \le \frac{p^d-1}{p-1}$

then the outgoing nieghbors of $A \in X_p^d$ are $A \cdot N_j$ for $1 \le j \le \frac{p^d-1}{p-1}$, where if $AN_j \notin X_p^d$ we fix it by recalling that $AN_j \in PGL_d\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$ and hence has a unique rep. in X_p^d.

example

$$\begin{pmatrix} 8 & 5 \\ & 4 \end{pmatrix}\begin{pmatrix} 1 & \\ & 2 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ & 8 \end{pmatrix} \underset{\text{primitivity}}{\equiv} \begin{pmatrix} 4 & 5 \\ & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ & 4 \end{pmatrix}$$

one only need to divide by p and reduce a_{ij} mod p^{n_i} to get the rep. in X_p^d

Claim: X_p^2 is a tree.

Instead of writing $N_1, \dots N_{p+1}$ we write

$N_0 = \begin{pmatrix} p & \\ & 1 \end{pmatrix}$ $N_1 = \begin{pmatrix} p & 1 \\ & 1 \end{pmatrix}$ --- $N_{p-1} = \begin{pmatrix} p & p-1 \\ & 1 \end{pmatrix}$ and $N_\infty = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$

① X_p^2 is symmetric, i.e. $A \mapsto B \iff B \mapsto A \quad \forall A, B \in X_p^2$.

Assume $A = \begin{pmatrix} p^m & a \\ 0 & p^n \end{pmatrix}$. Then $AN_0 = \begin{pmatrix} p^{m+1} & a \\ & p^n \end{pmatrix}$ and $(AN_0)N_\infty = \begin{pmatrix} p^{m+1} & pa \\ & p^{n+1} \end{pmatrix} \equiv \begin{pmatrix} p^m & a \\ & p^n \end{pmatrix} = A$

$\Rightarrow A \longmapsto AN_0 \longrightarrow A$.

Similarly $A \longmapsto AN_j = \begin{pmatrix} p^{m+1} & a + jp^m \\ & p^n \end{pmatrix}$ and $AN_1N_\infty = \begin{pmatrix} p^{m-1} & pa + jp^{m+1} \\ & p^{n+1} \end{pmatrix} \equiv \begin{pmatrix} p^m & \\ & p^n \end{pmatrix}$

So $A \longmapsto AN_j \longmapsto A$. for $j = 1, \dots p-1$ $\equiv A$.

Finally, $A \longmapsto AN_\infty = \begin{pmatrix} p^m & pa \\ & p^{n+1} \end{pmatrix} \overset{\text{fix}}{\equiv} \begin{pmatrix} p^m & pa \bmod p^m \\ & p^{n+1} \end{pmatrix}$ (+ maybe divide by p)

Write $a = jp^{m-1} + t \quad t \in \{0, \dots, p^{m-1} - 1\}$, then $pa \bmod p^m = pt$, then

$(AN_\infty)N_j = \begin{pmatrix} p^m & \overset{\bmod p^m}{pa} \\ & p^{n+1} \end{pmatrix} \begin{pmatrix} p & j \\ & 1 \end{pmatrix} = \begin{pmatrix} p^{m+1} & jp^m + pt \\ 0 & p^{n+1} \end{pmatrix} = \begin{pmatrix} p^{m+1} & pa \\ & p^{n+1} \end{pmatrix} = \begin{pmatrix} p^m & a \\ & p^n \end{pmatrix}$

example $\begin{pmatrix} 4 & 3 \\ & 2 \end{pmatrix} \xrightarrow{\infty} \begin{pmatrix} 4 & 6 \\ & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 2 \\ & 4 \end{pmatrix}$

$\begin{pmatrix} 4 & 1 \\ & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 2 \\ & 4 \end{pmatrix}$

③

Define level structure on $PGL_d(\mathbb{Z}[\frac{1}{p}])/$ on X_p^d

$\text{level}(A) = \log_p(\det A)$ for $A \in X_p^d$

on $PGL_d(\mathbb{Z}[\frac{1}{p}])$ the level of an element is the level of the $PGL_d(\mathbb{Z})$ - rep in X_p^d.

$$\text{level}\begin{pmatrix} p^m & a \\ & p^n \end{pmatrix} = m+n$$

② Claim: $\exists$ a path of length $= \text{level}(A)$ from I to A.

Proof: Instead we will show that there is such a path from A to I. By ① (Symmetry) the claim will follow.

$$A = \begin{pmatrix} p^m & a \\ & p^n \end{pmatrix}$$

if $m>0$ $\quad AN_\infty = \begin{pmatrix} p^m & pa \overset{\%}{\cancel{\text{mod}}}\, p^m \\ 0 & p^{n+1} \end{pmatrix} = \begin{pmatrix} p^{m-1} & a \text{ mod } p^{m-1} \\ & p^n \end{pmatrix}$ mod $\equiv$ % $\longrightarrow$ level $=(m+n$

if $m=0$ and $n>0$ $\quad AN_0 = \begin{pmatrix} 1 & 0 \\ & p^n \end{pmatrix}\begin{pmatrix} p & \\ & 1 \end{pmatrix} = \begin{pmatrix} p & \\ & p^n \end{pmatrix} = \begin{pmatrix} 1 & \\ & p^{n-1} \end{pmatrix} \rightarrow$ level $= n-1$

$m=n=0$ $\quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

③ Claim: For $A \neq Id$ in $(AN_i)_{i=0,\dots,p-1,\infty}$ there are p vertices at level $= \text{level}(A)+1$ and 1 vertex at level $= \text{level}(A) - 1$.

Proof:

Case 1 $\begin{pmatrix} p^n & a \\ & 1 \end{pmatrix}$ $n>0$ level n

$$\begin{pmatrix} p^n & a \\ & 1 \end{pmatrix} N_j = \begin{pmatrix} p^{n+1} & j p^n + pa \\ & 1 \end{pmatrix} \quad \text{level } n+1$$

$$\begin{pmatrix} p^n & a \\ & 1 \end{pmatrix} N_\infty = \begin{pmatrix} p^n & pa \\ & p \end{pmatrix} = \begin{pmatrix} p^{n-1} & a\,p^{n-1} \\ & 1 \end{pmatrix} \quad \text{level } n-1$$

Case 2 $\begin{pmatrix} 1 & 0 \\ & p^m \end{pmatrix}$ $m>0$ level m

$$\begin{pmatrix} 1 & \\ & p^m \end{pmatrix} N_j = \begin{pmatrix} p & j \\ & p^m \end{pmatrix} \quad \begin{cases} \text{level } m+1 & j \neq 0 \\ \text{level } m-1 & j=0 \end{cases}$$

$$\begin{pmatrix} 1 & \\ & p^m \end{pmatrix} N_\infty = \begin{pmatrix} 1 & 0 \\ 0 & p^{m+1} \end{pmatrix} \quad \text{level } m+1$$

Case 3 $\begin{pmatrix} p^n & a \\ & p^m \end{pmatrix}$ $p \nmid a$ level $= n+m$

Exercise

Cor: X_p^2 is a tree.

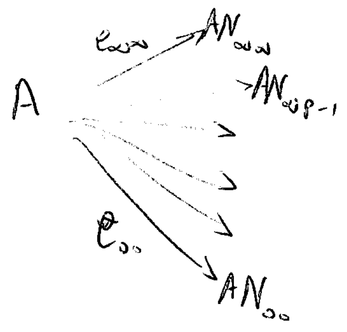
Lecture 10

The outgoing edges of I are $(N_i)_{i=1}^{\frac{p^d-1}{p-1}}$. For general A the outgoing edges are $(\text{fixed}(AN_i))_{i=1}^{\frac{p^d-1}{p-1}}$

(Bringing it back to X_p^d)

X_p^2 is a symmetric $(p+1)$-regular tree.

$$X_p^3 \quad \begin{array}{c}\begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix}\\ \shortparallel \\ N_{\infty\infty}\end{array} \quad \begin{array}{c}\begin{pmatrix}1&&\\&p&x\\&&1\end{pmatrix}\\ \shortparallel \\ N_{x\infty}\end{array} \quad \begin{array}{c}\begin{pmatrix}p&x&y\\&1&\\&&1\end{pmatrix}\\ \shortparallel \\ N_{xy}\end{array}$$



p^2+p+1 outgoing neighbors.

Unlike the case $d=2$, the out-edges here are not the same as in-edges. E.g.,

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \xrightarrow{e_{xy}} \begin{pmatrix}&&\\&&\\&&\end{pmatrix}$$

↳ matrix with $\det p^2$ $\longrightarrow$ after fixing $\det = p^{2-3m} \neq 1$

There is no going back.

However

$$\begin{tikzcd}
\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \arrow[r, "e_{\infty\infty}"] & \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \arrow[r, "e_{0,\infty}"] & \begin{pmatrix}1&&\\&p&\\&&p\end{pmatrix} \arrow[ll, bend left, "e_{0,0}"]
\end{tikzcd}$$

we got paths of length 3.

$$\begin{tikzcd}
\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \arrow[r, "e_{\infty,\infty}"] & \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \arrow[r, "e_{\infty,\infty}"] & \begin{pmatrix}1&&\\&1&\\&&p^2\end{pmatrix} \arrow[ll, dashed, bend left, "\nexists\,X"]
\end{tikzcd}$$

What are the triangles of $\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix}$

$$\begin{tikzcd}
& \begin{pmatrix}p&&\\&1&\\&&p\end{pmatrix} \arrow[dl, "e_{0,\infty}"'] & & \begin{pmatrix}1&&\\&p&x\\&&p\end{pmatrix} & \text{those don't go back} \\
\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \arrow[rr, "e_{\infty\infty}"] & & \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \arrow[ul, "e_{\infty 0}"'] \arrow[ll, dashed, bend left] \arrow[ur] \arrow[dl, "e_{0\infty}"] \arrow[dr, "e_{xy}"] & & \\
& \begin{pmatrix}1&&\\&p&\\&&p\end{pmatrix} \arrow[ul, "e_{\infty 0}"] & & \begin{pmatrix}p&x&y\\&1&\\&&p\end{pmatrix} \arrow[ulll, dashed, "\text{for } y=0"] & \text{those go back iff } y=0
\end{tikzcd}$$

In total the edge is contained in p triangles.

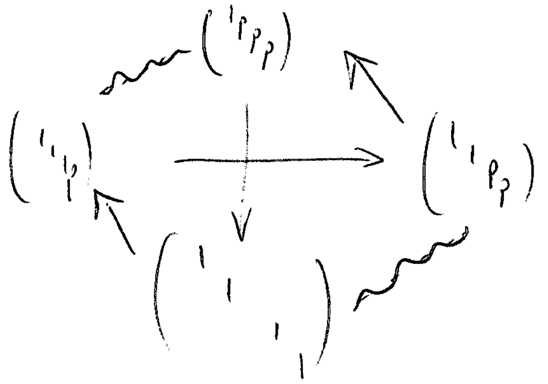
The building of $PGL_d(\mathbb{Z}[\frac{1}{p}])$ is defined as follows:

Vertices X_p^d

$(d-1)$-cells $\{v_1, \dots, v_d\}$ s.t. $\exists$ path $v_1 \to v_2 \to \dots \to v_d \to v_1$

pure $(d-1)$-dim. complex

$d=4$ thetraedra



one need to add two of the edges to get the thetraedra.

<u>2</u>

2nd defn: X_p^d is the flag complex of the graph with vertex set X_p^d and edges

$$A \longmapsto \mathrm{fix}(AN) \quad \text{for} \quad N = \begin{pmatrix} 1_{p_p} & * \\ 0 & \ddots\, 1_{p_1} \end{pmatrix}$$

p's or 1's on the diagonal and

if $a_{ii}=p$ $a_{jj}=1$ and $i<j$ then $a_{ij}\in\{0,\dots,p-1\}$ otherwise $a_{ij}=0$.

Apartments

The picture you get from restricting to diagonal vertices and edges.

Tits

"Field with one element $\mathbb{F}_1$"

$$X_1^d = \left\{ \begin{pmatrix} p^{n_1} & & 0 \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} : \begin{array}{l} \text{primitive,} \\ \min\{n_1,\dots,n_d\}=0 \end{array} \right\}$$

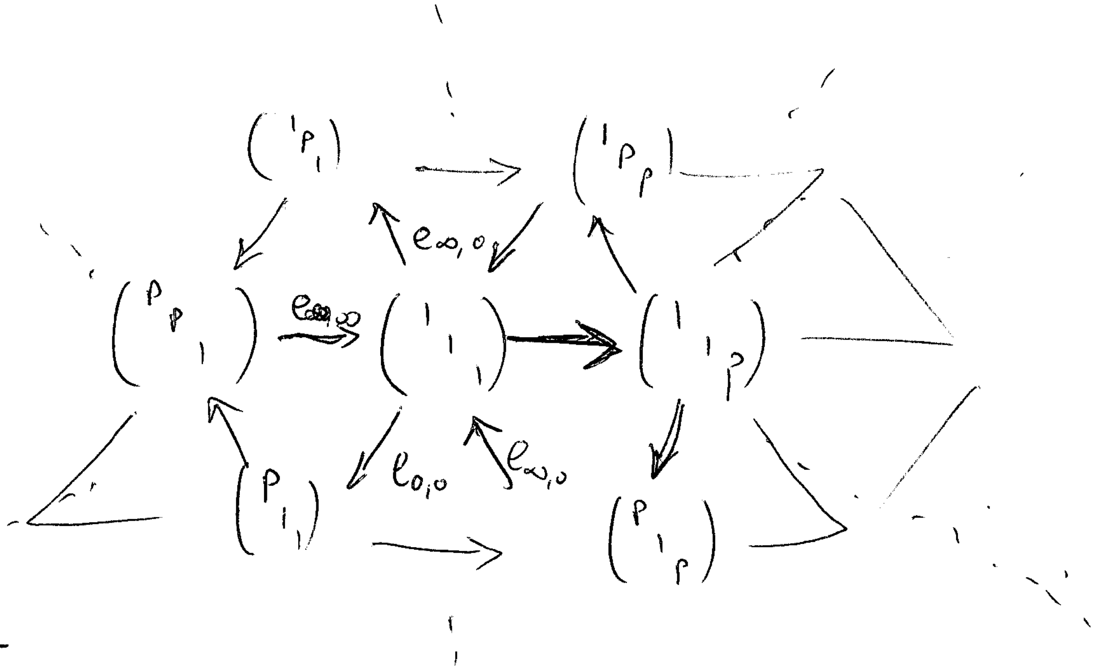
$$N = \left\{ \begin{pmatrix} p & & \\ & 1 & \\ & & \ddots \,1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & p & \\ & & 1 \ddots 1 \end{pmatrix} \dots \right\}$$

X_1^2

$$\begin{tikzcd}\cdots & \begin{pmatrix}1&\\&p^2\end{pmatrix} \arrow[r, shift left] & \begin{pmatrix}1&\\&p\end{pmatrix} \arrow[l, shift left] \arrow[r, shift right, "e_0"'] & \begin{pmatrix}1&\\&1\end{pmatrix} \arrow[l, shift right, "e_\infty"'] \arrow[r, shift left, "e_0"] & \begin{pmatrix}p&\\&1\end{pmatrix} \arrow[l, shift left, "e_\infty"] \arrow[r, shift left, "e_0"] & \begin{pmatrix}p^2&\\&1\end{pmatrix} \arrow[l, shift left, "e_\infty"] & \cdots\end{tikzcd}$$

line

X_1^3



Triangular ~~Hexagonal~~ tesselation of the plane.

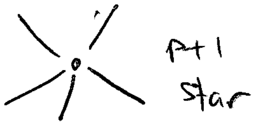
Building

local view - link

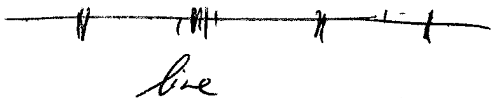
section - apartment

global view - entire building.

X_1^2, local



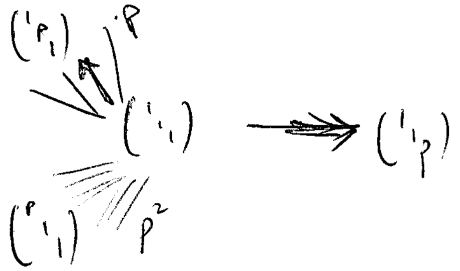
Section:



T_{p+1}

Section : Triangular tesselation

See picture.



Back to the group $G = PGL_d(\mathbb{Z}[\frac{1}{p}])$

G acts on the building

vertices $\longleftrightarrow G/K$ $\qquad K = PGL_d(\mathbb{Z})$

$G \circlearrowright G/H$ by $g \cdot g'H = gg'H$.

Facts:

G acts transitively on the vertices.

G acts transitively on 1-edges (edges coming from N_i $i=1,\dots,\frac{p^d-1}{p-1}$)

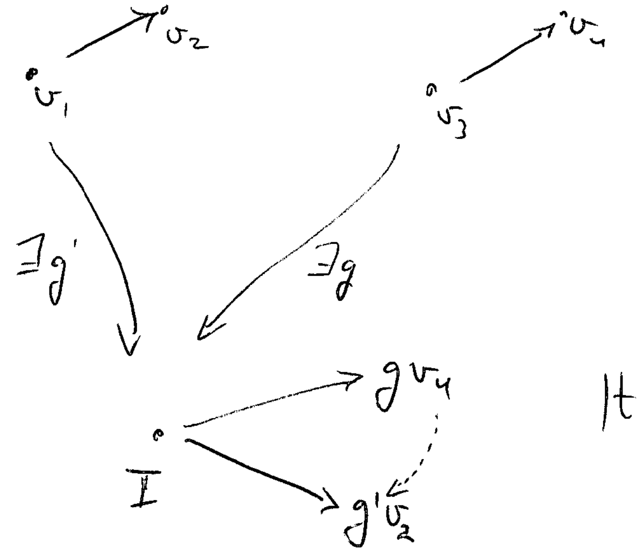
G acts transitively on $(d-1)$-cells.

Thus $X^{(d-1)} = \left\{ g\begin{pmatrix}1&&\\&\ddots&\\&&1\end{pmatrix}, g\begin{pmatrix}1&&\\&\ddots&\\&&p\end{pmatrix}, \dots, g\begin{pmatrix}1&&\\&p&\\&&\ddots p\end{pmatrix} \right\}_{g\in G}$

For any $g \in G$, the complex spanned by $g \cdot \left\{ \begin{matrix} \text{diagonal} \\ \text{matrices} \end{matrix} \right\}$ is isomorphic to the diagonal matrices = fundamental apartment

these are all called apartments. ~~(and they are all the apartments)~~
These are not all the " .

<u>Proof of transitive action on edges</u>



It is left to see that

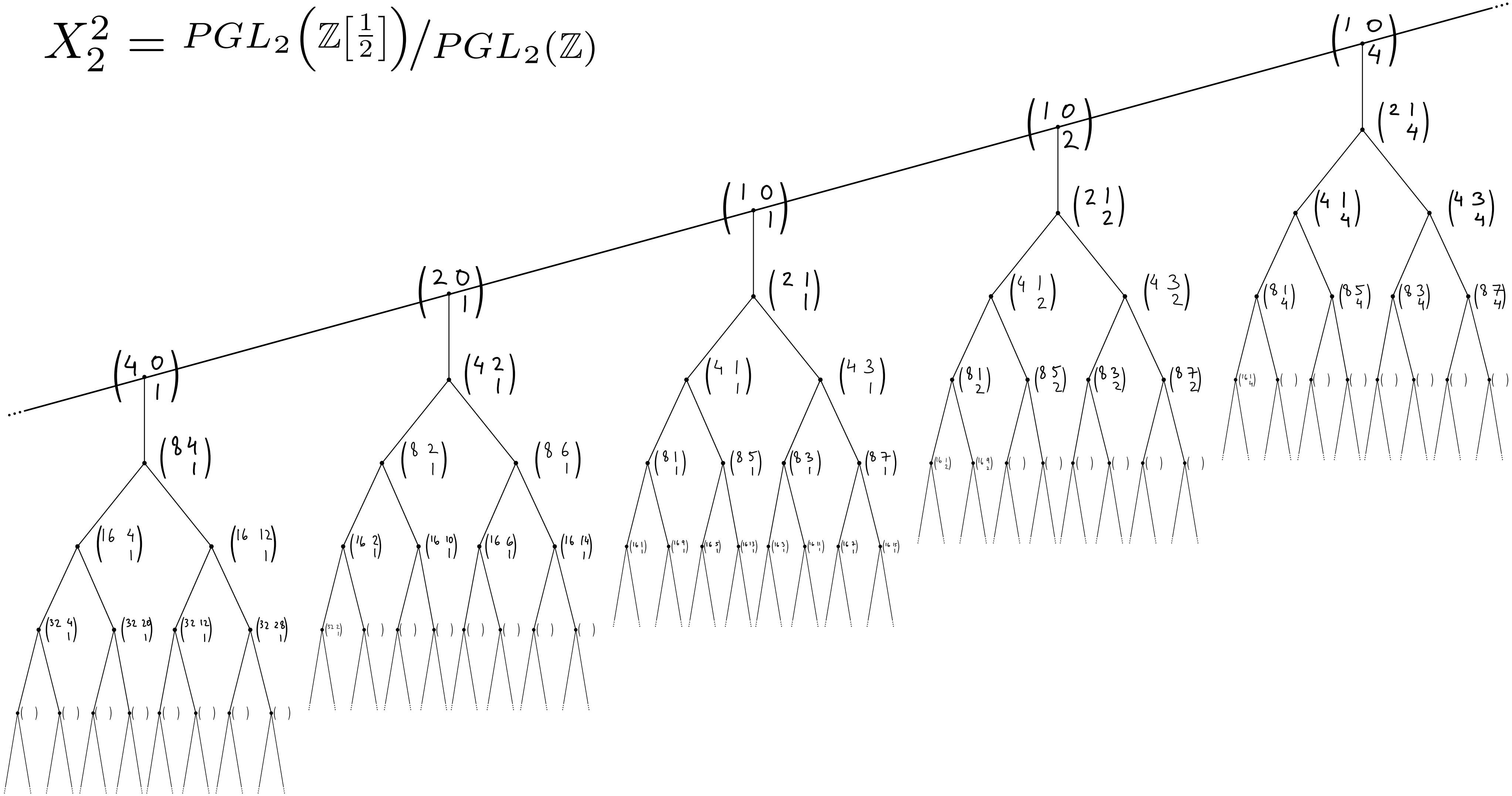
Stab(I) acts transitively on $N_1, \dots, N_{\frac{p^d - 1}{p-1}}$ from the ~~left~~ left

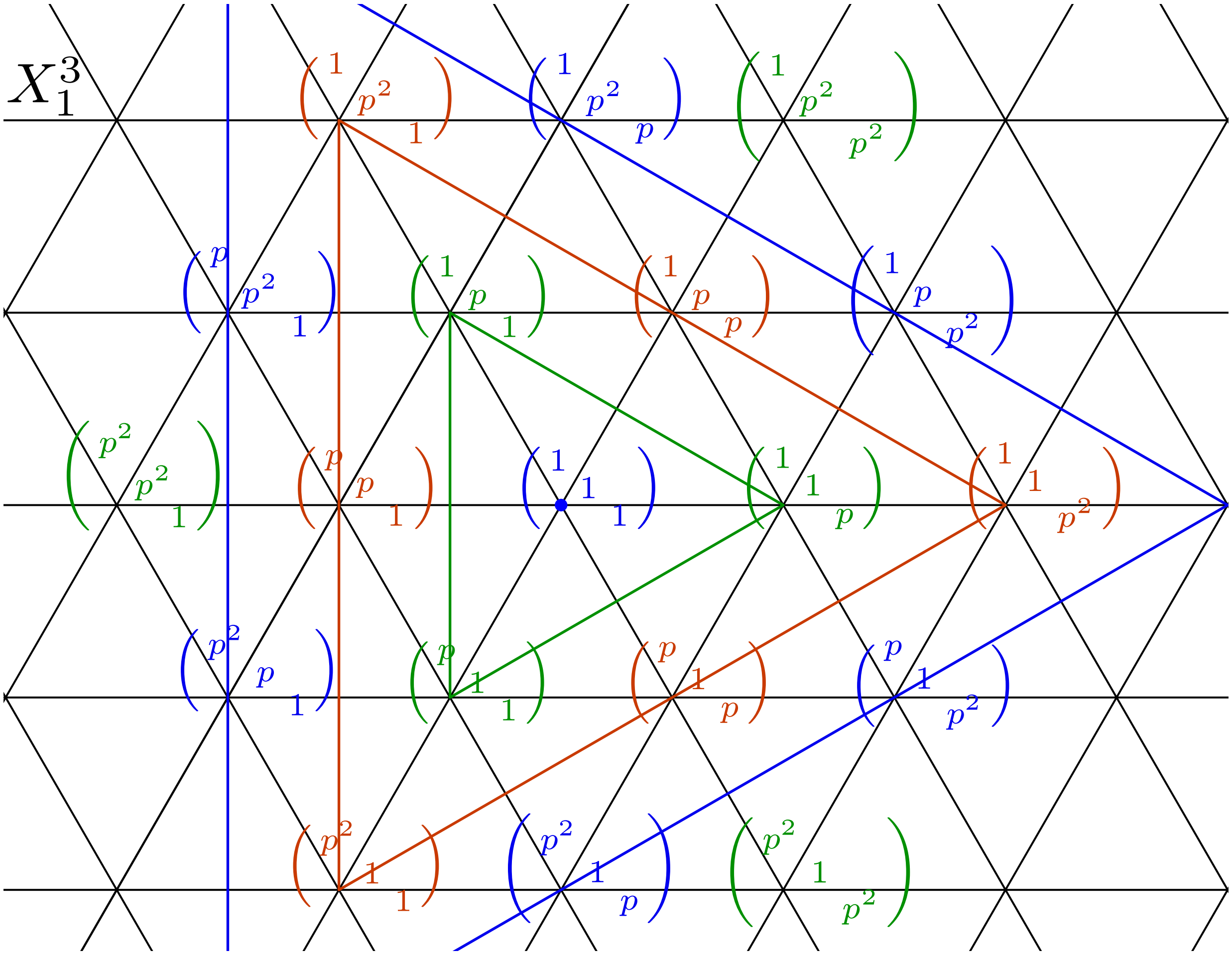
exercise.

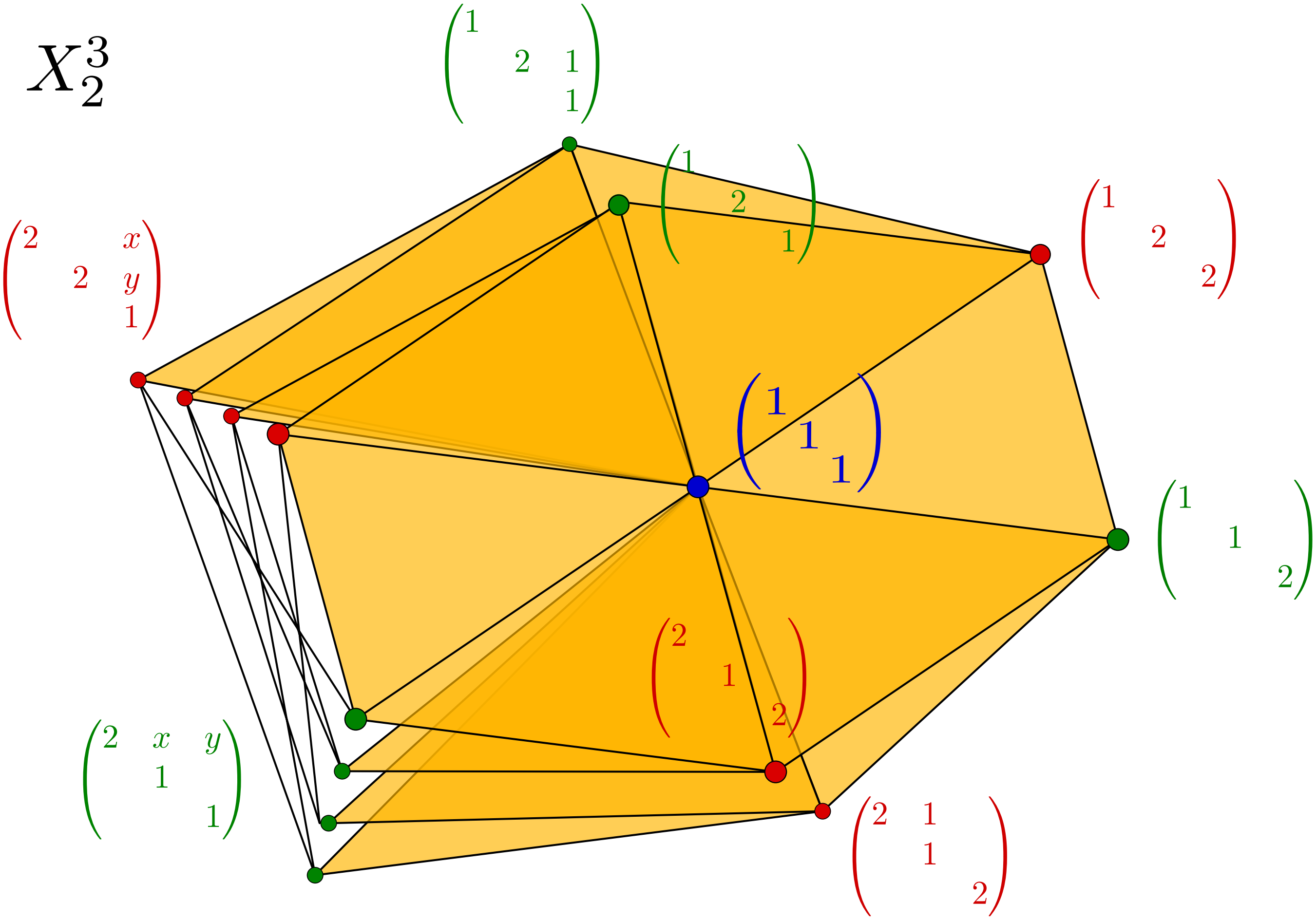
▨

$\mathrm{Stab}(I) = K = PGL_d(\mathbb{Z})$

$$X_2^2 = {}^{PGL_2\left(\mathbb{Z}[\frac{1}{2}]\right)}\big/_{PGL_2(\mathbb{Z})}$$




X_1^3
$\begin{pmatrix}1&&\\&p^2&\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&p^2&\\&&p\end{pmatrix}$
$\begin{pmatrix}1&&\\&p^2&\\&&p^2\end{pmatrix}$
$\begin{pmatrix}p&&\\&p^2&\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&p&\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&p&\\&&p\end{pmatrix}$
$\begin{pmatrix}1&&\\&p&\\&&p^2\end{pmatrix}$
$\begin{pmatrix}p^2&&\\&p^2&\\&&1\end{pmatrix}$
$\begin{pmatrix}p&&\\&p&\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix}$
$\begin{pmatrix}1&&\\&1&\\&&p^2\end{pmatrix}$
$\begin{pmatrix}p^2&&\\&p&\\&&1\end{pmatrix}$
$\begin{pmatrix}p&&\\&1&\\&&1\end{pmatrix}$
$\begin{pmatrix}p&&\\&1&\\&&p\end{pmatrix}$
$\begin{pmatrix}p&&\\&1&\\&&p^2\end{pmatrix}$
$\begin{pmatrix}p^2&&\\&1&\\&&1\end{pmatrix}$
$\begin{pmatrix}p^2&&\\&1&\\&&p\end{pmatrix}$
$\begin{pmatrix}p^2&&\\&1&\\&&p^2\end{pmatrix}$


X_2^3
$\begin{pmatrix}1&&\\&2&1\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&2&\\&&1\end{pmatrix}$
$\begin{pmatrix}2&&x\\&2&y\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&2&\\&&2\end{pmatrix}$
$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$
$\begin{pmatrix}1&&\\&1&\\&&2\end{pmatrix}$
$\begin{pmatrix}2&&\\&1&\\&&2\end{pmatrix}$
$\begin{pmatrix}2&x&y\\&1&\\&&1\end{pmatrix}$
$\begin{pmatrix}2&1&\\&1&\\&&2\end{pmatrix}$

Lecture 11

Local structure

$\mathrm{Star}(v) = \{\sigma : v \in \sigma\} \cong \mathrm{Cone}(\mathrm{link}(v))$

Cone() = 

$\mathrm{Cone}(X) = X \times I /_{(x,0) \sim (x',0)}$

Recall: d-cells containing $I = \mathbb{Z}^d$ correspond to chains

$$\mathbb{Z}^d \overset{p}{>} p^{n_1} L \overset{p}{>} p^{n_2} L_2 > \dots > p^{nd} L_d = p^{nd} \mathbb{Z}^d$$

$\longleftrightarrow$ d-cycles ~~of~~ 1-edges $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & p_1 \times \dots & \\ & & & \ddots \; 1 \end{pmatrix}$

Actually, we have $\mathbb{Z}^d > L_1 > L_2 > \dots > L_d = p\mathbb{Z}^d$

because if $pL_i \overset{p}{>} pL_{i+1}$ we get covol( ) growth of p^d which is impossible as the total change in co-vol is p^d!

d-cells containing I $\longleftrightarrow$ $\mathbb{Z}^d \overset{p}{>} L_1 \overset{p}{>} L_2 > \dots > L_d > p\mathbb{Z}^d$

$\Updownarrow$ $\underline{\text{IV}}$ iso-thm (correspondence thm)

$$\mathbb{F}_p^d = \mathbb{Z}^d/p\mathbb{Z}^d > L_1/p\mathbb{Z}^d > \dots > L_d/p\mathbb{Z}^d > p\mathbb{Z}^d/p\mathbb{Z}^d = 0$$

$\Updownarrow$

$\mathbb{F}_p^d \supsetneq V_1 \supsetneq V_2 \cdots \supsetneq V_{d-1} \supsetneq 0$ maximal flags in $\mathbb{F}_p^d$ ~~[illegible]~~

cells containing Id $\longleftrightarrow$ Flags in $\mathbb{F}_p^d$

dim $\longleftrightarrow$ length -2

(non-trivial subspaces of $\mathbb{F}_p^d$)

In particular $\mathbb{F}_p^d \supsetneq V \supsetneq 0$ are in correspondence with neighboring vertices to Id.

<u>Def:</u> The spherical building of $GL_d(\mathbb{F}_p)$ is:

vertices - non trivial subspaces of $\mathbb{F}_p^d$

edge - inclusion — $\{V_1, V_2\}$ is an edge if $V_1 \subsetneq V_2$ or $V_2 \subsetneq V_1$

general cells - flag complex. In particular $(d-2)$-cells are maximal flags.

$\cong$ link of I ~~in~~ the affine building of $PGL_d(\mathbb{Z}[\frac{1}{p}])$

X_p^2 = (p+1) reg tree $\longrightarrow$ $lk(Id) \cong$  (p+1)-points

$\longleftrightarrow$ non trivial subspace of $\mathbb{F}_p^2$ = (p+1)-points no inclusion. (lines)

$\mathbb{F}_p^3$; lines p^2+p+1

planes p^2+p+1

Every line is contained in (p+1)-planes

$\Longrightarrow$ Spherical ~~complex~~ building of $GL_3(\mathbb{F}_p)$ is a (p+1)-reg bip. graph with $2(p^2+p+1)$ vertices.

This is an excellent expander $\xrightarrow[\text{theory}]{\text{Garland}}$ Good $\overset{\text{Spec}}{\vee}$ expansion for X_p^2.

$G = PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts transitively on vertices by group action $G/K = GL_d(\mathbb{Z})$

1-edges, (d-1)-cell (top), apartments (by defn)

saw

<u>(d-1)-cells:</u> Since G acts trans. on vertices, it is left to show that $Stab(Id) = K = GL_d(\mathbb{Z})$ acts transitively on (d-1) cells containing I

Such $(d-1)$-cells correspond to maximal flags in $\mathbb{F}_p^d$.

The action of $GL_d(\mathbb{Z})$ is by its map to $GL_d(\mathbb{F}_p)$

$$A \longmapsto A \bmod p$$

$GL_d(\mathbb{F}_p)$ acts trans. on max flags in $\mathbb{F}_p^d$ (Convince yourself)

However $GL_d(\mathbb{Z}) \longrightarrow GL_d(\mathbb{F}_p)$ is not onto ($\det A = \pm 1$)

Nevertheless $SL_d(\mathbb{Z}) \twoheadrightarrow SL_d(\mathbb{F}_p)$ and the later acts transitively on max flags.

Lecture 12

Cells in the link of $I = \mathbb{Z}^d \longleftrightarrow \mathbb{Z}^d \supsetneq L_1 \supsetneq L_2 \supsetneq \dots \supsetneq L_j \supsetneq p\mathbb{Z}^d$

$$\Updownarrow$$

$$\mathbb{F}_p^d \supsetneq V_1 \supsetneq V_2 \supsetneq \dots \supsetneq V_j \supsetneq \{0\}$$

flags in $\mathbb{F}_p^d$!

We defined the spherical building of $\mathbb{F}_p^d$

cells $\cong$ Flags in $\mathbb{F}_p^d$. We got $\underset{\in X_p^d}{\mathrm{link}(v)} \cong$ spherical building of $\mathbb{F}_p^d$.

Claim: $G = PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts transitively on top $(d-1)$-cells.

Proof: We already know that G acts transitively on vertices so it suffices to show the stabilizer of a vertex acts transitively on $(d-1)$-cells containing it.

For example that $\mathrm{stab}_G(I) = K = PGL_d(\mathbb{Z})$ acts transitively on $(d-1)$-cells containing I.

$(d-1)$-cells containing $I \longleftrightarrow$ max flags in $\mathbb{F}_p^d$

Claim: K acts transitively on max flags by the mod p map.

and $PGL_d(\mathbb{F}_p)$ acts transitively on maximal flags in $\mathbb{F}_p^d$.

Thus if $GL_d(\mathbb{Z}) \twoheadrightarrow GL_d(\mathbb{F}_p)$ we are done.

This however is not the case. Instead look at

$$\underset{\substack{\swarrow \\ \text{comm.} \\ \text{ring}}}{SL_d^{\pm}(R)} = \{A \in M_d(R) : \det A = \pm 1\}$$

$SL_d^{\pm}(\mathbb{F}_p)$ acts transitively on maximal flags.

Now $SL_d^{\pm}(\mathbb{Z}) \twoheadrightarrow SL_d^{\pm}(\mathbb{F}_p)$ $\textcircled{*}$

Claim:
In general, if R is an Euclidean domain, then

$$SL_d^{\pm}(R) = \Big\langle \underbrace{\underset{\substack{\parallel \\ T_{a,i,j}}}{\begin{pmatrix} 1 & & a \\ & \ddots & \\ & & 1 \end{pmatrix}}, \underset{\substack{\parallel \\ S_{i,j}}}{\begin{pmatrix} 1 & & 1 \\ & \ddots & \\ 1 & & 1 \end{pmatrix}}, \underset{\substack{\parallel \\ M_i}}{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \; 1 \end{pmatrix}}}_{\textcircled{**}} \Big\rangle$$

then $\textcircled{*}$ follows

Proof: $A \in SL_d^{\pm}(R)$ by Euclid ~~algorithm~~ on first row $A = \left(\begin{array}{c|c} a & 0\,0\,0\,0 \\ \hline * & * \end{array}\right)$

but $a \in R^* \Rightarrow$ Euclid on first column $A = \left(\begin{array}{c|c} a & 0 \cdots 0 \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right)$

This action can be done using the matrices in $\textcircled{**}$ $\xrightarrow{\text{cont}}$ $\begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_d \end{pmatrix}$

$\prod a_i = \pm 1$

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & \ddots \; a_d \end{pmatrix}$$

$$\big\downarrow$$

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \xrightarrow{col} \begin{pmatrix} a & a \\ & b \end{pmatrix} \xrightarrow[T_{\frac{1-a}{a},1,2}]{col} \begin{pmatrix} 1 & a \\ \frac{b-ab}{a} & b \end{pmatrix} \xrightarrow{row} \begin{pmatrix} 1 & a \\ 0 & ab \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}$$

move all to the last diagonal → it must be 1 ▨

Cayley graph G, $S \subseteq G$ $\quad Cay(G; S) = (V, E)$

$V = G \quad E = \{(g, sg) : g \in G, s \in S\}$

This is an S-regular graph.

Too restrictive

All graphs are directed

Schreier graph

G group $\quad H \le G \quad S \subseteq G$

$Sch(G, H; S) = (V, E) \quad V = G/H \quad E = \{(gH, sgH) : g \in G, s \in S\}$

Too general

Hecke graph

$G \quad H \le G \quad S \subseteq G$

$Hec(G, H; S) = (V, E) \quad V = G/H \quad E = \{(gH, gsH) : g \in G, s \in S\}$

This has transitive G action.

Claim: $$X_p^d = \mathrm{Hecke}\Big(\underset{PGL_d(\mathbb{Z}[\frac{1}{p}])}{\underset{\parallel}{G}},\ \underset{PGL_d(\mathbb{Z})}{\underset{\parallel}{K}},\ \{\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right)\}\Big)$$

Catch: $S = \{s_1, \dots, s_m\}$

$$\begin{tikzcd} & gs_1K \\ gK \arrow[ur] \arrow[r] \arrow[dr] & gs_2K \\ & gs_mK \end{tikzcd}$$

maybe $gs_1K = g_2sK$

real problem if $gK = g'K$ it is not necessarily true that $g's_i = gs_i$ so we also need to go to

$gK \longrightarrow g's_iK$ for $1 \leq i \leq m$ and $g' \in G$ s.t. $gK = g'K$.

Define: S is K balanced if $KSK = SK$

First note that $\mathrm{Hecke}(G, K, S)$ we actually have

$$\begin{tikzcd} & {} \\ gkK = gK \arrow[ur] \arrow[r] \arrow[dr] \arrow[d] & gksK \\ {} & {} \end{tikzcd}$$

so, if $KSK = SK = \coprod_{s \in S'} sK$

balancification of S

$\Rightarrow$ edges of the graph are (gK, gsK) for $s \in S'$

[Outgoing neighbors of gK are $\{gsK\}$ for $s \in S'$]

$\boxed{\text{If } S \text{ is } K\text{-balanced then } S' = S}$ $\Rightarrow$ The graph is $|S|$-regular

---

Back to the claim:

For $G = PGL_d(\mathbb{Z}[\frac{1}{p}])$ $K = PGL_d(\mathbb{Z})$ $S = \{\begin{pmatrix}1&\\&p\end{pmatrix}\}$ we can take $S' = \{\begin{pmatrix}1&\\&p\end{pmatrix}, \begin{pmatrix}p&j\\&1\end{pmatrix} \; j=0,\dots,p-1\}$

and S' is K-balanced and gives the $(p+1)$-regular tree.

First check that $X_p^2 = Hec(G, K; S)$

need to show $K\begin{pmatrix}1&\\&p\end{pmatrix}K = \coprod_{j=0}^{p-1} \begin{pmatrix}p&j\\&1\end{pmatrix}K \amalg \begin{pmatrix}1&\\&p\end{pmatrix}K$

Recall: For $A \in PGL_d(\mathbb{Z}[\frac{1}{p}])$, the level of A is $\log_p \det A$ when A is scaled to be primitive.

In other words, if $A = \begin{pmatrix}p^m&a\\&p^n\end{pmatrix} \cdot k$, then level $= m+n$.
$\gcd(p^m, p^n, a) = 1$

Claim $K\begin{pmatrix}1&\\&p\end{pmatrix}K = \{g \in G : \text{level}(g) = 1\}$.

Proof:

(1) $K\begin{pmatrix}1 & \\ & p\end{pmatrix}K \subseteq$ level 1 because all the matrices on the left have det p. and are primitive.

(2) level 1 $= \coprod \begin{pmatrix}p & j\\ & 1\end{pmatrix}K \amalg \begin{pmatrix}1 & \\ & p\end{pmatrix}K$ because the general form is $\begin{pmatrix}p^m & a\\ & p^n\end{pmatrix}$ for $\gcd(p^m, p^n, a) = 1$. and the disjointness was left as an exercise

(3) $\coprod \begin{pmatrix}p & j\\ & 1\end{pmatrix}K \amalg \begin{pmatrix}1 & \\ & p\end{pmatrix}K \subseteq K\begin{pmatrix}1 & \\ & p\end{pmatrix}K$

we showed this. ~~when~~ $\underbrace{\begin{pmatrix} & 1\\ 1 & \end{pmatrix}}_{\in K}\begin{pmatrix}1 & \\ & p\end{pmatrix}K = \begin{pmatrix}p & \\ & 1\end{pmatrix}K$

$$\underbrace{\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}}_{\in K}\underbrace{\text{[illegible]}\begin{pmatrix}1 & \\ & p\end{pmatrix}}K = \begin{pmatrix}p & j\\ & 1\end{pmatrix}K.$$

Actually, $\mathrm{Hec}(G; K, g)$ if level $g = 1$ we get $\mathrm{Hec}(G, K, \begin{pmatrix}1 & \\ & p\end{pmatrix})$

if level$(g) > 1$ we get a disconnected graph
union of trees,

Note: In $\mathrm{Hec}(G, K, S)$ we have G-action on edges and vertices.

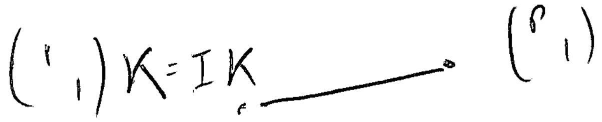
We will construct Ramanujan graphs as Hecke-Schreier graphs

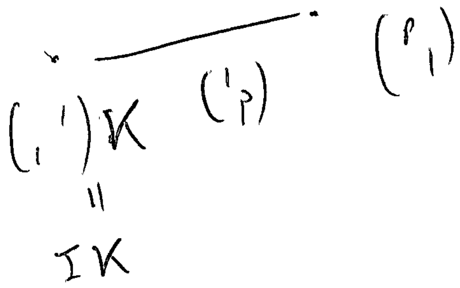
$$HS(G, H, K, S) = Sch(Hec(G, K, S), H, S)$$

$$V = {}_{H}\backslash G/K \qquad E = \{(HgK, HgsK)\}$$

Cayley and Schreier graphs are edge labeled by S

Hecke graphs are not.





The labels we got are related to the specific representatives we choose the edges are not labeled.

HW:

Show that in PGL_2 $\{g : level(g) = n\} = K\begin{pmatrix} 1 & \\ & p^n \end{pmatrix}K$.

This is not true in higher dim.

~~a combinatorial operator on G/K~~

Def: A combinatorial branching map on G-set X is a map $T : X \to 2^X$ such that $T(gx) = gT(x)$ $\forall x \in X$ $g \in G$.

If X is transitive, then pick $x_0 \in X \Rightarrow T(x_0)$ determine T

$$T(x) \underset{\substack{\downarrow \\ \exists g}}{=} T(gx_0) = gT(x_0)$$

Furthermore $T(x_0)$ is K-stable where $K = \mathrm{Stab}(x_0)$ because

$\forall k \in K$ $\quad kT(x_0) = T(kx_0) = T(x_0)$

Lecture

HW: $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$, $K = PGL_2(\mathbb{Z})$. Show

$\{A \in G : level(A) = m\} = K\begin{pmatrix}1 & \\ & p^m\end{pmatrix}K$.

Hecke graphs

Combinatorial operators $V = G/K$ $E = \{gK, gsK\}_{s \in S}$

Saw: From S we can create $S \subseteq S'$ s.t.

$KSK = S'K$

and then the out neighbors of gK are $\{gsK\}_{s \in S'}$

$Hecke(G, K, \{\begin{pmatrix}1 & \\ & p\end{pmatrix}\}) = X_p^2$ is $(p+1)$-reg.

$S = \{\begin{pmatrix}1 & \\ & p\end{pmatrix}\}$ $S' = \{\begin{pmatrix}1 & \\ & p\end{pmatrix}, \begin{pmatrix}p & j \\ & 1\end{pmatrix}\ j = 0, \dots, p-1\}$

Set $X = G/K$ a comb branching op. on X

a G-equi map $T : X \to 2^X$

$\forall x \in X\ \forall g \in G\ T(gx) = gT(x)$

Since $G \underset{\text{trans.}}{\curvearrowright} X$ $T(x_0)$ determines T (x_0 any fixed choice of element in X, e.g. K)

For $x_0 = K$,

Tx_0 must satisfy $kT(x_0) = T(x_0)$ $\forall k \in K$

since $kT(x_0) = T(kx_0) = T(x_0)$.

Actually, any choice of a K-fixed set $S \subseteq X$ determines a unique combinatorial branching operator with $Tx_0 = S$. (Check)

Rem: More generally given $G \curvearrowright X$ transitive action and $x_0 \in X$ we define $K = \mathrm{Stab}_G(x_0)$

---

directed Hecke graph ⟶ branching rule (out neighbors)

$$\begin{array}{lcl} K \text{ balanced set} & \longleftrightarrow & K \text{ fixed } \tilde{S} \text{ set.} \\ S \text{ s.t. } KSK = SK & & K\tilde{S} = \tilde{S} \\ \text{here } S \subseteq G & & \text{here } \tilde{S} \subseteq X = G/K \end{array}$$

$$S = \tilde{S}K \longleftarrow\!\!\shortmid \quad \tilde{S}$$

$$S \longmapsto SK/K$$

G equivariant branching operator on trans. G-set X

$$\Updownarrow$$

sets^K which are invariant w.r.t. the stabilizer of a fixed vertex $x_0 \in X$

$$\Updownarrow$$

(union of) double cosets of K.

<u>Once again</u>

$G \overset{\text{trans}}{\curvearrowright} X$, pick $x_0 \in X$. $K = \mathrm{stab}(x_0)$

Take some bi-K-inv set $M \subseteq G$ ($\Leftrightarrow M$ is a union of double K cosets)

decompose $M = \coprod_{s \in S} sK$ (thus defining S)

and then S is a K-balanced set → Hecke graph

$T x_0 = \{s x_0\}_{s \in S} \longrightarrow T(g x_0) = \{g s x_0\}_{s \in S}$
gives a branching operator.

Look at $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$, $K = PGL_2(\mathbb{Z})$

$X = G/K$ $(p+1)$-reg tree

What are comb. operators on X.

$Tx = B_r(x)$

$Tx = S_r(x)$

$Tx = \{y : dist(x,y) \in \{3,7,100\}\}$

$Tx =$ finite union of spheres   Those are all of them.

$X = T_k$

$G = Sym(X) = Aut(X)$

what comb. op. are there? Union of spheres

If $y \in Tx_0$ and $dist(x_0,y) = r$ then $S_r(x_0) \subseteq Tx_0$

Since $\forall y' \in S_r(x_0)$ $\exists k \in K = Stab_G(x_0)$ s.t. $ky = y'$

Hence $y' = ky \in T(x_0) = T(kx_0) = T(x_0)$

We want to study the behavior of T by spectral means.

Namely, define $A_T \circlearrowright L^2(\underset{G/K}{\overset{\|}{X}})$ by $(A_T f)(x) = \sum_{y \in T_x} f(y)$

Then, e.g., if $\mathrm{Spec}(A_T) = \{ \underset{\text{"if } X \text{ is finite"}}{\underset{\nwarrow}{|Tx_0|}}, \text{small e.v.} \}$, then T is rapidly mixing.

Now, we can ~~also~~ talk about polynomials $A_T^2 - A_T^3$ or, more generally, the ring of G-equiv. functions $\underset{\text{with finite support}}{\wedge}$ on $L^2(X)$.

For $X = T_k$, either $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$ or $G = \mathrm{Sym}(T_k)$, the regular adj. operator generates all these operators.

$\Longleftrightarrow \forall r \in \mathbb{N}$, the operator $A_r f(x) = \sum_{y \in S_r(x)} f(y)$ is a poly in A_1.

E.g. $A_2 = A_1^2 - k A_1^0$.

$A_3 = A_1^3 - \square A_1$

$A_4 = A_4^4 - \square A_1^2 + \square A_1^0$

Chebychev polynomials.

Ramanujan graphs and complexes - Lecture 14

December 11, 2017

Remainder Let X be a G-set. A (G-equivariant) branching operator on X is a $T : X \to \{\text{finite subsets of } X\}$ such that $g.T(x) = T(g.x)$ for all $x \in X$ and $g \in G$.

- If X is transitive, we showed that all branching operators arise as follows: Fix $x_0 \in X$. Define $K = \mathrm{Stab}_G(x_0)$. Choose some bi-Kinvariant set $M \subset G$, namely M is a union of double K-cosets KgK for various g, and decompose M as a disjoint union (define S) so that $M = \biguplus_{s \in S} sK$. Finally, set $T(x_0) = \{sx_0\}_{s \in S}$. In general $T(gx_0) = \{gsx_0\}_{s \in S}$.
- Those are equivalent to Hecke graphs - Indeed, X with T as adjacency operator is the Hecke graph of G with respect to K and S. Furthermore S is K-balanced, since $KSK = KM = M = SK$.
- Eventually, we want to understand double K cosets of G and their decomposition to right K-cosets.
- When $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$ and $K = PGL_2(\mathbb{Z})$ we already saw that double K cosets are the levels in G and equal $K \begin{pmatrix} 1 & \\ & p^{\ell} \end{pmatrix} K$. On T_k the branching operators are union of spheres.

Going to higher dimensions In higher dimensions there are much more ranching operators.

Here are some branching operators on X_p^3:

- Recall the $p^2 + p + 1$ outgoing neighbors of the identity. We can define Tx to be the outgoing neighbors of x. This is a minimal branching operator (it is not the union of smaller branching operators) which is equivalent to saying that it comes from a single double coset.
- $Tx =$ change triangle (distance 2 with respect to 1 operator) is not minimal. There are 6 of those vertices which can be splitted into 3 +3 which are forming two minimal branching operators. (See Figure 1). Algebraically, this means that $K \begin{pmatrix} 1 & & \\ & 1 & \\ & & p^2 \end{pmatrix} K \neq K \begin{pmatrix} 1 & & \\ & p & \\ & & p \end{pmatrix} K$ although both of them are of level 2. Actually, level 2 is the disjoint union of the last 2 double cosets (See Figure 1).

Theorem 0.1. *(Cartan decomposition)* $G = \biguplus K \begin{pmatrix} p^{n_1} & & & & \\ & p^{n_2} & & & \\ & & p^{n_3} & & \\ & & & \ddots & \\ & & & & p^{n_d} \end{pmatrix} K$ *, where the union is over* $0 = n_1 \leq n_2 \leq \ldots \leq n_d$. *In particular in* PGL_3*, the l-th level is composed of (*$1+\lfloor l/2 \rfloor$*)-double* K*-cosets. Can decompose* A_l *(the vertices at distance* l*) is a union of those branching operators.*

Proof. Let $g \in PGL_d(\mathbb{Z}[\frac{1}{p}])$. We need to get to $\begin{pmatrix} p^{n_1} & & & & \\ & p^{n_2} & & & \\ & & p^{n_3} & & \\ & & & \ddots & \\ & & & & p^{n_d} \end{pmatrix}$ with $n_1 = 0$, $n_i \leq n_{i+1}$ by applying K from the right and from the left. Then, we need to show that there is a unique choice of $n_1, \ldots, n_d$ to which we can arrive by such action. Scale g to be integer and primitive. There exists i, j such that p does not divide g_{ij}. Apply Euclid to the row of g_{ij} and get

$$\begin{pmatrix} * & * & * & * & * & * & * \\ & & & & & & \\ * & * & * & * & * & * & * \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * \\ & & & & & & \\ * & * & * & * & * & * & * \end{pmatrix}$$

Apply column operations to the first column and get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}.$$

Write $B = p^{n_2} C$ with C primitive and continue by induction. □

Exercise 0.2. Show that this is a disjoint union.

So far we only talked about operators on vertices. One can also talk about operators on cells in general.

For $\lambda = (0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_d)$ define T_λ to be the branching operator associated with

$$K \begin{pmatrix} p^{\lambda_1} & & & & \\ & p^{\lambda_2} & & & \\ & & p^{\lambda_3} & & \\ & & & \ddots & \\ & & & & p^{\lambda_d} \end{pmatrix} K.$$

Any branching operator on X_p^d is a union of these branching operators.

Theorem 0.3. *Surprising fact: all branching operators on X_p^d commute.*

For a graph we saw that every branching operator is a (Chebyschev) polynomial in A and all polynomials in a given operator commute.

Proof. Enough to prove for T_λ and T_μ. The statement is equivalent to showing that

$$K(p^\lambda)K(p^\mu)K = K(p^\mu)K(p^\lambda)K. \tag{0.1}$$

(This follows from the fact that we have the correspondence $X \to G/K$ given by $X \mapsto MK/K$, where $M \subset G$ such that $Mx_0 = S$. In this case $T_\lambda(x_0) = \{sx_0\}_{s \in S_\lambda}$ where $K(p^\lambda)K = \biguplus_{s \in S_\lambda} sK$. $T_\lambda T_\mu(x_0) = T_\lambda(\{sx_0\}_{s \in S_\mu} = \{stx_0\}_{t \in S_\lambda, s \in S_\mu}$which is mapped to $K(p^\mu)K(p^\lambda)K/K$. In the other direction, taking a family of cosets $R \subset G/K$ observe the subset of X defined by Rx_0. Then, $K(p^\lambda)K(p^\mu)Kx_0$ is a T_λ-neighbor of a T_μ-neighbor of x_0. We used here the fact that $(p^\mu)Kx_0$ is one T_μ neighbor of x_0 and that $K(p^\mu)Kx_0$ are all T_μ neighbors of x_0. Therefore

$$T_\lambda T_\mu = K(p^\lambda)K(p^\mu)K = K(p^\mu)K(p^\lambda)K = T_\mu T_\lambda.$$

Turning to prove (0.1) we use a trick of Gelfand. First observe that $(K(p^\lambda)K)^t = K^t(p^\lambda)^t K^t = K^t(p^\lambda)K^t = K(p^\lambda)K$. Here we already used Cartan's result. Similarly,

$$(K(p^\lambda)K(p^\mu)K)^t = K(p^\mu)K(p^\lambda)K.$$

On the other hand, the left hand $K(p^\lambda)K(p^\mu)K$ is a union of double K-cosets, so it is $\biguplus_{i=1}^m K(p^{\nu_i})K$ and we got that

$$(K(p^\lambda)K(p^\mu)K)^t = \left(\biguplus_{i=1}^m K(p^{\nu_i})K\right)^t = \biguplus_{i=1}^m (K(p^{\nu_i})K)^t = \biguplus_{i=1}^m K(p^{\nu_i})K = K(p^\lambda)K(p^\mu)K$$

and all together we get (0.1). □

From Ramanujan Graphs to Complexes

18.12.17

Reminder:

$G = PGL_d(\mathbb{Z}[\frac{1}{p}])$

$K = PGL_d(\mathbb{Z})$

There exists a correspondence between:

G-equivariant branching operators on $X = G/K$

and

Unions of double K-cosets in G

Another correspondence we have

$$X \longleftrightarrow G/K$$

subsets of X ⟷ right K-inv. subsets of G

So a double coset KgK gives a branching operator $X \longrightarrow 2^X$

$$(KgK)(\overbrace{\underset{\underset{G}{\cap}}{t}K}^{\overset{X}{\in}}) = \underbrace{tKgK}_{\substack{\text{right } K\text{-inv.}\\ \text{set}}} \Rightarrow \text{Subset of } X$$

This is well defined since

$$(KgK)(\overbrace{\underset{\overset{\cap}{G}}{tk}K}^{=tK\in X}) = t\underbrace{kK}_{=K}gK = tKgK$$

For $g \in G$ denote

H_g = the branch. op. corresponding to KgK

$H_g : X \longrightarrow 2^X$, or $H_g : L^2(X) \circlearrowleft$

<u>Reminder:</u> We saw that for our G, K, all H_g commute.

Also, the double K-cosets are parametrized by

$$g = \begin{pmatrix} p^{n_1} & & 0 \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} \qquad 0 = n_1 \leq n_2 \leq \dots \leq n_d$$

<u>Cor:</u> For every $v, w \in X^{(0)}$ there exists an apartment containing both.

Pf: We know $G \circlearrowright X^{(0)}$ transitively, so we can

take WLOG $v = \overset{\text{the "root"}}{v_0} = [I] (= K)$.

Then $\exists g$ s.t $w = gK$. write $g = kak'$

where $a = \begin{pmatrix} p^{n_1} & & 0 \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix}$ $0 = n_1 \leq \dots \leq n_d$.

The fundamental apt. was

$$\mathcal{A} = \left\{ a v_0 \;\middle|\; a = \begin{pmatrix} p^{n_1} & & 0 \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} \right\}_{0 = n_1 \leq \dots \leq n_d}$$

Now $k\mathcal{A}$ contains v and w since

$$v = v_0 \in \mathcal{A} \Rightarrow v_0 = k v_0 \in k\mathcal{A}$$

$$w = g v_0 = k a k' v_0 = k \underbrace{a v_0}_{\in \mathcal{A}} \in k\mathcal{A}$$

Tit's Axiom: $\forall \sigma, \tau$ chambers (top cells) $\exists$ apt. containing both

Ex: Prove it is true in our case

Comb. Operators on the Quotients of the building

If $H \leq G$ we can look at ${}_H\backslash X = {}_H\backslash {}^{G}/_{K}$

E.g. $G = \mathrm{Aut}(T_k)$ $K = \mathrm{stab}_G(\mathrm{root})$

$$X = {}^{G}/_{K} = T_k$$

For every $H \leq G$ ${}_H\backslash X$ is a k-reg graph.

Also the other way around is true: every k-reg graph is obtained as ${}_H\backslash X$ for some $H \leq \mathrm{Aut}(T_k)$.

Exercise (Clues: π_1, deck transformations)

Combinatorial ops. are well defined on such quotients.

Union of H-K double cosets ⟷ subsets of ${}_H\backslash X$

$$(KgK)(HxK) = \overbrace{HxKgK}$$

or, if $T : X \to 2^X$ is g-equivariant then for $Hx \in {}_H\backslash X$ we can define $T : {}_H\backslash X \longrightarrow 2^{{}_H\backslash X}$

by $T(Hx) = HT(x)$.

This is the same as saying that if Γ is a k-reg graph, take G-equ. cover map $p: T_k \to \Gamma$ and define a branch. op. on Γ by

$$\underbrace{T(v) = p(H(p^{-1}(v))) = \{\text{neighbors of } v \text{ in } \Gamma\}}_{\text{Adjacency Op.}}$$

Thus whenever $|H\backslash G/K| < \infty$ we have a fin. quot. of X with all Hecke ops. (H_g) defined on it.

Note: For our "usual" G, K, all H_g commute also on quotients. Thus simul. diag. on $L^2(H\backslash X)$.

Cor: They are all normal.

reason: for $a = \begin{pmatrix} p^{n_1} & & 0 \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix}$ $\quad H_a^* = H_{a^{-1}}$ (Exercise)

Ramanujan Complexes

Def: (Li , Lub-Samueles-Vishne)

A quotient $\Gamma \backslash X_p^d$ is ramanujan if

all simul. eigen values $(\lambda_1, \dots, \lambda_{d-1})$ of $A_1, \dots, A_{d-1}$

is either trivial or in $Spec_{(A_1, \dots, A_{d-1})}(X_p^d)$

Note: For $d=2$ we get the regular ramanujan def.

Reminder: For some ops. $A_1, \dots, A_m \circlearrowright V$

the sim. spectrum is

$$\{ (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \mid \exists v \neq 0 \text{ s.t. } A_i v = \lambda_i v \text{ for all } 1 \le i \le m \} \subseteq$$

$$\subseteq \quad sp(A_1) \times sp(A_2) \times \dots \times sp(A_m)$$

Thm: The operators $A_j = H_{\begin{pmatrix} \overbrace{p \cdots p}^{j \text{ times}} & 0 \\ 0 & \underbrace{1 \cdots 1}_{d-j \text{ times}} \end{pmatrix}}$

generate all Hecke ops. on G/K.

pf: Order all Heck ops. on G/K as follows: Take $H_{\vec{\lambda}} = K \begin{pmatrix} p^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & p^{\lambda_d} \end{pmatrix} K$

where $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_d = 0$. Then the order is by λ_1 and then by $m_\lambda := \{\# j \mid \lambda_j = \lambda_1\}$.

e.g. $\begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 2 & \\ & & & 0 \end{pmatrix} > \begin{pmatrix} 3 & & & \\ & 2 & & \\ & & 2 & \\ & & & 0 \end{pmatrix} > \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \lessgtr \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Def: For $v \in G/K$ the distance of v from I is

$\max\{\lambda_j\}$ with $g \in K \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_d} \end{pmatrix} K$

where $v = gK$.

Claim: $\forall \vec{\lambda}$ $\exists j$ s.t. if we take

$\vec{\lambda}' = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_j - 1, \lambda_{j+1}, \lambda_{j+2}, \dots, \lambda_d)$

then all vertices $A_j \begin{pmatrix} p^{\lambda_1'} & & \\ & \ddots & \\ & & p^{\lambda_d'} \end{pmatrix} K$

are either in the K orbit of

$\begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_d} \end{pmatrix} K$ or in smaller ones

w.r.t our ordering

with respect to

Thus by induction we can write the corresponding double cosets using $A_1, \dots, A_{d-1}$ and this "clean out" $K\begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_d} \end{pmatrix}K$

$$A_j = K\begin{pmatrix} p & & & & & \\ & \ddots & & & & \\ & & p & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}K = \left\{ \underbrace{\begin{pmatrix} p & * \cdots * & 0 & * & \cdots \\ & p & * \cdots * & * & * \\ & 1 & & 0 & \\ & & 1 & & \\ & & & p & * \;\; * \;\; \cdots \\ 0 & & & p & * \\ & & & & 1 \\ & & & & & 1 \ddots \end{pmatrix}}_{S_j} K \right\}$$

$\nearrow$ neighbors of Id of level j

$\uparrow$ in the graph sense

Thus $A_j(gK) = g S_j K$

In particular $A_j K = S_j K$ are all in one K-orbit.

For $g = \begin{pmatrix} p^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & p^{\lambda_d} \end{pmatrix}$ we get $A_j(gK) =$

$$= g S_j K = \begin{pmatrix} p^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & p^{\lambda_d} \end{pmatrix} \underbrace{\begin{pmatrix} p & & * \\ & p & \\ & & \ddots & p \end{pmatrix}}_{\circledast} K$$

Since $\lambda_1 \geq \lambda_2 \dots$ we can

do row operations to "kill"

all $*$'s in $\circledast$.

Thus $$g S_j K = K \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_d} \end{pmatrix} \begin{pmatrix} p & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1\ p & \\ 0 & & & & \ddots\ p \end{pmatrix} K$$

Therefore, A_j takes $\begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_d} \end{pmatrix}$

to double cosets corr. to various

$\vec{\lambda} = (\lambda_1 + m_1, \dots \lambda_d + m_d)$ where $m_i = 0/1$

and $\sum m_i = j$

So if we want to get some

$H_{\begin{pmatrix} p^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & p^{\lambda_d} \end{pmatrix}'}$ we look at

$$\vec{\lambda}' = (\lambda_1 - 1, \lambda_2 - 1, \dots \lambda_{m_\lambda} - 1, \lambda_{m_\lambda + 1}, \dots \lambda_d)$$

$$m_\lambda = \#\{\lambda_j \mid \lambda_j = \lambda_1\}$$

and then $A_{m_\lambda} \begin{pmatrix} p^{\lambda_1'} & & \\ & \ddots & \\ & & p^{\lambda_d'} \end{pmatrix} K$ contains elements from $K \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_d} \end{pmatrix} K$ and smaller double cosets.

Ramanujan graphs and complexes - Lecture 16 - December 24

December 30, 2017

Remainder - Hecke operators The Hecke operators of $B^d = X_p^d$ is the subalgebra of locally finite G-equivariants operators on vertices $T : B^0 \to 2^{B^0}$ (as a subalgebra of $Lin\left(L^2\left(B^0\right)\right)$). We have

$$\mathcal{H} = \mathrm{Span}_{\mathbb{C}}\left(\begin{array}{c} G-\text{equivariant branching} \\ \text{operators acting on } L^2\left(B^0\right)\end{array}\right) = \mathrm{Span}_{\mathbb{C}}\{KgK\,:\,g\in G\}.$$

Using Cartan, we also saw that this is the same as

$$\oplus_{0=\lambda_1\leq\lambda_2\leq\ldots\lambda_d}\mathbb{C}\cdot K(p^{\overrightarrow{\lambda}})K$$

and last week we saw that $A_1\ldots,A_d$ generate those, i.e.

$$\mathcal{H}=\mathbb{C}\{A_1,\ldots,A_{d-1}\}.$$

By Gelfand we know that

$$\mathcal{H}=\mathbb{C}[A_1,\ldots,A_{d-1}]$$

and in fact we have the following result:

Theorem 0.1. *(McDonald?, Satake?)* $\mathbb{C}[A_1,\ldots,A_{d-1}]\cong\mathbb{C}[x_1,\ldots,x_{d-1}]$.

Higher dimensions Change K. Take $\sigma\in B^{(j)}$. Then $G\sigma\subset B^{(j)}$. For $PGL_{2,3}$ the group acts transitively on each dimension. In higher dimensions, it does not. Take $X = G\sigma$. To study G-equivariant branching maps on X, we need to understand the action of K_σ-double cosets on K_σ-left costes ($G/K_\sigma\cong G_\sigma = X$), where $K_\sigma=\mathrm{Stab}_G(\sigma)$. We can take σ to be to be a set (cell), ordered set, pointed cells (cells with a chosen vertex), etc.... Every choice for the structure of σ gives different orbits and different stabilizers.

$G = PGL_2$ with $K_e = \mathrm{Stab}_G$(directed edge from $\begin{matrix}1&\\&p\end{matrix}$ to $\begin{matrix}1&\\&1\end{matrix}$) this is the same as oriented, pointed and ordered edge (but not as set of vertices). In this case $B^{(1)} = Ge$. Example of G-equivariant branching operator $B^{(1)} = T_{p+1}$ is $T(e_0) = K_{e_0}e$, where e_0 is the directed edge above. (This is in fact the general thing up to union of such things). For example, if we take $T(e_0) = K_{e_0}\cdot\left(\text{directed edge from }\begin{matrix}1&\\&1\end{matrix}\text{ to }\begin{matrix}p&\\&1\end{matrix}\right)$ and denoting this edge by e

$$\begin{tikzcd}
\begin{pmatrix} p & \\ & 1\end{pmatrix} & \begin{pmatrix} 1 & \\ & 1\end{pmatrix} \arrow[l, "e"'] \arrow[dl] & \begin{pmatrix} 1 & \\ & p\end{pmatrix} \arrow[l, "e_0"'] \\
\begin{pmatrix} p & 1\\ & 1\end{pmatrix} & & \\
\vdots & & \\
\begin{pmatrix} p & p-1\\ & 1\end{pmatrix} & &
\end{tikzcd}$$

$$\begin{aligned} T(e_0) &= \left\{ \left(\begin{pmatrix} p & j \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) \,:\, j = 0. \ldots, p-1 \right\} \\ &= \left\{ e \,:\, \mathrm{orig}(e) = \mathrm{term}(e_0),\, e \neq \mathrm{flip}(e_0) \right\}. \end{aligned}$$

$$T(\overline{e}) = \ldots = \left\{ e' \,:\, \mathrm{orig}(e') = \mathrm{term}(\overline{e}),\, e' \neq \mathrm{flip}(\overline{e}) \right\}.$$

This is a non-backtracking condition.

Note that T is not normal. Indeed T^*Te_0 contains $\begin{pmatrix} p & 1 \\ & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \\ & p \end{pmatrix}$ and TT^*e_0 does not.

Ramanujan graphs and complexes - Lecture 17 - December 25

December 30, 2017

Remainder We were looking on G-equivariant branching operators on X, where $G = PGL_d(\mathbb{Z}[\frac{1}{p}])$, $K = PGL_d(\mathbb{Z})$ and $G/K \cong B^{(0)}$.

For different X, change K. For example $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$, $K = PGL_2(\mathbb{Z}) = \mathrm{Stab}_G(v_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})$ and $G \circlearrowleft X_p^2 = T_{p+1}$. To study ranching operators on edges, not that $G \circlearrowleft B^{(1)}$ transitively, thus $B^{(1)} \cong G/K_e$, where $K_e = \mathrm{Stab}_K(e)$.

Then, equivariant branching operator on $B^{(1)}$ corresponds to double K_e cosets.

E.g. $T(v,w) = \{(w,u) : u \neq v\}$. This is known as the non-backtracking walk operator. We saw that T is not normal and that Hecke operators do not commute any more.

Another example is $T(v,w) = \{(w,v)\}$ which is the flipping operator.

Understanding branching operators algebraically $e_\infty : \begin{pmatrix} 1 & \\ & p \end{pmatrix} \to \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. We know that there exists S such that $K_{e_\infty} S K_{e_\infty} = \uplus_{s\in S} s K_{e_\infty}$ and $T(e_\infty) = (se_\infty)_{s\in S}$.

Claim 0.1. $K_{e_\infty} \begin{pmatrix} 1 & \\ & p \end{pmatrix} K_{e_\infty} = \biguplus_{j=0}^{p-1} \begin{pmatrix} p & j \\ & 1 \end{pmatrix} K_{e_\infty}$

Proof. $\begin{pmatrix} p & j \\ & 1 \end{pmatrix} K_{e_\infty} e_\infty = \begin{pmatrix} p & j \\ & 1 \end{pmatrix} e_\infty = \left[\begin{pmatrix} p & j \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & p \end{pmatrix} \to \begin{pmatrix} p & j \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} p & jp \\ & p \end{pmatrix} \to \begin{pmatrix} p & j \\ & 1 \end{pmatrix}\right.$
$\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \to \begin{pmatrix} p & j \\ & 1 \end{pmatrix}\right] = e_j$, so $K_{e_\infty} \begin{pmatrix} p & \\ & 1 \end{pmatrix} K_{e_\infty} e_\infty = K_{e_\infty} \left[\begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix} \underset{e_0}{\to} \begin{pmatrix} p & \\ & 1 \end{pmatrix}\right] \subset \{e_0, \ldots, e_{p-1}\}$,

where the last inclusion is obtained geometrically.

$\begin{pmatrix} 1 & j \\ & 1 \end{pmatrix} \in K_{e_\infty}$

$$\begin{pmatrix} 1 & j \\ & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & \\ & p \end{pmatrix} \to \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & pj \\ & p \end{pmatrix} \to \begin{pmatrix} 1 & j \\ & 1 \end{pmatrix}\right] = e_\infty$$

and

$$\begin{pmatrix} 1 & j \\ & 1 \end{pmatrix} e_0 = \left[\begin{pmatrix} 1 & j \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \to \begin{pmatrix} 1 & j \\ & 1 \end{pmatrix}\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \to \begin{pmatrix} p & j \\ & 1 \end{pmatrix}\right] = e_j.$$

□

$K_{e_\infty} = K_{v_0} \cap K_{v_\infty}$,where $v_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $v_\infty = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$. Therefore $K_{e_\infty} = PGL_2(\mathbb{Z}) \cap \begin{pmatrix} 1 & \\ & p \end{pmatrix} PGL_2(\mathbb{Z}) \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}$. Since

$$PGL_2(\mathbb{Z}) = \{A \in M_2(\mathbb{Z}) : \det A = \pm 1\}$$

and

$$\begin{pmatrix} 1 & \\ & p \end{pmatrix} PGL_2(\mathbb{Z}) \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ & p \end{pmatrix} \begin{pmatrix} n & m \\ k & l \end{pmatrix} \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} n & \frac{m}{p} \\ pk & l \end{pmatrix}, \det = \pm 1$$

where $m, n, k, l \in \mathbb{Z}$ satisfy $nl - km = \pm 1$. Therefore

$$K_{v_\infty} = \left\{ A \in \begin{pmatrix} \mathbb{Z} & \frac{1}{p}\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det A = \pm 1 \right\}.$$

For example $= \begin{pmatrix} 2 & \frac{1}{p} \\ p & 1 \end{pmatrix} \in K_{v_\infty}$ and $\begin{pmatrix} 2 & \frac{1}{p} \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & p \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ p & p \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & p \end{pmatrix} = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$ and $\begin{pmatrix} 2 & \frac{1}{p} \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} p & 1 \\ & p \end{pmatrix}$.

$$\text{Stab}e_\infty = K_{v_0} \cap K_{v_\infty} = \left\{ A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det = \pm 1 \right\}.$$

What happens in PGL_3

$$\begin{tikzcd}
 & \begin{pmatrix} 1 & & \\ & p & \\ & & p \end{pmatrix} = v_2 & \\
\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = v_0 \arrow[ur, "e_1"] \arrow[rr, "e_0"] & t_0 & \begin{pmatrix} 1 & & \\ & 1 & \\ & & p \end{pmatrix} = v_1 \arrow[ul]
\end{tikzcd}$$

$$\text{Sta}(e_0) = \left\{ A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \frac{1}{p}\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \frac{1}{p}\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det = \pm 1 \right\}$$
$$= \left\{ A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det = \pm 1 \right\}$$

$$\text{Stab}(e_1) = \left\{ A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \begin{pmatrix} \mathbb{Z} & \frac{1}{p}\mathbb{Z} & \frac{1}{p}\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det = \pm 1 \right\}$$
$$= \left\{ A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det = \pm 1 \right\}$$

and therefore

$$\text{Stab}(t_0) = \text{Sta}(e_0) \cap \text{Sta}(e_1) = \left\{ A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det = \pm 1 \right\}.$$

We saw that $\text{PGL}_d(\mathbb{Z}[\frac{1}{p}])$ acts transitively on $B^{(d-1)}$ (top cells).

Exercise 0.2. For $\sigma_0 \in B^{(d-1)}$, show that $\mathrm{Stab}(\sigma_0) = \left\{ \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & & \cdots & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & & & \\ \vdots & & \mathbb{Z} & & \vdots \\ & & & \ddots & \\ p\mathbb{Z} & & \cdots & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} : \det = \pm 1 \right\}$.

This is known as the Iwahori group of G.

An example for a branching operators on edges of $B(PGL_3)$. See illustration.

$C^0 \underset{\partial}{\overset{\delta}{\rightleftarrows}} C^1$. We saw that if G acts transitively on X , $x_0 \in X$ and $K = \mathrm{Stab}_G(x_0)$. Then $KgK : X \to 2^X$ defined by $(KgK)g'x_0 = g'KgKx_0 = g'Kgx_0$ is well defined. Indeed

$$(KgK)g'kx_0 = g'kKgx_0 = g'Kgx_0.$$

Assume $G \circlearrowright X$, $X = Gx_0 \uplus Gx_1$ and denote $K_i = \mathrm{Stab}_G x_i$. Now K_0gK_1 defines an equivariant branching operator $Gx_0 \to 2^{Gx_1}$ by

$$(K_0gK_1)(g'x_0) = g'K_0gK_1x_1 = g'K_0gx_1.$$

Ramanujan graphs and complexes

31/12/17

X transitive G-set. $x_0 \in X$, $K = \mathrm{Stab}_G x_0$

Every KSK induces a branching operator $T: X \to 2^X$. Specifically if $KSK = \coprod_{s \in S} sK$, then $T(gx_0) = \{gsx_0\}_{s\in S}$. We also denote $T \circlearrowright L^p(X)$ $(1 \le p \le \infty)$

By $Tf(x) = \sum_{y \in T(x)} f(y)$.

$$L^\infty(X) \cong L^\infty(G/K) = L^\infty(G)^K$$
$$\tilde{f}(gx_0) = f(gK) \longleftarrow\!\shortmid f$$

$L^\infty(G)$ is a G representation by right translation: $(gf)(g') = f(g'g)$

For any G-rep. V and $H \le G$ we write V^H the pointwise invariant vectors $V^H = \{v \in V : hv = v \ \forall h \in H\}$

So $L^\infty(G)^K = \{f \in L^\infty(G) : kf = f \ \forall k \in K\}$

$f(gk) = f(g) \ \forall k \in K \longleftrightarrow$ f can be thought of as a

$$\begin{tikzcd}
\text{Directly} & L^\infty(G)^K \arrow[r] & L^\infty(X) \\
& f \arrow[r, mapsto] \arrow[d, bend right, "?"'] \arrow[dr, phantom, "//"] & \tilde{f}(gx_0) = f(g) \arrow[d, "T"] \\
& L^\infty(G)^K \arrow[r, dash, "\cong"] & L^\infty(X)
\end{tikzcd}$$

Let $\alpha_S = \sum_{s \in S} s \in \mathbb{C}G$

Claim: ① If V is any G-rep, then α_S takes V^K to itself, namely $\alpha_S(V^K) \subseteq V^K$.

② For $V = L^\infty(G)$, α_S corresponds to T in the sense of the commutative diagram above (under

$$\begin{tikzcd}
V^K \arrow[loop below, "\alpha_S"] \arrow[r, phantom, "\simeq"] & L^\infty(G)^K \arrow[r, phantom, "\cong"] & L^\infty(X) \arrow[loop below, "T"]
\end{tikzcd}$$
).

Proof:

① We need to show for $v \in V^K$ ~~that~~ and $k \in K$ that $k\alpha_S v = \alpha_S v$.

$$k\alpha_S v = k\sum_{s \in S} sv = \sum_{s \in S} ksv$$

Since

$$k \coprod_{s\in S} sK = \coprod_{s\in S} ksK = KSK = \coprod_{s\in S'} sK$$

$\Rightarrow$ [illegible] $\forall s\in S \ \exists s'\in S$ and $k'\in K$ s.t.

$ks = s'k'$. Furthermore $s \mapsto s'$ is a permutation on S

$$\Rightarrow k\alpha_{\alpha} v = \sum_{s\in S} ksv = \sum_{s\in S} s'\underbrace{k'v}_{v} = \sum_{s\in S} s'v = \alpha_S v.$$

2) For $V = L^\infty(G)_{\text{right}}$ and $\tilde{f} \in L^\infty(X) \longmapsto f(g) := \tilde{f}(gx_0)$

$$\alpha_S f(g) = \sum_{s\in S} sf(g) = \sum_{s\in S} f(gs)$$

$$\widetilde{\alpha_S f}(gx_0) = \sum_{s\in S} f(gs) = \sum_{s\in S} \widetilde{f(gsx_0)} = T\tilde{f}(gx_0). \qquad \boxed{/\!/}$$

Now, for $\Gamma \leq G$ we look at $\Gamma\backslash X \cong \Gamma\backslash G/K$ we get

$$\begin{tikzcd} L^\infty(\Gamma\backslash X) \arrow[r, phantom, "\cong"] \arrow[d, "T"'] & L^\infty(\Gamma\backslash G/K) \arrow[r, phantom, "\overset{\Gamma}{\cong}"] & L^\infty(\Gamma\backslash G)^K \arrow[d, "\alpha_S"] \\ L^\infty(\Gamma\backslash X) \arrow[r, phantom, "\cong"] & L^\infty(\Gamma\backslash G/K) \arrow[r, phantom, "\cong"] & L^\infty(\Gamma\backslash G)^K \end{tikzcd}$$

Claim: For $V = L^\infty(\Gamma\backslash G)$ $\Gamma \leq G$ right translation. α_s corresponds to T under ~~V~~ $V^K \cong L^\infty(\Gamma\backslash X)$

The same proof as the one for ② works with g replaced by Γg. ▨

On the building $\alpha_s \circlearrowright L^2(G) \rightsquigarrow \alpha_s \circlearrowright L^2(\Gamma\backslash G)$

↪ a finite graph/complex

Why do we care?

Because we can decompose reps. to irreducible rep.

$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{i\in I}} V_i$. From this we get $L^2(\Gamma\backslash X) = L^2(\Gamma\backslash G)^K$

$= \widehat{\bigoplus_{i\in I}} V_i^K$

$$\begin{tikzcd}
L^2(\Gamma\backslash X) \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dr, phantom, "/\!/"] & L^2(\Gamma\backslash G)^K \arrow[r, phantom, "="] \arrow[d, "\alpha_s"'] \arrow[dr, phantom, "/\!/"] & \widehat{\bigoplus_{i\in I}} V_i^K \arrow[d, "\alpha_s|_{V_i^K}"] \\
L^2(\Gamma\backslash X) \arrow[r, phantom, "\cong"] & L^2(\Gamma\backslash G)^K \arrow[r, phantom, "\overset{!}{=}"] & \widehat{\bigoplus_{i\in I}} V_i^K
\end{tikzcd}$$

↙

Since $\alpha_s \in \mathbb{C}G$.
$\bigoplus V_i$ respects α_s.

On the building

$$\alpha_s \circlearrowright L^2(G) = \bigoplus_{V\in\hat{G}} V_E^K$$

$$\alpha_s \circlearrowright L^2(\Gamma\backslash G) = \bigoplus V_i^K$$

$\Gamma\backslash G/K$ is Ramanujan $\Longleftrightarrow$ every irreducible rep. $V \subseteq L^2(\Gamma\backslash G)$ is also in the regular rep.

$G \underset{\text{tran}}{\circlearrowright} X$, $x_0 \in X$ $K = \mathrm{Stab}_G(x_0)$

$T : X \to 2^X$ equi. branching operator, given by KSK, namely

$KSK = \coprod_{s \in S} sK$ and $T(gx_0) = \{gsx_0\}_{s \in S}$

For $\Gamma \leq G$ T descends to $\Gamma \backslash X \cong \Gamma \backslash G / K$. To understand the spectrum of $\underbrace{T \text{ on } \Gamma\backslash X}_{\text{meaning on } L^p(\Gamma\backslash X)}$ we use rep. theory.

We saw:

$$\begin{tikzcd}
L^\infty(\Gamma\backslash X) \arrow[d, "T"'] \arrow[r, phantom, "\cong"] & L^\infty(\Gamma\backslash G)^K \arrow[d, "\alpha_S"] \\
L^\infty(\Gamma\backslash X) \arrow[r, phantom, "\cong"] & L^\infty(\Gamma\backslash G)^K
\end{tikzcd}$$

action by right trans.

$L^\infty(\Gamma\backslash G)$ is a G-rep. by $gf(\Gamma x) = f(\Gamma x g)$

$\alpha_S = \sum_{s \in S} s \in \mathbb{C}G$.

Decompose $L^\infty(\Gamma\backslash G) = \bigoplus_{i \in I} V_i$

$\downarrow$ of G-rep $(= \mathbb{C}G$ modules)

α_S decomposes on it

The same holds for L^2

$$\begin{tikzcd}
L^2(\Gamma\backslash X) \arrow[r, phantom, "\cong"] \arrow[d, "T"'] & L^2(\Gamma\backslash G)^K \arrow[r, phantom, "\cong"] \arrow[d, "\alpha_s"] & \bigoplus_{i\in I} V_i^K \arrow[d, "\bigoplus_{i\in I} \alpha_s|_{V_i}"] \\
L^2(\Gamma\backslash X) \arrow[r, phantom, "\cong"] & L^2(\Gamma\backslash G)^K \arrow[r, phantom, "="] & \bigoplus_{i\in I} V_i^K
\end{tikzcd}$$

assume $\Gamma\backslash X$ is finite

$$\Rightarrow \quad \operatorname{Spec}\left(T \circlearrowright L^2(\Gamma\backslash X)\right) = \bigcup_{i\in I} \operatorname{Spec}\left(\alpha_s|_{V_i^K}\right).$$

Note: only the isomorphism type of the V_i matter.

If $\bigoplus_{i\in I} V_i^K \cong \bigoplus_{i\in I} W_i^K$ with $W_i \cong V_i$ as G-rep. $\forall i\in I$

Then

$$\begin{tikzcd}
\bigoplus_{i\in I} V_i^K \arrow[r, phantom, "\cong"] \arrow[d, "\bigoplus_{i\in I} \alpha_s|_{V_i}"] & \bigoplus W_i^K \arrow[d, "\bigoplus_{i\in I} \alpha_i|_{W_i^K}"] \\
\bigoplus_{i\in I} V_i^K \arrow[r, phantom, "\cong"] & \bigoplus_{j\in I} W_i^K
\end{tikzcd}$$

$\longrightarrow$ it is enough to understand those

$$\operatorname{Spec}\left(T \circlearrowright L^2(\Gamma\backslash X)\right) = \bigcup_{i\in I} \operatorname{Spec}\left(\alpha_s \circlearrowright W_i^K\right)$$

<u>Example</u>: $G = PGL_2(\mathbb{Z}[\tfrac{1}{p}])$ $\quad K = PGL_2(\mathbb{Z})$ $\quad X = T_{p+1},\ x_0 = \mathbb{I}$, $\quad T = \underset{K\begin{pmatrix}1&\\&p\end{pmatrix}K}{\overset{\parallel}{Adj}}$

$\Gamma \leq G$ a subgroup s.t. $\Gamma\backslash X$ is a finite graph ($(p+1)$-regular).

$$K\begin{pmatrix}1&\\&p\end{pmatrix}K = \begin{pmatrix}1&\\&p\end{pmatrix}K \uplus \biguplus_{j=0}^{p-1} \begin{pmatrix}p&j\\0&1\end{pmatrix}K.$$

<u>Facts</u>: Let $z_1, \underset{\frac{1}{z_1}}{\overset{\parallel}{z_2}} \in \mathbb{C}$. Define a rep. of G as follows:

$$\underset{\{z_1,z_2\}}{\overset{}{V_{\vec{z}}}} = \left\{ f : G \to \mathbb{C} : \begin{array}{l}\forall \text{ upper triangular } b \in G \quad \forall g \in G\\ f(bg) = \chi_{\vec{z}}(b) f(g)\end{array} \right\}, \text{ where}$$

$$\chi_{\vec{z}}\left(\begin{pmatrix}p^{m_1} & *\\ 0 & p^{m_2}\end{pmatrix}\right) = \left(\frac{z_1}{\sqrt{p}}\right)^{m_1}\left(\sqrt{p}\, z_2\right)^{m_2}$$

This is the induction from B to G of $\chi_{\vec{z}}$, denoted $\mathrm{Ind}_B^G(\chi_{\vec{z}})$.

$\chi_{\vec{z}}$ is a $\underbrace{\text{1-dim rep}}_{\mathrm{Hom}: G \to \mathbb{C}^*}$ of B upper triangular.

G acts on $V_{\vec{z}}$ by multiplication from the right:
$(gf)(x) = f(xg)$

<u>Fact</u>: For G, K, Γ as above if $L^2(\Gamma\backslash G) = \widehat{\bigoplus_{i \in I}} V_i$
then for all but finitely many $i \in I$ $\quad V_i^K = 0$.

Proof: Follows from the diagram since $\Gamma\backslash X$ is finite.

② Furthermore: If $V_i^K \neq 0$ and V_i is irreducible, then $V_i = V_{\vec{z}}$ for some $\vec{z}$ or $V_i \cong \mathbb{C}$ ~~and~~ the rep is either $g.\alpha = \alpha \; \forall \alpha \in \mathbb{C}$ or $g.\alpha = (-1)^{\text{level}(g)} \alpha \; \forall \alpha \in \mathbb{C}$.

③ If $V_i^K \neq 0$, then $\dim V_i^K = 1$, so α_S acts on V_i^K as a scalar, each V_i contributes one eigenvalue to Adj.

- If V_i is trivial $g.\alpha = \alpha$, then $V_i^K = V_i = \mathbb{C}$ $\quad \mathbb{C}_{triv}$
- If V_i is the determinant $g.\alpha = (-1)^{\text{level}(g)} . \alpha = (-1)^{\text{ord}_p(\det(g))} \alpha$, then $\quad \mathbb{C}_{det}$
  $V_i^K = V_i$ since $\det(g \in PGL_2(\mathbb{Z})) = \pm 1$

Let's compute α_S on those $\mathbb{C}_{triv}$ and $\mathbb{C}_{det}$.

$v \in V = \mathbb{C}_{\cancel{det}\, tri}$ s.t. $v \neq 0$. What is $\alpha_S v = \begin{pmatrix} 1 & \\ & p \end{pmatrix} v + \sum_{j=0}^{p-1} \begin{pmatrix} p & j \\ & 1 \end{pmatrix} v = (p+1) v$

triv action

α_S acts on $\mathbb{C}_{triv}$ by multiplication by $p+1$.

Whenever some $V_i \leq L^2(\Gamma\backslash G)$ is $\cong \mathbb{C}_{triv}$ $\quad (p+1) \in \text{Spec}(\text{Adj}_{\Gamma\backslash X})$

$V = \mathbb{C}_{det}$, $0 \neq v \in V$.

$$\alpha_S v = \binom{\ }{p} v + \sum_{j=0}^{p-1} \binom{p}{j} v = -(p-1) v$$

if $\mathbb{C}_{det} \leq L^2(\Gamma \backslash G)$ then $\Gamma \backslash X$ is bipartite.

We got $\mathbb{C}_{det}$, $\mathbb{C}_{triv}$ are the trivial eigenvalues.

In general an eigenvalue is trivial if it comes from a 1-dim representation

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~~If $V \cong V_{\vec{z}}$ and $V_{z}^{K} \neq 0$ then take $0 \neq f \in V_{\vec{z}}^{K}$ s.t. $f(g_0) = 1$~~

~~Then $\forall g \in G$ $g = b \cdot k$ with $b \in B$, $k \in K$. Now, we get~~

Let $f \in V_{\vec{z}}^{K}$. Then $\forall g \in G$ we can write $g = bk$ with $b \in B$ and $k \in K$. $\Rightarrow$ $f(g) = f(bk) = \chi_{\vec{z}}(b) f(k) = \chi_{\vec{z}}(b) f(I)$.

$\Rightarrow V_{\vec{z}}^{K}$ is $\leq$ one dim.

It is one dim by defining $f(bk) = \chi_{\vec{z}}(b)$ if it is well defined it is in $V_{\vec{z}}^{K}$.

It is well defined

$$bk = b'k' \Rightarrow b'^{-1}b = k'k^{-1} \in B\cap K$$ but if $g \in B\cap K$

then $\chi_{\bar z}(g) = 1$ since $g \in B\cap K \Rightarrow g - \begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}$

---

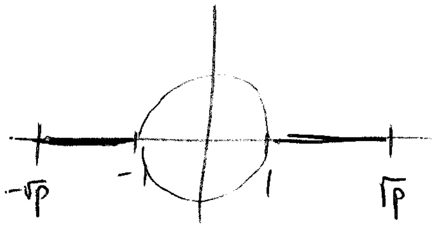
e.v. $\forall \vec z$ $\quad V_{\vec z}^{K} = \mathbb{C} f_{\vec z}$, where $f_{\vec z}(bk) = \chi_{\vec z}(b)$

$$\alpha_S f_{\vec Z} = \begin{pmatrix}1 & \\ & p\end{pmatrix} f_{\vec z} + \sum_{j=0}^{p-1} \begin{pmatrix}p & j \\ & 1\end{pmatrix} f_{\bar z} = \lambda f_{\bar z}$$

$$\alpha_S f_{\bar z}(I) = f_{\bar z}\begin{pmatrix}1 & \\ & p\end{pmatrix} + \sum_{j=0}^{p-1} f_{\bar z}\begin{pmatrix}p & j \\ & 1\end{pmatrix} = \chi_{\bar z}\begin{pmatrix}1 & \\ & p\end{pmatrix} + \sum_{j=0}^{p-1} \chi_{\bar z}\begin{pmatrix}p & j \\ & 1\end{pmatrix}$$

$$= \frac{\sqrt{p}\, z_2}{} + p\left(\frac{z_1}{\sqrt{p}}\right) = \sqrt{p}\,(z_1 + z_2).$$

Fact: if $V_{\bar z}$ has a unitary structure, then either $z_1 \in S^1$ or $z_1 \in \pm[1, \sqrt{p}]$ (We have such a structure from our Hilbert space)

$\Rightarrow$ 

Case a $\quad z_1 \in S^1$

Case b $\quad z_1 \in \pm[1, \sqrt{p}]$

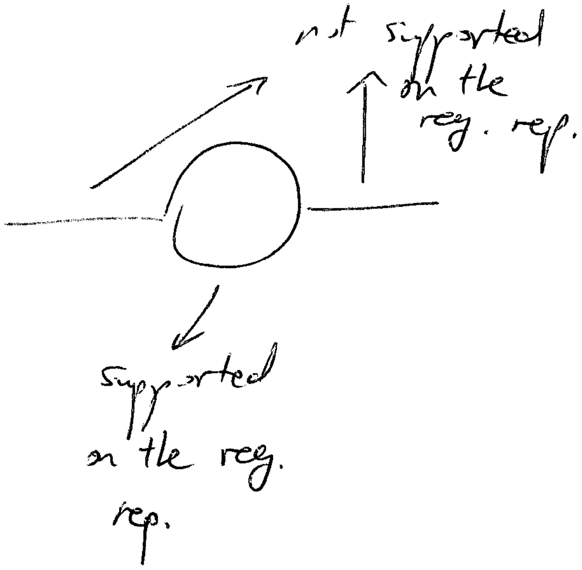
Case a $\lambda = 2\,Re(z_1)\sqrt{p} \in [-2\sqrt{p}, 2\sqrt{p}]$ (Ramanujan)

Case b $z_1 = 1 \to 2\sqrt{p}$

$z_1 = \sqrt{p} \to p+1$

$\lambda \in^{I} [2\sqrt{p}, p+1]$

$\vec{z}$ - Satake parameter.



NBRW on $PGL_2(\mathbb{Z}[\frac{1}{p}]) \equiv G$

$\mathcal{B} = T_{p+1} \Rightarrow$ [illegible] $\mathcal{B}^1_{\text{directed}} = G/E \qquad E = \left\{\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in K\right\}$

For $\Gamma \leq G \quad X = \Gamma\backslash\mathcal{B}$ finite $\quad X^1_I \longleftrightarrow \Gamma\backslash G/E$

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{i\in I}} V_i \longrightarrow L^2(X^{(1)}_I) = L^2(\Gamma\backslash G)^E = \bigoplus_{i\in I} V^E$$

decomp of $L^2(\Gamma\backslash G)$
as a G-rep.

We know the unitary rep: $\mathbb{C}_{triv}$, $\mathbb{C}_{det}$ $\quad g\alpha = (-1)^{ord_p(\det g)} \cdot \alpha$

$$V_{\vec{z}} = \left\{ f : G \to \mathbb{C} : \begin{array}{c} \forall \text{upper triangular } b \ \forall g \\ f(bg) = \chi_{\vec{z}}(b) f(g) \end{array} \right\}$$

$\chi_{\vec{z}}\left(\begin{pmatrix} p^{n_1} & * \\ & p^{n_2} \end{pmatrix}\right) = \left(\frac{z_1}{\sqrt{p}}\right)^{n_1} (z_2\sqrt{p})^{n_2} \qquad \chi_{\vec{z}} : B \longrightarrow \mathbb{C}^*$ is a character. $\quad (z_1^{-1} = z_2)$

either: ⓐ $|z_1| = 1$
ⓑ $z_1 \in \pm[1, \sqrt{p}]$

X is Ramanujan if every $V_{\vec{z}}$ which occurs in $L^2(\Gamma\backslash G)$ is of type ⓐ

NBRW:

$$T(\overset{\cdot}{v} \longrightarrow \overset{\cdot}{w}) = \{\overset{\cdot}{w} \longrightarrow \overset{\cdot}{u} \; : \; u \neq v\}$$

$e_0 = \{\begin{pmatrix}1 & \\ & p\end{pmatrix} \longrightarrow \begin{pmatrix}1 & \\ & 1\end{pmatrix}\}$

T as double E-coset:

$$T\left(\begin{pmatrix}1 & \\ & p\end{pmatrix} \to \begin{pmatrix}1 & \\ & 1\end{pmatrix}\right) = \left\{\begin{pmatrix}1 & \\ & 1\end{pmatrix} \rightrightarrows \begin{pmatrix}p & j \\ & 1\end{pmatrix}\right\}_{j=0,\dots,p-1} = E\begin{pmatrix}p & \\ & 1\end{pmatrix}E$$

$$T(ge_0) = gE\begin{pmatrix}p & \\ & 1\end{pmatrix}\cancel{E}e_0 = gE\begin{pmatrix}p & \\ & 1\end{pmatrix}e_0$$

$$L^2(\Gamma\backslash G) \rightsquigarrow \underset{\circlearrowleft \, T}{L^2(X_I^{(1)})} \overset{\text{depends on } e_0}{\cong} L^2(\Gamma\backslash G)^E = \bigoplus_{i\in I} V_i^E \cong \underset{\circlearrowleft \, \alpha_S}{\bigoplus_{i\in I} V_{\vec{z}_i}^E}$$

$$\alpha_S = \sum_{j=0}^{p-1}\begin{pmatrix}p & j \\ & 1\end{pmatrix}$$

$f \in V_{\vec{z}}^K$ $\qquad f(g) = f(bk) = \chi_{\vec{z}}(b)\, f(1)$

some $b\in B$ $k\in K$

$f \in V_{\vec{z}}^E$ $\qquad f(g) = f(b\,t_i\,e) = \chi_{\vec{z}}(b)\, f(t_i)$ $\qquad \dim V_{\vec{z}}^E \le |B\backslash G/E|$ + group in place of E.

if $G = \coprod_{i=1}^{r} B\,t_i E$

For our E: $G = BE \longleftrightarrow B$ acts transitively on directed edges.

$$\begin{tikzcd}
v \arrow[r] \arrow[d, "\exists g"'] & w \\
\begin{pmatrix}1&\\&1\end{pmatrix} \arrow[r] & \begin{pmatrix}p&j\\&1\end{pmatrix} \text{ or } \begin{pmatrix}1&\\&p\end{pmatrix}
\end{tikzcd}$$

in $B \cap K = \left\{\begin{pmatrix}1&k\\&1\end{pmatrix}\right\} \Rightarrow$ doesn't take $\begin{pmatrix}1&\\&p\end{pmatrix}$ to $\begin{pmatrix}p&j\\&1\end{pmatrix}$

We got Edges $= B\left(\begin{pmatrix}1&\\&1\end{pmatrix} \to \begin{pmatrix}1&\\&p\end{pmatrix}\right) \amalg B\left(\begin{pmatrix}1&\\&1\end{pmatrix} \to \begin{pmatrix}p&\\&1\end{pmatrix}\right)$

$\Longleftrightarrow G$ is a disjoint union $G = BE \amalg B\begin{pmatrix}&1\\1&\end{pmatrix}E$

$\Longrightarrow \dim V_{\bar z}^{E} \leq 2$ and each $f \in V_{\bar z}$ is determined by $f(I)$ and $f\left(\begin{pmatrix}&1\\1&\end{pmatrix}\right)$

Actually $\dim V_{\bar z}^{E} = 2$. Take $t_1 = \begin{pmatrix}1&\\&1\end{pmatrix}$ $t_2 = \begin{pmatrix}&1\\1&\end{pmatrix}$ $i,j = 1,2$

$f_i(t_j) = \delta_{ij}$ [HW: Show f_i are well defined]

NBRW space

$$(\alpha_S f_1)(t_1) = \sum \begin{pmatrix}p&j\\&1\end{pmatrix} f_1(I) = \sum f_1\left(\begin{pmatrix}p&j\\&1\end{pmatrix}\right) = \sum \chi_{\bar z}\left(\begin{pmatrix}p&j\\&1\end{pmatrix}\right)$$

$$= \sum_{j=0}^{p-1} \frac{z_1}{\sqrt{p}} = \sqrt{p}\, z_1$$

$$\alpha_s f_1 \left(t_2\right) = \sum \binom{p\ j}{1} f_1 \left(1\ {}^1 \right) = \sum f\left(\left(p_j^{\,1}\right)\right)$$

$G = PGL_2(\mathbb{Z}[\tfrac{1}{p}])$

$K = PGL_2(\mathbb{Z})$

$e_0 = \left[\begin{pmatrix}1&\\&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&\\&1\end{pmatrix}\right]$

$\underline{E} = \mathrm{Stab}_G\, e_0 = \left\{ \begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ p\mathbb{Z} & \mathbb{Z}\end{pmatrix} \in K \right\}$

$\mathcal{B}_{\underline{I}}^1 = Ge_0 \implies L^\infty(\mathcal{B}_{\underline{I}}^1) \cong L^\infty(G)^E$

$\Gamma \leq G$ s.t. $X = \Gamma \backslash \mathcal{B}$ is a finite graph. Then

$$L^2(X_{\underline{I}}^1) \cong L^2(\Gamma\backslash G)^E = \bigoplus_{i \in I} V_i^E$$

$\circlearrowright$ $\alpha_s \in \mathbb{C}G$ $\circlearrowright$

T NBRW operator α_s

$$\mathrm{Spec}(T \circlearrowright X_{\underline{I}}^1) = \bigcup_{i \in I} \mathrm{Spec}(\alpha_s \circlearrowright V_i^E)$$

$\mathcal{B}$ - triangular matrices $\chi_{\vec{z}} : \mathcal{B} \to \mathbb{C}^\times$ hom $V_{\vec{z}} = \mathrm{Ind}_B^G \chi_{\vec{z}}$

$V_{\vec{z}} = \{ f : G \to \mathbb{C} : f(bg) = \chi_{\vec{z}}(b) f(g) \}$.

$V_{\vec{z}}^E = (\mathrm{Ind}_B^G \chi_{\vec{z}})^E = \{ f : G \to \mathbb{C} : f(bge) = \chi_{\vec{z}}(b) f(g) \ \forall b \in B, g \in G, e \in E \}$.

$\dim V_{\vec{z}}^E \leq |{}_B\backslash G/_E|$ Actually there is equality.

We saw $B'_{\underline{I}} = Be_0 \amalg \underbrace{B\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}e_0}$

$e_1 = \left[\begin{pmatrix} p & \\ & 1\end{pmatrix} \to \begin{pmatrix} 1 & \\ & 1\end{pmatrix}\right]$

$$\Longrightarrow \quad G = B\underbrace{\begin{pmatrix} 1 & \\ & 1\end{pmatrix}}_{t_1}E \amalg B\underbrace{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}_{t_2}E.$$

We took $f_i \in V_{\vec{z}}^{E}$ defined by $f_i(t_j) = \delta_{ij}$ (HW: well defined)

those are indep.

Pause: For PGL_d and $K_{top} = $ Iwahori $= \left\{\begin{pmatrix} \ddots & \mathbb{Z} \\ p\mathbb{Z} & \ddots \end{pmatrix} \in K\right\} \equiv \text{Stab}_p(\text{ordered chamber})$

we have $G = \coprod_{\sigma \in S_d} B\sigma K_{top}$ $\quad \dim V_{\vec{z}}^{K} = d!$

Iwahori - Bruhat decomposition $f_\sigma(\tau) = \delta_{\sigma,\tau}$ $\quad \sigma, \tau \in S_d$ the extension to a function in $V_{\vec{z}}^{K_{top}}$ is well defined

For NBRW $\alpha_s = \sum_{j=0}^{p-1} \begin{pmatrix} p & j \\ & 1\end{pmatrix}$

because

$$\begin{pmatrix} p & j \\ & 1\end{pmatrix} e_0 = \begin{pmatrix} p & j \\ & 1\end{pmatrix}\left[\begin{pmatrix} 1 & \\ & p\end{pmatrix} \to \begin{pmatrix} 1 & \\ & 1\end{pmatrix}\right] = \left[\begin{pmatrix} p & pj \\ & p\end{pmatrix} \to \begin{pmatrix} p & j \\ & 1\end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 & \\ & 1\end{pmatrix} \to \begin{pmatrix} p & j \\ & 1\end{pmatrix}\right]$$

$$\alpha_s f_i(t_1) = \sum_{j=0}^{p-1}\left(\begin{pmatrix} p & j \\ & 1\end{pmatrix} f_i\right)\begin{pmatrix} 1 & \\ & 1\end{pmatrix} = \sum_{j=0}^{p-1} f_i\begin{pmatrix} p & j \\ & 1\end{pmatrix} = \sum \chi_{\vec{z}}\left(\begin{pmatrix} p & j \\ & 1\end{pmatrix}\right) f_i\Big(\overset{t_1}{\begin{pmatrix} 1 & \\ & 1\end{pmatrix}}\Big)$$

$$= z_1\sqrt{p}\, f_i(t_1) = z_1\sqrt{p}\, \delta_{i,1}$$

$$(\alpha_s f_i)(t_2) = \sum_{j=0}^{p-1} f_i\left(\begin{pmatrix} & 1\\ p & j\end{pmatrix}\right) = \circledast$$

$g \in b t_i E \iff g e_0 = b t_i e_0$

$$\begin{pmatrix} & 1\\ p & j\end{pmatrix} e_0 = \left[\begin{pmatrix} p & p\\ & p j\end{pmatrix} \to \begin{pmatrix} & 1\\ p & j\end{pmatrix}\right] = \left[\begin{pmatrix} 1 & \\ & 1\end{pmatrix} \to \begin{pmatrix} & 1\\ p & j\end{pmatrix}\right]$$

$j \neq 0$

$$\begin{pmatrix} & 1\\ p & j\end{pmatrix}\underbrace{\begin{pmatrix} -j & -m\\ p & \bar{j}\end{pmatrix}}_{\in K} = \begin{pmatrix} p & \bar{j}\\ 0 & 1\end{pmatrix}$$

write $\bar{j} = j^{-1} \pmod p$

so $j\bar{j} = mp + 1$ for some $m \in \mathbb{Z}$

$$\| $$

$$\begin{pmatrix} p & \bar{j}\\ & 1\end{pmatrix} e_0$$

$j = 0$

$$\left[\begin{pmatrix} 1 & \\ & 1\end{pmatrix} \to \begin{pmatrix} 1 & \\ & p\end{pmatrix}\right]$$

$$\|$$

$$\begin{pmatrix} 1 & \\ & p\end{pmatrix} e_1$$

$$\circledast \quad \alpha_s f_i(t_2) = \sum f_i\left(\begin{pmatrix} & 1\\ p & j\end{pmatrix}\right) = f_i\left(\begin{pmatrix} 1 & \\ & p\end{pmatrix} t_2\right) + \sum_{j=1}^{p-1} f_i\left(\begin{pmatrix} p & \bar{j}\\ & 1\end{pmatrix} e_0\right)$$

$$= \sqrt{p}\, z_2\, \delta_{i,2} + (p-1)\frac{z_1}{\sqrt{p}}\,\delta_{i,1}$$

$$\text{So } \left[\alpha_s \vec{\sigma} V_{\vec{z}}^{E}\right]_{(f_1, f_2)} = \begin{pmatrix} \sqrt{p}\, z_1 & \frac{p-1}{\sqrt{p}} z_1 \\ 0 & \sqrt{p}\, z_2\end{pmatrix} \longrightarrow \operatorname{Spec}\left(\alpha_s \vec{\sigma} V_{\vec{z}}^{E}\right) = \{\sqrt{p}\, z_1, \sqrt{p}\, z_2\}$$

For Ramanujan graphs $|z_1| = |z_2| = 1$

$L^2(X_{\pm}^1) = L^2(\Gamma \backslash G)^E = \bigoplus V_{\vec{z}}^E$ Peres & Lubetzky got this decomposition by elementary methods.

<u>Rem</u>: If V is a unitary rep of $G = PGL_2$ with $V^K \neq 0$ then

$V \cong \mathbb{C}_{triv} \longrightarrow k = p+1$

or

$V \cong \mathbb{C}_{det} \longrightarrow -k = -p-1$

or

$V \cong V_{\vec{z}} \longrightarrow$ non & trivial spec

If $V^E \neq 0$ then either

$V^K \neq 0 \Rightarrow$ we know (HW compute NBRW on $\mathbb{C}_{triv}$ and $\mathbb{C}_{det}$)

or

$V^K = 0 \Rightarrow V$ is the Steinberg rep. or Steinberg rep. $\otimes \mathbb{C}_{det}$

$\downarrow$ $\downarrow$

Spec 1   Spec -1

those two correspond to cycles in the graph.

In ONB $\left[\alpha_s \circlearrowright V_{\underline{z}}\right] = \begin{pmatrix} \sqrt{p}\, z_1 & (p-1) z_1 \\ & \sqrt{p}\, z_2 \end{pmatrix} = W$

$$\|W\|_{op} = \sqrt{\lambda_{max}(WW^*)} = \quad \left\|\begin{pmatrix} \sqrt{p} & p-1 \\ & \sqrt{p} \end{pmatrix}\right\|_{op} = \sqrt{\left\|\begin{pmatrix} \sqrt{p} & p-1 \\ & \sqrt{p} \end{pmatrix}\begin{pmatrix} \sqrt{p} & \\ p-1 & \sqrt{p} \end{pmatrix}\right\|} = \sqrt{\left\|\begin{pmatrix} p+(p-1)^2 & \sqrt{p}(p-1) \\ \sqrt{p}(p-1) & p \end{pmatrix}\right\|}$$

$\downarrow$

take $z_1 = 1 = z_2$

this is the Harish-Chandra function

$$= \sqrt{\left\|\begin{pmatrix} (p-1)^2+p & \sqrt{p}(p-1) \\ \sqrt{p}(p-1) & p \end{pmatrix}\right\|_{op}} = p$$

$\downarrow$

the worst
for p-reg
branching operator.

However

$$\|W^\ell\|_{op} = \sqrt{\lambda_{max}\left(W^\ell W^{*\ell}\right)} \neq \sqrt{\lambda_{max}\left((WW^*)^\ell\right)} = p^\ell$$

$$W^\ell = \begin{pmatrix} p^{\ell/2} & \approx p^{\ell/2} \cdot \ell \\ & p^{\ell/2} \end{pmatrix} \Rightarrow \|W^\ell\|_{op} \sim p^{\ell/2} \cdot$$

Non transitive action

$G \circlearrowright X \qquad X = \coprod_{i=1}^{r} Gx_i \qquad K_i = \mathrm{stab}\, x_i$

$T : X \to 2^X$ finite G- eq. branching operator

T is defined by Tx_i

For $i,j \in \{1, \dots, r\}$ take $S_{ij} \subseteq G$ such that

$$Tx_i = \bigcup_{j=1}^{r} \{ sx_j : s \in S_{ij} \}$$

Claim: $K_i S_{ij} K_j = \coprod_{s \in S_{ij}} sK_j$. ~~and vice versa~~

vice versa:

Any S_{ij} like this defines a G-equiv branching operator on X, by $Tx_i = \bigcup_{j=1}^{r} K_i S_{ij} x_j$

$$L^2(X) = L^2\Big(\coprod_{i=1}^{r} Gx_i\Big) = \bigoplus_{i=1}^{r} L^2(Gx_i) = \bigoplus_{i=1}^{r} L^2(G)^{K_i} = \bigoplus$$

$$L^2(\Gamma\backslash X) = L^2\Big(\coprod_{i=1}^{r} \Gamma\backslash Gx_i\Big) = \bigoplus_{i=1}^{r} L^2(\Gamma\backslash Gx_i) = \bigoplus_{i=1}^{r} L^2(\Gamma\backslash G)^{K_i} =: \bigoplus_{\mu\in I} \bigoplus_{i=1}^{r} V_\mu^{K_i}$$

$\circlearrowright$

T

$\uparrow$

$L^2(\Gamma\backslash X) = \bigoplus_{\mu\in I} V_\mu$

$\swarrow$

decomp.
as G-rep.

For a rep V of G define

$$V^{(K_1,\dots,K_r)} = \bigoplus_{i=1}^{r} V^{K_i}.$$

Define $\alpha_S \in M_r(\mathbb{C}G)$ $\quad (\alpha_S)_{ij} = \sum_{s\in S_{ij}} s$

<u>Claim</u>: ① α_S preserves $V^{(K_1,\dots,K_r)}$

② For $V = L^2(\Gamma\backslash X)$ we have a commutative diagram

$$\begin{tikzcd}
L^2(\Gamma\backslash X) \arrow[r, phantom, "\cong"] \arrow[d, "T"'] \arrow[dr, phantom, "//"] & L^2(\Gamma\backslash G)^{(K_1,\dots K_r)} \arrow[r] \arrow[d, "\alpha_S"] \arrow[dr, phantom, "//"] & \bigoplus_{\mu\in I} V_\mu^{(K_1,\dots K_r)} \arrow[d, "\oplus\alpha_S"] \\
L^2(\Gamma\backslash X) \arrow[r, phantom, "\cong"] & L^2(\Gamma\backslash G)^{(K_1,\dots K_r)} \arrow[r] & \bigoplus_{\mu\in I} V_\mu^{(K_1,\dots K_r)}
\end{tikzcd}$$

example: $SL_2(\mathbb{Z}[\frac{1}{p}]) \circlearrowright B(PGL_2) = T_{p+1}$

$$\begin{matrix} SL_2(\mathbb{Z}[\frac{1}{p}]) \\ \rotatebox{90}{$\not\in$} \\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{matrix}$$

Now G doesn't act transitively. There are 2 orbits of vertices. (even and odd spheres).

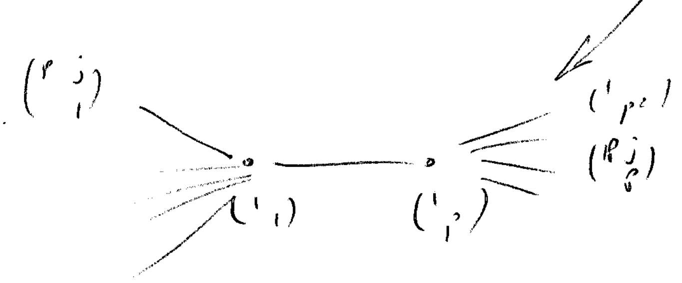
Adj is now from even to odd and odd to even

$K_{even} = SL_2(\mathbb{Z})$

$K_{odd} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/p \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ in $SL_2(\mathbb{Z})$

$K_{odd} S K_{even}$

$$\begin{pmatrix} 0 & S_{odd, even} \\ S_{even, odd} & 0 \end{pmatrix}$$



$\mathrm{Spec} = \{ \pm\sqrt{p}(z_1 + z_2) \}$

Ramanujan graphs and complexes

15/1/18

$\mathbb{Q}_{10}$ $+, \times$

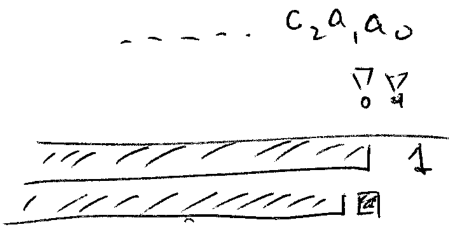
we showed it is a ring

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}_{10} \searrow \text{not comparible.}$$
$$\mathbb{Q} \subseteq \mathbb{R} \nearrow$$

showed $\sqrt{2} \notin \mathbb{Q}_{10}$

Claim: If $\alpha \in \mathbb{Q}_{10}$ and $\overset{\text{right most digit}}{\text{rmd}}(\alpha) \in \{1,3,7,9\} \in (\mathbb{Z}/10)^*$ then α is invertible.

Proof: By multiplication $\overset{\text{by } 10^k}{\vee}$ can assume $\alpha = \dots a_2 a_1 a_0$ $a_0 \neq 0$



$\blacksquare$

~~[illegible]~~

also $2,4,5,6,8 \in \mathbb{Q}_{10}^\times$

Spoiler: $\mathbb{Q}_{10}$ is not a field.

Claim: If p is prime, then $\mathbb{Q}_p$ is a field.

Proof: $\mathrm{rnd}(\alpha) \in (\mathbb{Z}/p)^* = (\mathbb{Z}/p\mathbb{Z} \setminus \{0\})$

$\boxed{\text{p-adic integers}}$ $p \in \mathbb{N}$

$\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p :$ with no digits ~~[illegible]~~ to the right of the decimal point$\}$

$\mathbb{Z}_p$ is a ring subring of $\mathbb{Q}_{10}$ which is compact and uncountable

" A seq of numbers in $\mathbb{Q}_p$ converges if every digit eventually stabilizes (to an admissible limit) "

X

↓

not exactly true

~~$\mathbb{Q}_{10}$ is not compact~~

$\mathbb{Z}_{10}$ has a lot ~~most~~ of $\mathbb{Q}$

$\left\{\frac{a}{b} : (b,10) = 1\right\} \subseteq \mathbb{Z}$

also, every $\alpha \in \mathbb{Z}_{10}$ with $\alpha_0 \in (\mathbb{Z}/10)^\times$ is in $\mathbb{Z}_{10}^\times$

$\rightarrow$ If p is prime $\mathbb{Z}_p^* = \{\alpha \in \mathbb{Z}_p : \alpha_0 \neq 0\}$

We get a decomposition of $\mathbb{Q}_p^\times$

every $\alpha \in \mathbb{Q}_p^\times$ can be written uniquely as $p^n u$ $n \in \mathbb{Z}$, $u \in \mathbb{Z}_p^\times$.

In particular $GL_1(\mathbb{Q}_p) / GL_1(\mathbb{Z}_p) = \mathbb{Q}_p^\times / \mathbb{Z}_p^* \cong \mathbb{Z}$

$\mathbb{Q}_p^\times = \langle p \rangle \times \mathbb{Z}_p^*$

Hensel's lemma

For $f(x) \in \mathbb{Z}[x]$, when does f have a solution in $\mathbb{Q}_{10}$? $\mathbb{Z}_{10}$?

Observe: If f has a sol in $\mathbb{Z}$ (in $\mathbb{Q}$) then it has a sol in $\mathbb{Z}_m$ (in $\mathbb{Q}_m$) for every m (including $m = \infty \Rightarrow \mathbb{Q}$ $\mathbb{Q}_\infty = \mathbb{R}$ $\mathbb{Z}_\infty = \mathbb{Z}$)

Since $\mathbb{Z} \hookrightarrow \mathbb{Z}_m$, $\mathbb{Q} \hookrightarrow \mathbb{Q}_m$ as rings.

Deep question: other direction.

Claim: TFAE

① $f(x)$ has a solution in $\mathbb{Z}_{10}$

② $\exists a_k \in \mathbb{Z}$ s.t. $f(a_k) \equiv 0 \pmod{10^k}$ $\forall k$

(3) " " " " and $a_k \equiv a_{k-1} \bmod 10^{k-1}$

e.g. $\{a_k\} = \{7, 67, 667, 6667, \ldots\}$

for $f(x) = 3x-1$ $f(a_k) \equiv 0 \pmod{10^k}$

indeed $f(x)$ has a root in $\mathbb{Z}_{10}$ $\ldots 6667$

Proof:

$3 \Rightarrow 2$ is obvious

$1 \Rightarrow 3$ there is a ring homo. $\mathbb{Z}_{10} \xrightarrow{\text{mod } 10^k} \mathbb{Z}/10^k$

$3 \Rightarrow 1$ define α is the obvious way as the limit of a_k. $(\alpha \bmod 10^k = a_k)$. Then $f(\alpha) \bmod 10^k = f(a_k) = 0 \ (10^k)$

$\Rightarrow f(\alpha) = 0$.

$2 \Rightarrow 3$ diagonal argument. ∎

Hensel's lemma

Let $f \in \mathbb{Z}[x]$. If $\exists\, a_0 \in \{0, \ldots, 9\}$ s.t. $f(a_0) \equiv 0 \bmod 10$ and $f'(a_0) \in (\mathbb{Z}/10)^*$, then f has a root in $\mathbb{Z}_{10}$.

$$\left(\sum_{i=0}^{n} a_i x^i\right)' = \sum_{i=0}^{n} i a_i x^{i-1}$$

examples:

$f(x) = mx - 1 \qquad m \in \mathbb{Z} \qquad (m, 10) = 1$

take $a_0 =$ inv of m in $\mathbb{Z}/_{10}$

$f(a_0) = m \cdot a_0 - 1 \equiv 0 \mod 10$

$f'(a_0) = m \in (\mathbb{Z}/_{10})^*$

$\Rightarrow \exists x \in \mathbb{Z}_{10}$ s.t. $f(x) = 0$. This is of course $\frac{1}{m}$.

$f(x) = x^2 + x + 8$

$a_0 = 1 \qquad f(a_0) = 0 \mod 10$

$f'(a_0) = 2 \cdot a_0 + 1 = 3 \in (\mathbb{Z}/_{10})^*$

$x^2 + x + 8$ has a solution in $\mathbb{Z}_{10}$, but not in $\mathbb{R}$. $\Rightarrow$

we cannot embed $\mathbb{Q}_{10}$ in $\mathbb{R}$.

Since the sol of

$x^2 + x + 8$ are $\frac{-1 \pm \sqrt{-31}}{2} \quad \Rightarrow \quad \sqrt{-31} \in \mathbb{Q}_{10}$

$f(x) = x^2 + 31$

$f'(x) = 2x \notin (\mathbb{Z}/_{10})^*$

$\Rightarrow$ Hensel's lemma is not iff.

<u>Proof</u>: We will construct $\{a_k\}_{k=1}^{\infty}$ s.t. $f(a_k) \equiv 0 \pmod{10^{k+1}}$ and $a_k \equiv a_{k-1} \pmod{10^k}$ by induction. The lemma we proved before, then proves the existence of a solution.

Taylor $f(x+h) = f(x) + f'(x)h + \underbrace{\frac{f''(x)}{2}}_{\in \mathbb{Z}} h^2 + \dots + \underbrace{\frac{f^{(k)}(x)}{k!}}_{\in \mathbb{Z}} h^k$, where $k = \deg f$.

We have a_{k-1} s.t. $f(a_{k-1}) \equiv 0 \ (10^k)$.

We try to construct $a_k = a_{k-1} + d \cdot 10^k$ s.t. $f(a_k) \equiv 0 \ (10^{k+1})$

$$(10^{k+1}) \quad 0 \overset{\exists d?}{\equiv} f(a_k) = f(a_{k-1} + d\cdot 10^k)$$

$$= \sum_{j=0} f^{(j)}(a_{k-1}) \cdot (d \cdot 10^k)^j \equiv f(a_{k-1}) + f'(a_{k-1})(d \cdot 10^k)$$

~~$\equiv f'(a_{k-1}) d\cdot 10^k$ choosing $d =$ inv of $f'(a_{k-1}) \in \mathbb{Z}/10^*$~~

~~we got~~

we need $f'(a_{k-1}) d 10^k \equiv -f(a_{k-1}) \ (10^{k+1})$

by assumption $f(a_{k-1}) \equiv 0 \mod 10^k \Rightarrow$ we need

$$f'(a_{k-1}) d \equiv -\frac{f(a_{k-1})}{10^k} \ (10)$$

since $f'(a_{k-1}) \in (\mathbb{Z}/10)^*$ we can take

$$d \equiv -\frac{1}{f'(a_0)} \cdot \frac{f(a_{k-1})}{10^k} \ (10)$$

$\big(f'(a_{k-1}) = f'(a_0) \mod 10\big)$

∎

$\sqrt{m} \in \mathbb{R}$ $m \geq 10$

~~$m \in \mathbb{Z}_{10} \iff \begin{cases} m \equiv 1 \mod 8 \\ m \equiv 1,4 \mod 5 \end{cases}$~~

For p prime $p \neq 2$. $\sqrt{m} \in \mathbb{Z}_p \iff \sqrt{m} \in \mathbb{F}_p$.

$f(x) = x^2 - m$ take $a_0 = \sqrt{m} \in \mathbb{F}_p$ then $f'(a_0) = 2a_0 \neq 0 \mod p$

~~some~~ use $\Rightarrow$ Hensel's lemma.

$\mathbb{Q}_{10}$ is not a field

$\mathbb{Z}_{10}$ has 0-divisors.

$f(x) = x^2 - x$

$a_0 \in \{0, 1, 5, 6\} \Rightarrow f(a_0) = 0 \mod 10$

$f'(a_0) = 2a_0 \qquad a_0 = 6 \Rightarrow f'(a_0) \not\equiv$ [illegible] mod 10

$\Rightarrow$ Hensel $\exists\ \alpha \in \mathbb{Z}_{10}$ with $\alpha_0 = 6$ s.t. $\alpha(\alpha - 1) = 0$

$\Rightarrow$ α [illegible] is a non trivial 0-divisor.

Ramanujan graphs and complexes 15/1/18

$\mathbb{R}$ - sequences of the form $a_n a_{n-1} a_{n-2} \cdots\cdots a_0 a_{-1} a_{-2}$ $a_i \in \{0,1,\dots,9\}$ $a_n \neq 0$

$n \in \mathbb{N}$

① Some numbers have two representatives $7.3 = 7.29999$

② signs $\pm$ $(+0 = -0)$

$\mathbb{Q}_{10}$: $\{ \cdots\cdots a_1 a_0 . a_{-1} a_{-2} a_{-3} \cdots\; a_{n+2}\, a_{n+1}\, a_{n}$: $n \in \mathbb{Z}$, $a_i = 0, \dots, 9$, $a_n \neq 0$ $\} \cup \{0\}$

$+, \times$: as before (addition and multiplication are easier since there is a right most num)

$\mathbb{R} = \{$things like $\pm 723.45107\dots\}$

$\mathbb{Q}_{10} = \{$things like $\dots\dots 3154.327\}$

$|\mathbb{Q}_{10}| = |\mathbb{R}| = 2^{\aleph_0}$

mirror (up to countable set)

~~however~~ carry still goes to the left.

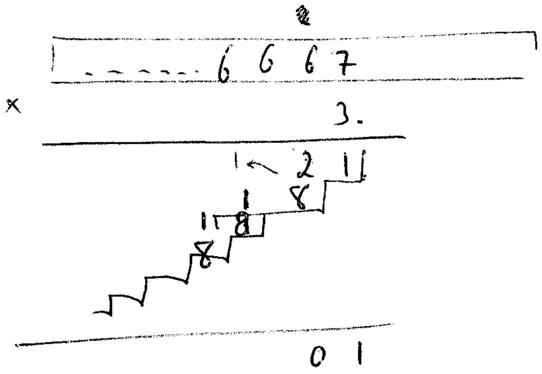
$\mathbb{Q}_{10}, +$ is a group! 0 is + neutral

$$\begin{array}{r} \ldots\ldots 99999. \\ + \qquad\qquad 1 \\ \hline 0 \end{array}$$

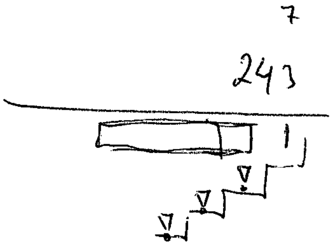
$$\begin{array}{r} \ldots\ldots 7541.327 \\ + \qquad\qquad ? \\ \hline 0 \end{array} \longrightarrow \quad \ldots\ldots 2458.673$$

$+, \times$ are distributive, commutative, associative

$\times$-inverses



For any $n \in \mathbb{N}$ $(\mathbb{Z}\setminus\{0\})$ $\frac{1}{n} \in \mathbb{Q}_{10}$



$3 \in (\mathbb{Z}/10)^{*} \longrightarrow$ multi by 3 mod 10 is bij on $\{0,\ldots,9\}$

$\exists$ unique inverse to $\frac{1}{243}$. in $\mathbb{Q}_{10}$.

If $(n,10)=1$ then $n \in \mathbb{Q}_{10}$ is unique

$\frac{1}{5}$

$$\begin{array}{c} \boxed{0.2} \\ 5 \\ \hline 1 \end{array}$$

Proof: $\forall\ n \in \mathbb{N}$ write $n = 2^a 5^b m$ where $(m,10)=1$

$$\frac{1}{n} = 0.5^a \cdot 0.2^b \cdot \frac{1}{m}$$

Is $\mathbb{Q}_{10}$ a field?

Is $\mathbb{Q}_{10} \cong \mathbb{R}$

Does $\mathbb{Q}_{10} \hookrightarrow \mathbb{R}$?, $\mathbb{R} \hookrightarrow \mathbb{Q}_{10}$?

$\downarrow$

no $\sqrt{2} \notin \mathbb{Q}_{10}$

there is no solution to $x^2 = 2 \mod 10$

$\rightarrow$ no: $\sqrt{-31} \in \mathbb{Q}_{10}$ (Exercise

$\hookrightarrow$ not periodic.

$$\begin{array}{r} \cdots\cdots 2j \times \\ \cdots\cdots 2j \times \\ \hline 9999969. \end{array} = -31.$$

$\mathbb{Q}_7 \ni \sqrt{2}$ $\mathbb{Q}_7$ is not the same as $\mathbb{Q}_{10}$.

Lecture 2

26/3/18

from last week $\begin{cases} G = PGL_d(\mathbb{Q}_p), \quad K: PGL_d(\mathbb{Z}_p). \\ (\nabla)K = (\nabla)K \quad \text{if } A \in GL_d(\mathbb{Z}_p) \text{ is triang. then the diag. elements are units } (\mathbb{Z}_p^\times). \end{cases}$

Lattices

$$\mathbb{Z}^d \le \mathbb{R}^d \begin{cases} \nearrow \mathbb{Z}_p^d \le \mathbb{Q}_p^d : \text{submodule over int. ring inside vector space.} \\ \searrow SL_d(\mathbb{Z}) \le SL_d(\mathbb{R}) : \text{discrete cocpt/cofinite sgp. of top. gp.} \end{cases}$$

we will use this notion.

$\mathbb{Z}_p^+ \le \mathbb{Q}_p^+$ not discrete, not cocpt. $[\mathbb{Q}_p^+/\mathbb{Z}_p^+ \simeq \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, discrete$]$

$\mathbb{Z}_p^\times \le \mathbb{Q}_p^\times$ similar.

$\mathbb{Z} \le \mathbb{Q}_p$ neither.

cocpt lattice!

$\underbrace{\mathbb{Z}[\tfrac{1}{p}]^\times}_{\overset{\shortparallel}{\{\pm p^l \mid l \in \mathbb{Z}\}}} \le \mathbb{Q}_p^\times. \qquad \mathbb{Q}_p^\times / \mathbb{Z}[\tfrac{1}{p}]^\times \simeq \mathbb{Z}_p^\times/\{\pm 1\}$ cpt.

$\mathbb{Z}[\frac{1}{p}] \le \mathbb{Q}_p$ not discrete $(p^l \to 0)$.

$PGL_d(\mathbb{Z}[\frac{1}{p}]) \le PGL_d(\mathbb{Q}_p)$. <u>cocompact</u>: we saw $PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts trans. on $(\mathcal{B}_p^d)^0 = PGL_d(\mathbb{Q}_p)/PGL_d(\mathbb{Z}_p)$,

so $PGL_d(\mathbb{Z}_p)PGL_d(\mathbb{Z}[\frac{1}{p}]) = PGL_d(\mathbb{Q}_p)$.

$\rightarrow PGL_d(\mathbb{Z}[\frac{1}{p}]) \backslash PGL_d(\mathbb{Q}_p) \simeq PGL_d(\mathbb{Z}_p) \cap PGL_d(\mathbb{Z}[\frac{1}{p}]) \backslash PGL_d(\mathbb{Z}_p)$ cpt.

<u>not discrete</u>: $\begin{pmatrix} 1 & p^l \\ & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$.

$$\Gamma(n) := PGL_d(\mathbb{Z}[\tfrac{1}{p}])(n) = \{A \in \underbrace{PGL_d(\mathbb{Z}[\tfrac{1}{p}])}_{\updownarrow} \mid A \equiv I \ (\text{mod } n)\} = \ker\left(PGL_d(\mathbb{Z}[\tfrac{1}{p}]) \to PGL_d(\mathbb{Z}/n\mathbb{Z})\right)$$

$(p \nmid n)$

Primitive p-matrices in $M_d(\mathbb{Z})$

$\Gamma(n) \le PGL_d(\mathbb{Q}_p)$. <u>not discrete</u>: $\begin{pmatrix} 1 & p^l \\ & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$.

<u>cocpt</u>: finite index in $\Gamma(1) = PGL_d(\mathbb{Z}[\frac{1}{p}])$. This bounds # orbits in $\mathcal{B}^0$.

<u>another argument</u>: $\underset{PGL_d(\mathbb{Z}[\frac{1}{p}]) + \text{cong. cond.}}{\overset{}{\Gamma(n)}} \cap \underset{\text{stab}(v_0) = PGL_d(\mathbb{Z}_p)}{K} = PGL_d(\mathbb{Z})(n) = \ker(PGL_d(\mathbb{Z}) \to PGL_d(\mathbb{Z}/n))$ is finite,

but discrete $\cap$ cpt = finite.

$PU_d(\mathbb{Z}[\frac{1}{p}])(n) \leq PU_d(\mathbb{Q}_p) \underset{p\equiv 1(4)}{\cong} PGL_d(\mathbb{Q}_p)$ a discrete cocompact lattice.

$U_d(R) = \{A \in M_d(R[i]) \mid A^*A = I\}$ } Preview

$\uparrow$ com. ring

$(A^*)_{i,j} = \overline{A_{ji}}$

$\overline{a+bi} = a-bi$

<u>The Thing:</u> $\mathbb{Z}[\frac{1}{p}] \underset{\text{lattice}}{\leq} \mathbb{R} \times \mathbb{Q}_p$.

$\alpha \longmapsto (\alpha, \alpha)$

<u>Proof:</u> <u>discrete:</u> we'll show 0 is not an acc. point.

$\alpha_i \to 0 \iff \underset{\text{in } \mathbb{R}}{\alpha_i \to 0}$ & $\underset{\text{in } \mathbb{Q}_p}{\alpha_i \to 0}$.

$\alpha \in \mathbb{Z}[\frac{1}{p}] \to$ if $\underset{\text{in } \mathbb{R}}{\alpha_i \to 0}$ Then $\alpha_i = \frac{m_i}{p^{l_i}}$, $l_i \to \infty$ and then $\alpha_i \not\to 0$ in $\mathbb{Q}_p$

Likewise: $\mathbb{Z}[\frac{1}{pq}] \overset{\text{disc.}}{\hookrightarrow} \mathbb{R} \times \mathbb{Q}_p \times \mathbb{Q}_q$.

<u>cocpt:</u> $\mathbb{Z}[\frac{1}{p}] + \underbrace{([0,1] + \mathbb{Z}_p)}_{\text{cpt.}} = \mathbb{R} \times \mathbb{Q}_p$. why? If $(\underset{\mathbb{R}}{\overset{\cap}{\alpha}}, \underset{\mathbb{Q}_p}{\overset{\cap}{\beta}})$, sub. $\underbrace{\beta \bmod 1}_{\overset{\cap}{\mathbb{Z}[\frac{1}{p}]}}$ and get $(\underset{\mathbb{R}}{\overset{\cap}{r}}, \mathbb{Z}_p)$.

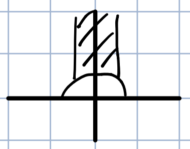
Now sub. $\underset{\mathbb{Z}}{\overset{\cap}{\lfloor r \rfloor}}$ and get $(\underbrace{r - \lfloor r \rfloor}_{\overset{\cap}{[0,1]}}, \mathbb{Z}_p)$.

<u>Likewise:</u> $\underset{\text{non-cocpt lattice}}{PGL_d(\mathbb{Z}[\frac{1}{p}])} \leq \underbrace{PGL_d(\mathbb{R} \times \mathbb{Q}_p)}_{PGL_d(\mathbb{R}) \times PGL(\mathbb{Q}_p)}$

<u>Fact:</u> $\exists \underset{\text{of finite volume}}{Y} \subseteq PGL_d(\mathbb{R})$ s.t. $PGL_d(\mathbb{Z})Y = PGL_d(\mathbb{R})$ [$PGL_d(\mathbb{Z})$ is a lattice in $PGL_d(\mathbb{R})$].

<u>Example:</u> $d=2$, $PGL_d(\mathbb{R}) \simeq \mathfrak{h} \times PO(2)$

$Y =$



$\times PO(2)$.

New, take $PGL_d(\mathbb{Z}[\frac{1}{p}]) \leq \underbrace{PGL_d(\mathbb{R}) \times PGL_d(\mathbb{Q}_p)}$. It is discrete (by same argument).

<u>Cofinite:</u> $PGL_d(\mathbb{Z}[\frac{1}{p}]) \cdot (Y \times PGL_d(\mathbb{Z}_p)) =$

If $(A,B) \in PGL_d(\mathbb{R}) \times PGL_d(\mathbb{Q}_p)$, since $PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts trans. on $PGL_d(\mathbb{Q}_p)/PGL_d(\mathbb{Z}_p)$ we can move (A,B) to $(C, PGL_d(\mathbb{Z}_p))$. Now, $\exists D \in PGL_d(\mathbb{Z})$ s.t. $DC \in Y$ and $D \in PGL_d(\mathbb{Z}) \leq PGL_d(\mathbb{Z}_p)$

$\to$ Now I'm in $Y \times PGL_d(\mathbb{Z}_p)$

① Any $\overset{\text{loc. cpt.}}{\vee}$ topological gp. G has a unique $\overset{\text{regular}}{\vee}$ measure μ $\overset{\text{up to scalars}}{\vee}$ satisfying $\mu(Ag) = \mu(A)$ $\forall g\in G$ $A\subseteq G$ (Haar Measure).

② $\mu(Ag) = \Delta_G(g)\,\mu(A)$. Δ_G - the modular function of G.

③ $\overset{\sigma\text{-cpt}}{G} \underset{\text{trans.}}{\circlearrowright} \overset{\text{haus.}}{X}$. Does X have a G-inv measure?

$X \cong \Gamma\backslash G$ for some sgp $\underset{\text{"}\mathrm{stab}_G(x_0\in X)}{\overset{\text{closed}}{\Gamma}} \le G$. Is there G-inv measure on $\Gamma\backslash G$?

$\mu(Ag) = \mu(A)$, $A \subseteq \Gamma\backslash G$

Answer: if and only if $\Delta_G|_\Gamma \equiv \Delta_\Gamma$.

$\Delta_G \equiv 1$ for abelian/cpt/discrete/$GL_n(\mathbb{R}, \mathbb{Q}_p, \dots)$.

$$\cancel{\mu(G)} = \mu(gG) = \Delta(g)\cancel{\mu(G)}$$

Finally, if $\Gamma \underset{\text{discrete}}{\le} GL_d(\mathbb{R}, \mathbb{Q}_p, \dots)$, then by there is a unique G-inv measure μ on $\Gamma\backslash G$. Say Γ is a lattice if $\mu(\Gamma\backslash G) < \infty$.

①

[illegible] 9/4

G Top gp.

X Top space

$G \circlearrowright X$ e.g. $SL_2(\mathbb{Z}) \leq SL_2(\mathbb{R}) \circlearrowright \mathbb{H}$ | $K = SO(2)$

$SL_d(\mathbb{Z}) \leq GL_d(\mathbb{Q}_p) \circlearrowright \mathcal{B}_d^p$ ← take a geometric realization | $K = GL_d(\mathbb{Z}_p)$

We are interested in quotients!

For $\Gamma \leq G$, we consider $\Gamma \backslash X$

Assume $G \underset{\text{trans.}}{\circlearrowright} X$ (otherwise, decompose to orbits)

Choose some $x_0 \in X$ and set

$$K = \mathrm{Stab}_G(x_0)$$

[Assume: All is T_2 and σ-cpt.]

$\Rightarrow X \underset{\text{homom.}}{\cong} G/K$

And then $\Gamma \backslash X = \Gamma \backslash G/K$

One possibility: take $X = G$, so $\Gamma \backslash G$

Haar Thm! $\exists !$ A measure on G, s.t.

$$\forall A \subseteq G \ \forall g \in G$$

$$\circledast \ \mu(Ag) = \mu(A) \quad \text{(the right Haar measure)}$$

Note that in general $\mu(gA) \neq \mu(A)$

There exists (the modular func. of G):

$$\Delta_G : G \xrightarrow{\text{hom}} \mathbb{R}_{>0} \quad \text{s.t.}$$

$$\forall A \subseteq G \quad \mu(gA) = \Delta_G(g) \cdot \mu(A)$$

Note If G is abelin / cpt. / discrete / GL_n, then

$\Delta_G \equiv 1$; we say that G is unimodular.

Example For $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \leq GL_2\begin{pmatrix}\mathbb{R} \\ \mathbb{Q}_p\end{pmatrix}$

$$\Delta_B \neq 1$$

$\circledast$ I.e. $\int_G f(g)\, d\mu(g) = \int_G f(gg') \, d\mu(g)$

now, given $X \curvearrowleft G$ is there a G-inv. measure on X?

<u>Answer</u> Write $H\backslash G$, $H = \mathrm{Stab}_G(x_0)$

Then there exists a G-inv. μ iff

$\Delta_G|_H \equiv \Delta_H$

unique upto constant

<u>"Counter example"</u> $G = SL_2(\mathbb{R}) \overset{\text{Möbius}}{\circlearrowleft} \mathbb{R}\cup\{\infty\}$

If we had such a μ we could conclude:

$$\mu((0,1)) \overset{x\mapsto x+1}{=} \mu((a,a+1)) \overset{x\mapsto \frac{1}{x}}{=} \mu((1,\infty]) =$$

$$\sum_{a=1}^{\infty} \mu((a,a+1)) = \infty$$

(Assume $\Delta_G|_\Gamma \equiv 1$)

We say Γ is a lattice if Γ is discrete and $\mu(\Gamma\backslash G) < \infty$

$\nearrow$ ∃! G-inv. measure

(We say that Γ is cofinite)

(If $\Gamma\backslash G$ is cpt. $\Rightarrow$ also cofinite)

<u>Examples</u> $\mathbb{Z} \leq \mathbb{R}$

$SL_d(\mathbb{Z}) \leq SL_d(\mathbb{R})$ ← not co-cpt.

We saw last time:

$SL_d(\mathbb{Z}[\frac{1}{p}]) \leq SL_d(\mathbb{Q}_p)$

is cofinite, but not discrete!

If it were discrete, then $\forall$ cpt. $S \subseteq SL_d(\mathbb{Q}_p)$ we would have $|\Gamma \cap S| < \infty$, but

discrete $\longrightarrow$ $\Gamma \cap SL_d(\mathbb{Z}_p) = SL_d(\mathbb{Z})$ ← infinite

Why cocpt?

Hint If $H \leq G$ and $\overset{\text{cpt.}}{S} \leq G$ s.t. $HS = G$

then H is cocpt.

But $SL_d(\mathbb{Z}[\frac{1}{p}]) \overset{\text{trans.}}{\curvearrowright} \mathcal{B}_d^p = GL_d(\mathbb{Q}_p)/SL_d(\mathbb{Z}_p)$

$\Rightarrow SL_d(\mathbb{Z}[\frac{1}{p}])\, SL_d(\mathbb{Z}_p) = SL_d(\mathbb{Q}_p)$

The same is true for $\Gamma(N) = \{A \in \Gamma : A \equiv I \pmod{N}\}$ $p \nmid N$

Here is a way to "make it discrete":

We saw $\Gamma \overset{\text{lattice}}{\leq}_{\circledast} PSL_d(\mathbb{R}) \times PSL_d(\mathbb{Q}_p)$

Since: If $\underset{\text{finite vol.}}{Y \text{ is}}$ s.t. $PSL(\mathbb{Z})\, Y = PSL_d(\mathbb{R})$,

then $\Gamma(Y \times PSL_d(\mathbb{Z}_p)) = G_{\mathbb{R}} \times G_{\mathbb{Q}_p}$

i.e.: $\forall\ (A, B)$

$\exists C \in PSL_d(\mathbb{Z}[\frac{1}{p}])$

s.t. $CA \in Y$, $CB \in G_{\mathbb{Z}_p}$

Claim If $\Gamma \overset{\text{discrete}}{\leq} G_{\mathbb{Q}_p}$ then Γ is a lattice iff

$$\sum_{v \in \Gamma\backslash(\mathcal{B}_d^p)^0} \frac{1}{\mathrm{Stab}_\Gamma(v)} < \infty$$

v is a rep. for $p^{-1}(v)$

where
$$\begin{tikzcd} \mathcal{B} \arrow[d, "p"] \\ \Gamma\backslash\mathcal{B} \end{tikzcd}$$

⊛ via the diagonal embedding

In particular, if $|\Gamma \backslash \mathcal{B}^0| < \infty$ then Γ is a lattice; this, however, is immediate. The thm holds for infinite converging sums.

Moshe Morgenstern calls these: "Ramanujan Diagrams."

Exercise: Γ is $\overset{\text{cocpt.}}{\overbrace{a}}$ lattice iff $|\Gamma \backslash \mathcal{B}^0| < \infty$

Pf of thm: For $\underset{\text{s.t. } \Delta_{G/H} \equiv \Delta_H}{\underbrace{H \leq G}}$ what is μ for $H \backslash G$?

Define a map $P : \overset{*}{C_c}(G) \to C_c(H \backslash G)$

$$(Pf)(Hg) = \int_H f(hg)\, d\mu_H(h)$$

Fact: P is surjective ("Fact" has a technical pf)

Define for $f \in C_c(H \backslash G)$, $\tilde{f} \in C_c(G)$ representative

Define: $\int_{H \backslash G} f \, d\mu_{H \backslash G} = \int \tilde{f} \, d\mu_G$

This is well defined iff $\Delta_{G/H} \equiv \Delta_H$ ← $\boxed{\text{Exercise}}$

We need $\int_{\Gamma \backslash G} 1 \, d\mu_{\Gamma \backslash G} = \mu(\Gamma \backslash G) < \infty$

Let $\{v_i\}$ be rep. to $\Gamma \backslash \mathcal{B}^0$. Take $\{g_i\} \subseteq G_{\mathbb{Q}_p}$

s.t. $g_i v_0 = v_i$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = G_{\mathbb{Z}_p}$$

$$\tilde{f}(g) = \begin{cases} \frac{1}{|\Gamma_{v_i}|} & \text{if } g \in g_i K \Leftrightarrow g v_0 = v_i \\ 0 & \text{else} \end{cases}$$

Claim: $\int_\Gamma \tilde{f}(\gamma g) \, d\mu_\Gamma(\gamma) \equiv 1$

---

* Continuous with cpt. support; we can't write L^2 since we don't have a measure a-priory

Since

$$= \sum_{\gamma\in\Gamma} \tilde{f}(\gamma g) = \sum_{\gamma\in\Gamma} \begin{cases} \frac{1}{|\Gamma_{x_i}|} & \gamma v_0 = v_i \\ 0 & \text{else} \end{cases} = 1$$

We are finished with measures!

So far, we considered ($G = PGL_d$)

$$G_{\mathbb{Z}[\frac{1}{p}]} \quad G_{\mathbb{Z}_p} \quad G_{\mathbb{Q}_p}$$

and could not find co-cpt. lattices.

We move on to study $U_d(\mathbb{Q}_p) \circlearrowright \mathcal{B}(U) \equiv U_d(\mathbb{Q}_p) / \underbrace{U_d(\mathbb{Q}_p)}_{\text{max. cpt. subgp.}}$

For a comm. ring R, $\overbrace{}^{\text{with 2 invertible}}$ define

$$U_d(R) = \{ A \in M_d(R[\underset{\sqrt{-1}}{\overset{\shortparallel}{i}}]) \mid A^*A = I \}$$

(If R has a sq. root of -1, we add another one!)

(Notation: $U(d) \equiv U_d(R)$)

Fact: If R has $\sqrt{-1}$ (e.g. $\mathbb{C}$, $\mathbb{Q}_p$, $\mathbb{Z}_p$) for $p \equiv 1 \pmod 4$

then $U_d(R) \cong GL_d(R)$

pf: $\boxed{\text{Exercise}}$

Claim: $U_d(\mathbb{Z}[\frac{1}{p}]) \overset{\text{co-cpt. lattice}}{\Gamma} U_d(\mathbb{Q}_p)$

also $\Gamma(N) = \{ A \in \Gamma \mid A \equiv I \pmod N \}$

therefore, if $p \equiv 1 \pmod 4$

$$\Gamma \backslash \mathcal{B}(U_d(\mathbb{Q}_p)) = \Gamma \backslash \mathcal{B}^p_d \quad \text{is a finite complex!}$$

Because!

$U_d(\mathbb{Z}[\frac{1}{p}]) \leq U_d(\mathbb{Q}_p) \times U_d(\mathbb{R})$ is a co-cpt. lattice!

discrete - easy, as seen previously.

Why is it co-cpt.?

Take $S = U_d(\mathbb{Z}_p) \times U_d(\mathbb{R})$

cpt × cpt = cpt by Tychonoff

it is enough (and necessary) to show that

$$U_d(\mathbb{Z}[\tfrac{1}{p}]) \backslash (\mathcal{B}_d^p)^0 \text{ is finite}$$

$U_d(\mathbb{Z}[\frac{1}{p}])$ is discrete (in $U_d(\mathbb{Q}_p)$)

since it is disc. in $U_d(\mathbb{Q}_p) \times \overbrace{U_d(\mathbb{R})}^{cpt}$

$\Rightarrow$ you can "throw out" the cpt. part.

①

16/4/2018 שיעור [illegible] [illegible]

We saw $\mathbb{Z}[\tfrac{1}{p}] \overset{\text{co-cmp lattice}}{\leq} \mathbb{R} \times \mathbb{Q}_p$

Since $\overbrace{[0,1] \times \mathbb{Z}_p}^{\text{cpt}} \times \mathbb{Z}[\tfrac{1}{p}] = \text{everything}$

We moved to $GL_d(\mathbb{Z}[\tfrac{1}{p}]) \underset{\substack{\text{discrete}\\ \text{co finite}}}{\leq} GL_d(\mathbb{R}) \times GL_d(\mathbb{Q}_p)$

$$\overset{\leq}{\text{cofinite}} \ \text{co-cpt}, \quad GL_d(\mathbb{Q}_p)$$

$\rightarrow$ not discrete since

intersection with $\rightarrow \cap\, GL_d(\mathbb{Z}_p)$ is infinite in $GL_d(\mathbb{Z})$

We considered then

$U_d(\mathbb{Z}[\tfrac{1}{p}]) \underset{\text{discrete}}{\leq} U_d(\mathbb{R}) \times U_d(\mathbb{Q}_p)$

$U_d(\mathbb{Z}[\tfrac{1}{p}]) \underset{\text{disc}}{\leq} U_d(\mathbb{Q}_p)$

⊛ Are they co-cpt.? co-finite?

<u>Recall</u> If R is a com. ring

$U_d(R) = \{ A \in M_d(R[i]) \mid A^* A = I \}$ $\qquad (A^*)_{ij} = \bar{A}_{ji}$

e.g. $U(d) = U_d(\mathbb{R})$

⊛ <u>Answer</u> They are co-cpt.

As for the second lattice - this is a deep result, and follows from "Strong Approximation in Alg. groups"

For our purposes, we will "check by hand"

---

cofinite = quotient has finite volume

discrete ≡ no acc. point ; not even outside the gp!

<u>Claim:</u> If $\exists \varepsilon \in R$ s.t. $\varepsilon^2 = -1$ and 2 is invertible

then $U_d(R) \cong GL_d(R)$

e.g.: $\mathbb{C}$, $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{F}_p \leftarrow p \equiv 1 \pmod 4$

<u>Pf:</u> Observe: $R[i] = R[x]/(x^2+1) =$

$$i \longmapsto x \longmapsto \dots$$

$$\dots \to R[x]/(x-\varepsilon)(x+\varepsilon) \overset{CRT}{=} R[x]/(x-\varepsilon) \times R[x]/(x+\varepsilon) \cong R \times R$$

$$x \longmapsto (x,x) \longmapsto (\varepsilon, -\varepsilon)$$

Next, $M_d(R[i]) \cong M_d(R \times R) \cong M_d(R) \times M_d(R)$

$$(a_{ij} + b_{ij} i) \mapsto (a_{ij} + b_{ij}\varepsilon, a_{ij} - b_{ij}\varepsilon) \mapsto (a_{ij} + b_{ij}\varepsilon), (a_{ij} - b_{ij}\varepsilon)$$

$$A + Bi \xmapsto{\textcircled{e}} (A + B\varepsilon, A - B\varepsilon)$$

$A, B \in M_d(R)$

Note: $U_d(R) \in M_d(R[i])^\times = GL_d(R[i])$

$A + Bi \in U_d(R) \iff (A+Bi)(A+Bi)^* = I$

$\iff (A+Bi)(A-Bi)^t = I$

$\textcircled{e}$ $\iff A - Bi = ((A-Bi)^t)^{-1}$

Take $(X, Y) \in M_d(R) \times M_d(R)$, when is $(X,Y) \in e(U_d(R))$?

Iff: $Y = (X^t)^{-1}$

So, $U_d(R) \overset{\text{by definition}}{\subseteq} M_d(R[i]) \cong M_d(R) \times M_d(R)$

$$\begin{tikzcd} U_d(R) \arrow[d, phantom, "\shortparallel"] & \\ \{(X, (X^t)^{-1}) : X \in GL_d(R)\} \arrow[r, bend left, "\pi_1"] \arrow[r, phantom, "\cong"] & GL_d(R) \end{tikzcd}$$

$$\begin{tikzcd} U_d(R) \arrow[d, phantom, "\ni"] & GL_d(R) \arrow[d, phantom, "\ni"] \\ A + Bi \arrow[r, mapsto] & A + B\varepsilon \\ \frac{X + (X^t)^{-1}}{2} + \frac{X - (X^t)^{-1}}{2\varepsilon} i & X \arrow[l, mapsto] \end{tikzcd}$$

A more convenient gp.: The Unitary Similitudes gp.

$$GU_d(R) = \{ A \in M_d(R[i]) \mid A^*A = \lambda I \text{ for } \lambda \in R^{\times \circledast} \}$$

Home work: $PGU_d(\mathbb{R}) \cong PU_d(\mathbb{R}) \cong PSU_d(\mathbb{R})$ (for Reals)

In general, over fields, PGU, PU, PSU differ by finite index.

Home work: If $\sqrt{-1}, \frac{1}{2} \in R$ then

$$PGU_d(R) \cong PGL_d(R)$$

by $A + Bi \mapsto A + B\varepsilon$ / mod scalars,

and $GU_d(R) \overset{\text{"more or less"}}{\cong} R^\times \times GL_d(R)$

Since $\Gamma = GU_d(\mathbb{Z}[\frac{1}{p}]) \underset{\text{discrete}}{\leq} GU_d(\mathbb{Q}_p)$,

we should have: $\underset{U_d(\mathbb{Z}_p)}{\Gamma \cap \overset{\shortparallel}{K}}$ finite.

(a sanity check).

Indeed, $\Gamma \cap K = GU_d(\mathbb{Z}[\frac{1}{p}] \cap \mathbb{Z}_p) = GU_d(\mathbb{Z})$

We compute $GU_d(\mathbb{Z})$: .... these "permutation matrices"

with entries $\pm 1, \pm i$, So

$$|GU_d(\mathbb{Z})| = d! \cdot 4^d$$

Harder result: If $d = 2$, $p \neq 2$,

$$PGU_2(\mathbb{Q}_p) \cong PGL_2(\mathbb{Q}_p) \quad \text{even if } p \not\equiv 1 \pmod 4$$

(Show that they are isomorphic to some quternion algebra)

$\circledast$ Same as requiring $\lambda \in R[i]^\times$:

$$\lambda I = A^*A = (A^*A)^* = (\lambda I)^* = \bar{\lambda} I$$

①

שיעור ט אחרי - 30/4

LPS construction: $p \equiv 1 \pmod 4$ exis

$\varepsilon = \sqrt{-1}$ in $\mathbb{Z}_p \subseteq \mathbb{Q}_p$

$$U_d(\mathbb{Q}_p) \xrightarrow{\sim} GL_d(\mathbb{Q}_p) \curvearrowright \mathcal{B}_d^p = \mathcal{B}(PGL_d(\mathbb{Q}_p))$$

Isomorphism → $A + Bi \longmapsto A + B\varepsilon$ [denote: $x \mapsto \tilde{x}$]

Lattices: $U_d(\mathbb{Z}[\frac{1}{p}]) \leq U_d(\mathbb{Q}_p)$

Recall that the vertices are

$$(\mathcal{B}_d^p)^0 = PGL_d(\mathbb{Q}_p) / _{PGL(\mathbb{Z}_p)} \cong PGU_d(\mathbb{Q}_p) / _{PGU_d(\mathbb{Z}_p)}$$

induced by the above iso.

Also, $\dfrac{PGU_d(\mathbb{Q}_p)}{(\mathbb{Z}_p)} \cong \dfrac{PGL_d(\mathbb{Q}_p)}{(\mathbb{Z}_p)}$

L.P.S.: $\Gamma = PGU^{+}_{2,R}(\mathbb{Z}[\frac{1}{p}])(2) \leftarrow A \equiv I \pmod 2$

$\det A > 0$ (and real)

Claim: Γ acts simply on (vertices of)

$$T^0_{p+1} = PG(L/U)_2(\mathbb{Q}_p) / PG(L/U)_2(\mathbb{Z}_p)$$

Corollary Γ is a lattice (and thus also $PGU_d(\mathbb{Z}[\frac{1}{p}])$)

(we saw: $vol(\Gamma\backslash G) = \sum_{v \in FD(\Gamma \curvearrowright G)} \frac{1}{Stab_\Gamma(v)}$

in particular, if $|\Gamma\backslash\mathcal{B}^0| < \infty$, then Γ is cofinite)

or: since then we have $\Gamma K = G$, where

$K = PG(L/U)_2(\mathbb{Z}_p)$

$\Rightarrow \Gamma\backslash G \cong {}_{\Gamma\cap K}\backslash K$ cpt.

Γ w/o projection

Proof $\quad \mathrm{Stab}_\Gamma(v_0) = \Gamma \cap GU_2(\mathbb{Z}_p)$

$= GU_2^+\,\mathbb{Z}[\tfrac{1}{p}](2) \cap GU_2(\mathbb{Z}_p)$

$= GU_2^+(\mathbb{Z}[\tfrac{1}{p}] \cap \mathbb{Z}_p)(2)$

$= GU_2^+(\mathbb{Z})(2)$

Note: what are the units of the intersection? →

$=\{A \in M_2(\mathbb{Z}[i]) \mid A^*A = I,\ A \equiv I \pmod 2\}$

$=\left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \right\}$

(Since: $GU_2(\mathbb{Z}) = \left\{ \begin{bmatrix} \pm1/\pm i & 0 \\ 0 & \pm1/\pm i \end{bmatrix}, \begin{bmatrix} 0 & \pm1/\pm i \\ \pm1/\pm i & 0 \end{bmatrix} \right\}$
and one can eliminate 30 of the 32 possibilities.)

$\Rightarrow$ In ⓅGU we have $\mathrm{Stab}_\Gamma(v_0) = \{I\}$

Transitive: (we take $p=5$ e.g.)

For $A \in PGU_d(\mathbb{Z}[\tfrac{1}{p}])$, $B \in PGL_d(\mathbb{Z}[\tfrac{1}{p}])$

We define: $\quad \mathrm{level}_p(B) = \mathrm{ord}_p \det(B) - d \min_{i,j}\{\mathrm{ord}_p(B_{ij})\}$

(invariant: $\mathrm{level}_p(B) = \mathrm{level}_p(pB)$)

And define: $\mathrm{level}_\pi(A) = \mathrm{ord}_\pi \det(A) - d \min \mathrm{ord}_\pi(A_{ij})$

where $\pi\bar{\pi} = p$ (we choose to call one such prime π, and the other $\bar{\pi}$)

$5 = (1+2i)(1-2i)$

Claim ⊛ For $A \in PGU_d(\mathbb{Z}[\tfrac{1}{p}])$,

$\mathrm{level}_\pi(A) = \mathrm{level}_p(\tilde{A})$ $\quad$ (where $\widetilde{X+Yi} = X+Y\varepsilon$)

$= 1\text{-}\mathrm{dist}(v_0, Av_0)$

1-dist is dist along edges of color 1

position of right-most digit of p-adic number
↓

⊛ ~~Assume~~ $\mathrm{val}(\tilde{\pi}) = \mathrm{ord}_p(\tilde{\pi}) = 1$ [choose $\varepsilon = \sqrt{-1}$ s.t. it holds]

Example $\sqrt{-1} \equiv 7 \, (25)$ ⟵ one such choice of 2 !

$\mathbb{Q}_5[i]$ $\quad$ $\mathbb{Q}_5 (\mathrm{mod}\, 25)$

$5 \longmapsto \tilde{5} = 5$

$1+2i \longmapsto 15 \rightsquigarrow \mathrm{val} = 1$

$1-2i \longmapsto 12 \rightsquigarrow \mathrm{val} = 0$

In general $\mathbb{Q}_p[i]$

$\pi \longmapsto \tilde{\pi}$

$\bar{\pi} \longmapsto \tilde{\bar{\pi}}$

val's 0,1

$p = \pi \bar{\pi} \longmapsto \underset{\text{val } 1}{\boxed{p}} = \tilde{p} = \widetilde{\pi \bar{\pi}} = \tilde{\pi} \tilde{\bar{\pi}}$

① We show that $\mathrm{level}_\pi(A) = \mathrm{level}_p(\tilde{A})$!

$$\begin{tikzcd}
GL_d(\mathbb{Q}_p) \arrow[r] \arrow[d, "\det"'] \arrow[dr, phantom, "\text{// commutes, since det respects ring hom.}"] & GL_d(\mathbb{Q}_p) \arrow[d, "\det"] \\
\mathbb{Q}_p[i]^\times \arrow[r] \arrow[d, "\mathrm{ord}_\pi"'] \arrow[dr, phantom, "\text{// commutes by choice of } \sqrt{-p}"] & \mathbb{Q}_p^\times \arrow[d, "\mathrm{ord}_p"] \\
\mathbb{Z} \arrow[r, "\mathrm{id}"'] & \mathbb{Z}
\end{tikzcd}$$

Similarly, for $\min_{i,j} \mathrm{ord}_\pi(A_{ij})$

② We show that $\mathrm{level}_p(\tilde{A}) = 1 - \mathrm{dist}(v_0, \overset{\overset{\tilde{A} v_0}{\shortparallel}}{A v_0})$

Take $B \in PGL_d(\mathbb{Z}[\tfrac{1}{p}])$. Want to show:

$$\mathrm{level}_p(B) = 1 - \mathrm{dist}(v_0, B v_0)$$

For $k, k' \in K = PGL_d(\mathbb{Z}_p)$, we observe $\mathrm{level}(k B k') = \mathrm{level}_p(B)$
by ultrametric triangle inequality
$[\, \mathrm{val}(a+b) \geq \min(\mathrm{val}(a), \mathrm{val}(b)) \,]$

$val((kBk')_{ij}) \geq \min_{i,j} val(B_{ij})$ since $val(k_{ij}) \geq 0$

OTOH, $k^{-1}, (k')^{-1} \in K$, so also

$val(B_{ij}) \geq \min_{i,j} val(kBk')_{i,j}$

$\Rightarrow \min_{i,j} val\, B_{ij} = \min_{i,j} val(kBk')_{ij}$

$val(det(kBk')) = val(det\, B)$

$\mathbb{Z}_p^\times \ni det\, k \Rightarrow val = 0$

$\Rightarrow$ $level_p$ is K-bi-invariant. Also, $1\text{-}dist(v_0, Bv_0)$ is:

$$1\text{-}dist(v_0, kB\underbrace{k'v_0}_{v_0}) \underset{G\text{-action}}{=} 1d(k^{-1}v_0, Bv_0) = 1d\, v_0, Bv_0)$$

(Cartan)
We know, $PGL_d(\mathbb{Q}_p) = \coprod_{0=\lambda_1\leq\lambda_2\leq\dots} K \underset{[p^{\vec{\lambda}}]}{\begin{bmatrix} p^{\lambda_1} & & & \\ & p^{\lambda_2} & & \\ & & \ddots & \\ & & & p^{\lambda_d} \end{bmatrix}}$

$lev_p(p^{\vec{\lambda}}) = \sum_{i=1}^{d} \lambda_i$, so we need to show

$1d(v_0, [p^{\vec{\lambda}}]) = \sum \lambda_i$

There is a $\sum\lambda_i$-path of color 1 from $v_0 = I$ to $[p^{\vec{\lambda}}]$.

There is no shorter one, by determinant considerations.

To finish LPS, we need to show that

$\Gamma = PGU_2^+(\mathbb{Z}[\frac{1}{p}])(2)$ has (at least) $p+1$ elements of lev_π 1

(so gv_0 is a neighbor of v_0)

This suffices,

Since, if $gv_0 = g'v_0$ for g, g' of lev_π 1

then $g^{-1}g'v_0 = v_0$ $\Rightarrow$ $g^{-1}g' \in \text{Stab}_p(v_0) = \{1\}$

$\Rightarrow$ $g = g'$

(we didn't use the level at the end). We will use Jacobi

Example $p=5$

$$\Pi = \begin{bmatrix} 1 \pm 2i & \\ & 1 \mp 2i \end{bmatrix}, \begin{bmatrix} 1 & \mp 2 \\ \pm 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \pm 2i \\ \mp 2i & 1 \end{bmatrix}$$

If $\underbrace{a^2+b^2+c^2+d^2}_{\in \mathbb{Z}} = p$ then

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$ has det>0 and is unitary (in GU^+)

$$\| $$

$$\begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix}$$

The only thing we can do is

$$\begin{bmatrix} - v_1 - \\ - \alpha v_2 - \end{bmatrix}, \text{ for } \alpha \in \mathbb{C}^\times, |\alpha| = 1$$

7.5.18

Parzanchevski – From Ram. to Cpx.

$$p \equiv 1 \bmod 4$$

$$\Gamma = PGU_2^+(\mathbb{Z}[\tfrac{1}{p}])(2)$$

$$= \ker \varphi : PGU_2^+(\mathbb{Z}[\tfrac{1}{p}]) \to M_2(\mathbb{Z}/2\mathbb{Z})$$

$$A \mapsto A \ (\mathrm{mod}\ 2)$$

<u>Claim:</u> Γ acts simply transitively on the vertices of the $(p+1)$-regular tree

$$PGU_2^+(R) = \left\{ A \in M_2(R[i]) \,\middle|\, \begin{array}{l} A^*A = \lambda I, \ \det A \in R_{>0} \\ \lambda \in R^\times \end{array} \right\} \Big/ \{\text{scalar matrices}\}$$

$$\Gamma \hookrightarrow PGU_2(\mathbb{Q}_p) \cong PGL_2(\mathbb{Q}_p) \circlearrowright \underbrace{Tree_{p+1}}_{T_{p+1}}$$

we saw that, say v_0 is the root of T_{p+1}, $\mathrm{Stab}_{v_0}\Gamma = \Gamma \cap PGU_2(\mathbb{Z}_p) =$

$$= PGU_2^+(\mathbb{Z})(2) = \{I, -I\} = \{I\}$$

Also, we saw $p = \pi \cdot \bar{\pi}$ where $\pi \in \mathbb{Z}[i]$

Define $\sim ; \ \mathbb{Q}_p[i] \longrightarrow \mathbb{Q}_p$

$$\widetilde{a+bi} = a + b\underset{\substack{\wedge \\ \mathbb{Z}_p}}{\sqrt{-1}}$$

Then WLOG $val(\tilde{\pi}) = 1$, $val(\tilde{\bar{\pi}}) = 0$

we saw that for $A \in PGU_2(\mathbb{Z}[\frac{1}{p}])$

We proved:

$$\underset{\substack{\shortparallel \\ \mathrm{ord}_\pi(\det(A))}}{lev_\pi A} = lev_p \tilde{A} = 1 - \underset{\substack{\uparrow \\ \text{distance of color } 1}}{dist}(v_0, Av_0)$$

with A integral $\overset{(\text{Gaussian})}{+}$ primitive we need to show:

$$\left|\{A \in \Gamma \mid lev_{\pi} A = 1\}\right| = p+1$$

$$\Downarrow$$

Γ acts transitively + simply on the tree.

If $Av_0 \doteq Bv_0$ same neighbor, $A^{-1}Bv_0 = v_0$

$$\Downarrow$$

$$A = B$$

<u>Claim:</u> $PGU_2^{+}(\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\alpha} & \bar{\beta} \end{pmatrix} \middle| \ (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\} \right\}$

If $A = \begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix}$

then $\|v_1\| = \|v_2\|$ & $\langle v_1, v_2 \rangle = 0$

and we get the claim.

$\langle v_1, v_2 \rangle = 0 \Rightarrow v_2 \in v_1^{\perp} \cong \mathbb{C} \Rightarrow v_2$ det. up to const. of norm 1

However $\det\begin{pmatrix} -v_1- \\ -\theta v_2- \end{pmatrix} = \theta \det\begin{pmatrix} -v_1- \\ -v_2- \end{pmatrix}$

so $\det \in \mathbb{R}_{>0}$ for a unique θ.

Claim: For $A \in \Gamma$, scale by $\pi, \bar{\pi}$ to be integral & primitive

Then $A^* A = p^{\ell} \cdot I$ with $\ell = \mathrm{lev}_{\pi} A$.

Reason: $\mathrm{ord}_{\pi}(\det A) = \mathrm{lev}_{\pi} A$ translation by A

$$\mathrm{lev}_{\pi} A^* = 1\text{-dist}(v_0, A^* v_0) \overset{\downarrow}{=} 1\text{dist}(Av_0, v_0) \underset{\nearrow}{=} 1\text{-dist}(v_0, Av_0)$$

the dist in dim2 is symmetric

$$A^* A = p^{\ell} I \Rightarrow \det(A^* A) = p^{2\ell} \Rightarrow$$

$$\Rightarrow \underset{\substack{\| \\ 2\,\mathrm{lev}\,A}}{\mathrm{ord}_{\pi}(\det(A^*A))} = 2\ell$$

One needs to show: $|\{A\in\Gamma \mid A^*A = pI\}| = p+1$

$$\parallel$$

$$|\{(\alpha,\beta)\in\mathbb{Z}[i] \mid |\alpha|^2+|\beta|^2=p\}|$$

$\alpha \equiv 1 \quad \beta \equiv 0$

$\uparrow \bmod 2$

<u>Jacobi:</u> For $p \equiv 1 \bmod 4$ $\exists$ $8(p+1)$ sol. to $|\alpha|^2+|\beta|^2=p$.

Look at

$\alpha \bmod 2$	$\lvert\alpha\rvert^2 \bmod 4$
0	0
1	1
i	1
$1+i$	2

$|\alpha|^2+|\beta|^2 = p \equiv 1 \bmod 4 \Rightarrow$

α	β mod 2
0	1
0	i
$\boxed{1}$	$\boxed{0}$
i	0

Thus only $\frac{1}{4}$ of Jacobi's sol sat.

our congruence conditions.

$$8(p+1) \xrightarrow[\beta=0]{\alpha=1} 2(p+1) \xrightarrow[\text{Projectivising}]{\text{mod scalars}} p+1 \text{ sol's} \in PGL$$

Let's write $S_p \subseteq GL_2^+(\mathbb{Z}[\frac{1}{p}])$ these $p+1$ matrices.

<u>Claim:</u> $\Gamma = \langle S_p \mid A = A^{*^{-1}} \rangle = \langle \text{half of } S_p \mid \rangle$

meaning Γ is free over $\frac{p+1}{2}$ gen.

From Bass-Serre Theory if we show that no edge flipping $\in \Gamma \curvearrowright T_{p+1}$ then Γ is free.

If γ flips e, since γ fips an edge at $v_0 \to \gamma \in S_p$ and $\gamma^2 = id$

since $\mathrm{stab}_{v_0}\Gamma = \{id\}$

$$\gamma^*\gamma = p\cdot id \Rightarrow \gamma = \gamma^{-1} = \frac{\gamma^*}{p} \underset{\substack{\uparrow \\ \text{Projective}}}{=} \gamma^*$$

$$\Rightarrow \quad \begin{matrix} \bar{\beta} = \tilde{\beta} \not\Rightarrow \beta = 0 \\ \alpha = \bar{\alpha} \Rightarrow \alpha \in \mathbb{R} \end{matrix} \quad \Rightarrow \gamma = \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} \in PGO$$

$$\Downarrow$$

$$\alpha = 1$$

$$\gamma = id$$

Also $\mathrm{Cay}(\overset{\substack{PGL_2(\mathbb{Q}_p) \\ \rotatebox{90}{$\leq$}}}{\Gamma}, S_p) = T_{p+1} = PGL_2(\mathbb{Q}_p) / PGL_2(\mathbb{Z}_p)$

with the right edges.

Furthermore, for $\begin{matrix} 2 \mid n \\ p \nmid n \end{matrix}$ $\quad \underset{A \equiv I(n)}{\Gamma(n)} \leq_{f.i.} \overset{\text{LPS Lattice}}{\Gamma(2)}$

$$\Downarrow$$

$\Gamma(n)$ also a lattice.

We get $\Gamma(n) \backslash \mathcal{B}_p^2 = T_{p+1}$ = finite $p+1$ reg graph

But $\Gamma(n)\backslash B_p^2 \cong \Gamma(n)\backslash Cay(\Gamma(2), S_p) = Cay(\Gamma(n)\backslash\Gamma(2), S_p)$

AND THESE ARE

Ramanujan Graphs!!!

Actually, $\Gamma(2q)\backslash\Gamma(2) = PSU_2(\mathbb{F}_q)$

or $PGU_2(\mathbb{F}_q)$

q a prime

And we get the $X^{p,q}$ LPS graphs.

①

$p \equiv 1 \pmod 4$ 7/5/2018 א״ו כ״ב אייר

A Review of Recent lectures

$\overset{A \equiv I \pmod 2}{\downarrow}$

$$\Gamma = PGU_2^+(\mathbb{Z}[\tfrac{1}{p}])(2) \xhookrightarrow[\text{discrete}]{} PGU_2(\mathbb{Q}_p) \cong PGL_2(\mathbb{Q}_p) \circlearrowleft T_{p+1}$$

Claim: Γ acts simply trans. on the (vertices of) the $(p+1)$-reg tree.

$$PGU_2^+(\overset{\mathbb{Q} \text{ or } \mathbb{R}}{\overset{\subseteq}{R}}) = \{A \in M_2(R[i]) \mid A^*A = \overset{\overset{GU}{\downarrow}}{\lambda I},\ \lambda \in R^\times,\ \overset{\overset{GU+}{\downarrow}}{\det A \in R_{>0}}\}/\text{scalars}$$

We saw: $Stab_\Gamma(v_0) = \Gamma \cap PGU_2(\mathbb{Z}_p) = PGU_2^+(\mathbb{Z})(2) = \{I, -I\} = \{I\}$

" " : $p = \pi \cdot \bar\pi$, $\pi \in \mathbb{Z}[i]$

$$\mathbb{Q}_p[i] \xrightarrow{x \mapsto \tilde{x}} \mathbb{Q}_p$$

$$\widetilde{a+bi} = a + b\sqrt{-1}$$

WLOG, $val(\tilde\pi) = 1$, $val(\tilde{\bar\pi}) = 0$

(recall: $\tilde\pi, \tilde{\bar\pi} \in \mathbb{Z}_p$, $\widetilde{\pi\bar\pi} = p$)

(p=5) example $\pi = 1+2i$ $\bar\pi = 1-2i$ $1 \pm \sqrt{-1} \in \mathbb{Q}_5$

We saw for $A \in PGU_d(\mathbb{Z}[\frac{1}{p}]) \subseteq M_d(\mathbb{Z}[\frac{1}{p}, i])$

(Level) $lev_\pi A = lev_p \tilde{A} = 1\text{-}dist(v_0, Av_0)$

$\nearrow$ $\uparrow$

(distance along edges of color 1)

$ord_\pi(\det(A))$, where A is $\underset{\underset{\text{Gaussian integer}}{\Uparrow}}{\text{integral}}$, primitive

Need to Show: $\#\{A \in \Gamma : lev_\pi(A) = 1\} = p+1$

$\Rightarrow$ Γ acts simply trans. on the tree

[If $Av_0 = Bv_0$ is the same neighbor, $A^{-1}Bv_0 \Rightarrow A^{-1}B = I$.]

Claim $PGU_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : (\alpha, \beta) \in \mathbb{C}^2 \setminus (0,0) \right\}$

Pf: In one direction:

$$A^* A = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \\ & |\alpha|^2 + |\beta|^2 \end{pmatrix} = \lambda I \,,$$

$$\det A = |\alpha|^2 + |\beta|^2 > 0$$

OTOH, If $A = \begin{pmatrix} -v_1- \\ -v_2- \end{pmatrix}$, then $\|v_2\| = \|v_1\| = 0$

$\langle v_1, v_2 \rangle = 0 \Rightarrow v_2 \in v_1^\perp \cong \mathbb{C}$

$\Rightarrow v_2$ is determined up to a constant of norm 1.

However, $\det \begin{pmatrix} -v_1- \\ -\theta v_2- \end{pmatrix} = \theta \cdot \det \begin{pmatrix} -v_1- \\ -v_2- \end{pmatrix}$,

so $\det A \in \mathbb{R}_{>0}$ for a unique θ.

(LPS used the language of quarternions.)

Claim: For $A \in \Gamma$, scale A to be integral and prim. (scale by $\pi, \bar{\pi}$), and then:

$$A^* A = p^\ell I \quad \text{with } \ell = \mathrm{lev}_\pi A$$

Pf: $\mathrm{lev}_\pi(A) = \mathrm{ord}_\pi(\det(A))$

$\mathrm{lev}_\pi(A^*) = \mathrm{ord}_\pi(\overline{\det(A)}) = \mathrm{ord}_{\bar{\pi}}(\det(A))$

and $A^* A = p^\ell I \Rightarrow \det(A^* A) = p^{2\ell}$

$\Rightarrow \mathrm{ord}_\pi(\det(A^* A)) = 2\ell$

$\quad\quad \overset{\shortparallel}{\mathrm{ord}_\pi(\det(A))} + \mathrm{ord}_\pi(\det(A^*)) = 2\, \mathrm{ord}_\pi \det(A)$

$$= 1\text{-}\mathrm{dist}(v_0, A^* v_0) \overset{\text{translation by } A}{=} 1\text{-}\mathrm{dist}(A v_0, v_0) = 1\text{-}\mathrm{dist}(v_0, A v_0) \overset{\text{In dimension 2, all edges have color 1}}{=} \mathrm{ord}_\pi(\det(A))$$

[In dimension 3, the following has different orders for $\pi, \bar{\pi}$: $\begin{bmatrix} 1+2i & & \\ & 1-2i & \\ & & 1-2i \end{bmatrix}$; $p=5$]

Recall We need to show: $\#\{A \in \Gamma : A^*A = pI\} = p+1$

$$\overset{\parallel}{\#\{(\alpha,\beta) \in \mathbb{Z}[i] : |\alpha|^2+|\beta|^2 = p,\ \alpha \equiv 1,\ \beta \equiv 0 \pmod 2\}}$$

Homework! (check: no common factors, so no non-primitive possibilities.)
[If α, β have common factor, its square must divide p]

4-squares Thm — **Jacobi's Theorem** For ~~odd~~ prime, p, there exist $8(p+1)$ solutions to $|\alpha|^2 + |\beta|^2 = p$

We will prove a generalization for higher dimensions… eventually

for $\alpha \in \mathbb{Z}[i]$, Consider:

$\alpha \pmod 2$	$\lvert\alpha\rvert^2 \pmod 4$
0	0
1	1
i	1
$1+i$	2

But $|\alpha|^2+|\beta|^2 \equiv p \equiv 1 \pmod 4 \Rightarrow$ The possibilities are:

α	β	(mod 4)
0	1	
0	i	
$\boxed{1}$	$\boxed{0}$	
i	0	

$\Rightarrow$ only $\frac{1}{4}$ of Jacobi's solutions satisfy the congruence condition.
[The $8(p+1)$ solutions are distributed evenly, by symmetry.]

$\Rightarrow 2(p+1)$

… but we forgot to mod by scalars

$$\xrightarrow[\text{mod}\{\pm I\}]{} p+1 \text{ solutions in } PGL,$$

We write $S_p \subseteq GU_2^+(\mathbb{Z}[\frac{1}{p}])$ for these $p+1$ matrices.

Claim $\Gamma = \langle S_p \mid A = (A^*)^{-1} \rangle$

(or: "Half of S_p, w/o conjugates" and no relations)

So, Γ is a free gp. on $\frac{p+1}{2}$ elements.

This follows for Bass-Serre Thm, if we show:

no edge flipping:

If γ' flips e, then some γ flips an edge at v_0 (so $\gamma \in S_p$) and $\gamma^2 = id$ (, since $Stab_\Gamma(v_0) = I$)

$$\xcancel{\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} = \begin{matrix} \alpha^2 - \bar{\beta}^2 & -(\alpha + \bar{\alpha})\beta \\ \ldots & \end{matrix}}$$

No need to compute, since:

$$\gamma^* \gamma = p I \Rightarrow \underset{GU \nearrow}{\gamma^{-1} =} \frac{\gamma^*}{p}$$

In our gp. $\gamma = \gamma^* \Rightarrow \bar{\beta} = -\bar{\beta} \Rightarrow \beta = 0 \Rightarrow \gamma = \begin{bmatrix} \alpha & \\ & \bar{\alpha} \end{bmatrix}$

Example for $p = 5$, $S_5 = \left\{ \begin{pmatrix} 1 \pm 2i & \\ & 1 \mp 2i \end{pmatrix}, \right.$

$\left. \begin{pmatrix} 1 & \pm 2i \\ \mp 2i & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm 2 \\ \mp 2 & 1 \end{pmatrix} \right\}$

$\Rightarrow \alpha = \bar{\alpha} \Rightarrow \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}$ with $\alpha \in \mathbb{Z} \Rightarrow \alpha^2 = p \Rightarrow \gamma = I$

Since S_p is free, we have that the following Cayley graph is a <u>tree</u>

subgp.
$\downarrow$
$\Gamma \leq PGL_2(\mathbb{Q}_p)$

$$Cay(\Gamma, S_p) = PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p) \quad \text{(with the right edges)}$$

Furthermore, for any n s.t. $2|n \wedge p \nmid n$, consider

A normal subgp. $\longrightarrow$ $\underset{\{A \equiv I \pmod{n}\}}{\Gamma(n)} \overset{\text{f.i.}}{\trianglelefteq} \Gamma(2) \leftarrow$ LPS lattice

We get: $\Gamma(n)\backslash B_p^2 = T_{p+1}$

So $\Gamma(n)$ is a lattice in $PGL_2(\mathbb{Q}_p)$

$\Rightarrow \Gamma(n) \backslash B_2^p$ is a finite $(p+1)$-reg graph

$= \Gamma(n) \backslash Cay(\Gamma(2), S_p) \underset{\nearrow}{=} Cay(\Gamma(n)\backslash\Gamma(2), S_p) \leftarrow$ finite p-reg graph

easy to see.

<u>Point</u>: These are Ramanujan!

In fact, $\Gamma(n)\backslash\Gamma(2) = \begin{cases} PSL_2(\mathbb{Z}/n) \\ \text{or:} \\ PGL_2(\mathbb{Z}/n) \end{cases}$ (If $q \equiv 1 \pmod 4$)

And if $n = 2q$ (q prime)

$= \begin{cases} PSL_2(\mathbb{F}_q) \\ \text{or} \\ PGL_2(\mathbb{F}_q) \end{cases} \begin{matrix} \cong \\ \cong \end{matrix} \begin{matrix} PSL_2(\mathbb{F}_q) \\ PGL_2(\mathbb{F}_q) \end{matrix}$

These are the LPS $X^{p,q}$ graphs

<u>Bass-Serre</u> If $\Gamma \underset{\text{s.t.}}{\circlearrowright}$ verts of tree, then

$$\Gamma = \mathbb{Z} \times \dots \times \mathbb{Z} \times \mathbb{Z}/2 \times \dots \times \mathbb{Z}/2$$

①

שיעור לב אלכ' - 21/5/18

We saw LPS Lattices, defined as $\leftarrow \mathcal{B}$, the building

$$\underset{\text{even } N}{PGU_2^+(\mathbb{Z}[\tfrac{1}{p}])(N)} \le \overbrace{PGU_2(\mathbb{Q}_p) \underset{\text{if } p \equiv 1 (\text{mod } 4)}{\cong} PGL_2(\mathbb{Q}_p)}$$

Lattice; normal subgp of $\Gamma(2)$

$\Gamma(2)$ acts simply transitively on (verts) of $\mathcal{B}$

$\Rightarrow \mathcal{B} = Cay(\Gamma(2), S) \quad (|S| = p+1)$

$\Rightarrow \Gamma(N) \backslash \mathcal{B} = Cay(\Gamma(N) \backslash \Gamma(2), S) \equiv X^{p,N} \in$ the LPS construction

We still need to show: for $N = 2q$,

$$\Gamma(N) \backslash \Gamma(2) = \begin{cases} PGL_2(\mathbb{F}_q) \\ \text{or} \\ PSL_2(\mathbb{F}_q) \end{cases}$$

$X^{p,N}$ is Ramanujan - This follows from deep results by Deligne... we will not see the details.

Where does Jacobi's thm. come from?

We want a proof that generalizes to higher dim.

What happens in $U(3)$?

For U_2:

$$\left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \middle| \begin{array}{l} \alpha, \beta \in \mathbb{Z}(i) \\ |\alpha|^2 + |\beta|^2 = p \end{array} \right\}$$

First we asked for: $A \in M_2(\mathbb{Z}[i])$ s.t. $A^* A = pI$

We saw: for any α, β s.t. holds, $\exists! A \in PGU_2^+$ s.t.

$$A = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

As for $U(3)$:

$A \in M_3(\mathbb{Z}[i])$ $\quad A^* A = p I$

$$A = \begin{pmatrix} \alpha & \beta & \gamma \\ | & | & | \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 + |\gamma|^2 = p$$

Jacobi's 6-square thm., $\exists\ 2(p^2+1)$ solutions

Or: searched for a pattern/rule for such matrices.

We will see: <u>Siegel, Mass, Formula</u>

First: <u>Golden Gates</u> (LPS)

We saw: $S = \{ A \in PGU_2^{+}(\mathbb{Z}[\tfrac{1}{p}])(2) \mid A^* A = pI \}$

$|S| = p+1 \quad \mathcal{B} = \mathrm{Cay}(\Gamma(2), S)$

e.g. $p = 5$, $S = \left\{ \begin{bmatrix} 1+2i & \\ & 1-2i \end{bmatrix}^{\pm 1}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}^{\pm 1} \right\}$

No real difference $\rightarrow$ $PU(2) = \{ A \in M_2(\mathbb{C}) \mid A^* A = I \} / \text{scalars}$

$$\begin{aligned} &\ \ \ \Vert \\ PGU(2) &= \{ \qquad\qquad \mid A^* A = \lambda I \} / \text{scalars} \\ &\qquad\qquad\qquad \Updownarrow \\ &\qquad\qquad\quad \frac{A}{\sqrt{\lambda}} \in U \end{aligned}$$

e.g. $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \in U(2)$

Let $S \subseteq \overset{=G}{\overline{PU(2)}}$ s.t. S is finite, symmetric.

<u>Define</u> (a lie sp)! $T_S : L^2(PU(2)) \circlearrowleft$ (PU(2) is uncountable)

by $(T_S f)(x) = \sum_{s \in S} f(sx)$

$k = |S| \in Spec(T_S)$ since $T_S \mathbb{1} = k \mathbb{1}$

<u>Claim</u>: If $\langle S \rangle = G$ then $k \notin e.v.(T_S|_{L^2_0(G)})$

(where $L^2_0(G) = \mathbb{1}^\perp \leftarrow$ orth. complement)

<u>Pf</u> If $T_S f = kf$, assume f is continuous

(else, approximate by cont. functions...)

by compactness, choose $x_0 \in G$ s.t. $|f(x_0)|$ maximal.

$\Rightarrow$ all $f(sx_0) = f(x_0)$ $\forall s$

$\Rightarrow f(\langle S \rangle x_0) \equiv$ const

$\Rightarrow f \equiv$ const

<u>Note</u> It's possible that $k \in spec(T_S|_{\mathbb{1}^\perp})$

s.t. $\lambda_i \longrightarrow k$ (k is an acc. point)

We call this "Amenable S"

So - $k \in ev\ \mathbb{1}^\perp \Rightarrow$ disconnected

$k \in sp\ \mathbb{1}^\perp \Rightarrow$ Amenable

$k \in sp\ \mathbb{1}^\perp \Rightarrow$ Connected

expander: $sp\ \mathbb{1}^\perp << k$

Ram : $sp\ \mathbb{1}^\perp << 2\sqrt{k-1}$

<u>LPS</u> $\forall$ symmetric $S \subseteq G = PU(2)$ $\max\{|\lambda| : \lambda \in spec(T_S|_{\mathbb{1}^\perp})\} \geq 2\sqrt{k-1}$

(Similar to Alon-Boppana)

For $S = \{A \in PGU_2^+(\mathbb{Z}[\frac{1}{p}])(2) \mid A^* A = pI\}$

$\max\{|\lambda| : \lambda \in spec(T_S|_{\mathbb{1}^\perp})\} = 2\sqrt{k-1}$

"Ramanujan generators for PU(2)"

Write $\lambda_S = \max\{|\lambda| : \lambda \in \operatorname{spec}(T_S|_{\mathbb{1}^\perp})\}$

normalize Haar measure: $\mu(G) = 1$

<u>We ask</u>: for what $\varepsilon > 0$ is $G = \bigcup_{s \in S} B_\varepsilon(s)$

or: for what ε is $\mu(G \setminus \bigcup_{s\in S} B_\varepsilon(s)) = o(1)$

$o(1)$, as a func. of $k = |S|$

<u>Thm</u>: $\mu(G \setminus \bigcup_{s\in S} B_\varepsilon(s)) \leq \frac{\lambda_S^2}{k^2 \mu(B_\varepsilon)}$

(recall: $\mu_\varepsilon \equiv \mu(B_\varepsilon) \gtrsim \varepsilon^3$)

<u>Pf</u>: Take $f = \mathbb{1}_{B_\varepsilon(1)} - \underset{\text{constant}}{\mu_\varepsilon \mathbb{1}} \in \mathbb{1}^\perp$

$$\|f\|^2 \int f^2 d\mu = \int_G \left(\mathbb{1}_{B_\varepsilon(1)} - 2\mu_\varepsilon \mathbb{1}_{B_\varepsilon(1)} + \mu_\varepsilon^2 \mathbb{1}\right) d\mu$$

$$= \mu_\varepsilon - 2\mu_\varepsilon^2 = \mu_\varepsilon(1-\mu_\varepsilon)$$

$$\lambda_S^2 \mu_\varepsilon(1-\mu_\varepsilon) \geq \int_G (T_S f)^2 = \int_G \sum_{s\in S} f(s^{-1}g)^2$$

$$= \int_G \Big(\sum_{s\in G} \overbrace{\mathbb{1}_{B_\varepsilon(1)}(s^{-1}g)}^{=\mathbb{1}B_\varepsilon(g)} - \mu_\varepsilon\Big)^2 \geq \int_{G \setminus \bigcup B_\varepsilon(s)} (k \cdot \mu_\varepsilon)^2 d\mu$$

$\forall g \notin \bigcup_{s\in S} B_\varepsilon(s)$

$$= k^2 \mu_\varepsilon^2 \mu(G \setminus \bigcup B_\varepsilon(s))$$

$\Rightarrow \lambda_S^2 (1-\mu_\varepsilon) \geq k^2 \mu_\varepsilon \mu(G \setminus \bigcup B_{\varepsilon}(s))$

$\Rightarrow \mu(G \setminus \bigcup B_\varepsilon(s)) \leq \frac{\lambda^2(1-\mu_\varepsilon)}{k^2 \mu_\varepsilon} \lesssim \frac{\lambda^2}{k^2 \mu_\varepsilon}$

for small ε, similar size

What we really want to do:

Take $S_1 = S$, $S_2 = S \cdot S$, $S_3 = S \cdot S \cdot S$

Given $\varepsilon > 0$, for which ℓ is $\bigcup B_\varepsilon (S_\ell) = G$

If we were dealing with multi-sets, $|S_\ell| = k^\ell$

so that $\lambda_{S_\ell} = \lambda_S^{\ell}$, $T_{S_\ell} = T_S^{\ell}$

But S is symmetric, so we have repititions (and obtain the identity often in $S \cdot S \cdot S$..)

So we can use Chebyshev poly. for $T_{S_2} = T_S^2 - kI$...

calculate the backtracking...

__Claim__ If $\mu_\varepsilon \overset{\circledast}{\geq} \frac{\lambda_S}{k}$ then $G = \bigcup_{s \in S} B_{2\varepsilon}(s)$

__Homework!__ __Prove__

use previous thm "twice".

Why is Ram. optimal?

$\lambda_S = 2\sqrt{k-1}$ (for the LPS generators)

take $S^{(\ell)}$ = words of length ℓ in S

so $T_{S^{(\ell)}} \underset{\uparrow}{\longleftrightarrow} A^{(\ell)}$ in Ram. graphs

correspond spectrally

$\Rightarrow \lambda_{S^{(\ell)}} \approx (\ell+1) k^{\ell/2}$ (Ori doesn't recall the exact formula)

this can be obtained by nonbacktracking analysis

For Ram. generators, $|S^{(\ell)}| = k(k-1)^{\ell-1} \approx (k-1)^\ell$

$\circledast$ Like saying: $\varepsilon \approx \sqrt[3]{\frac{\lambda_S}{k}}$

Ⓖ

$\Rightarrow \quad \mu(G \setminus \bigcup B_\varepsilon S^{(\ell)}) \approx \dfrac{\ell^2 k^\ell}{(k-1)^{2\ell} \mu_\varepsilon} \gtrsim \dfrac{\ell^2}{(k-1)^\ell \varepsilon^3}$

for $\ell \to \infty$, for which $\overset{\mu_\varepsilon = \mu_\varepsilon(\ell)}{\cancel{\varepsilon = \varepsilon(\ell)}}$ does this $\longrightarrow 0$?

we need: $\mu_\varepsilon \gg \dfrac{\ell^2}{(k-1)^\ell} \approx \dfrac{\log^2_{(k-1)} |S^{(\ell)}|}{|S^{(\ell)}|}$

If $\bigcup_{s \in S^{(\ell)}} B_\varepsilon(s) = G$, by volume considerations,

$$\mu_\varepsilon \gtrsim \frac{1}{|S^{(\ell)}|}$$

① $\mathfrak{C}$

שיעור: פ אורי 28/5/2018

Γ = Lattice in $G = PGU_d(\mathbb{Q}_p) \cong PGL_d(\mathbb{Q}_p) \circlearrowright \mathcal{B}_p^d$

(e.g.: $\Gamma = PGU_d(\mathbb{Z}[\tfrac{1}{p}])(N)$)

$X = \Gamma \backslash \mathcal{B}_p^d$ is a finite complex

Any G-invariant branching operator, T, on $\mathcal{B}_p^d$,

① induces $T|_X$

② given as a lin comb. of double cosets of stabilizers.

Ramanujan: $Spec(T|_X) \subseteq Spec(T|_{\mathcal{B}_p^d}) \cup \{\text{trivial spec}\}$

For simplicity, fix a cell σ; T acts on $G\sigma \subseteq \mathcal{B}_p^d$

$K_\sigma = Stab_G(\sigma)$

Then $L^2(G\sigma) \cong L^2(G/K_\sigma) = \underbrace{L^2(G)}_{\substack{\uparrow \\ G\text{-rep., by } (gf)(x) = f(xg)}}{}^{K_\sigma}$

and $T|_X \circlearrowright L^2(\Gamma\backslash G\sigma) = L^2(\Gamma\backslash G/K_\sigma) = L^2(\Gamma\backslash G)^{K_\sigma}$

$$\left(\begin{array}{ll} \text{e.g} & \sigma = v_0 \\ & K = K_{v_0} = \begin{smallmatrix}\max \\ \text{cpt}\end{smallmatrix} = PG\overset{U}{L}_d(\mathbb{Z}_p) \\ & L^2(\text{vertices } (\Gamma\backslash\mathcal{B})) \cong L^2(\Gamma\backslash G)^K \end{array} \right)$$

If $\underset{\text{as } G\text{-reps}}{L^2(\Gamma\backslash G) \cong} \hat{\bigoplus}_i V_i$, $\quad L^2(\Gamma\backslash G/K_\sigma) = \hat{\bigoplus} \underset{\substack{\circlearrowleft \\ T}}{V_i^{K_\sigma}}$ (each summand is T-stable)

$Spec(T|_X) = \bigcup_{V_i \in L^2(\Gamma\backslash G)} Spec(T|_{V_i^{K_\sigma}})$

Corr: If every irr. $V \leq L^2(\Gamma\backslash G)$ is either "trivial" or (weakly) contained in $L^2(G)$

then: X is Ramanujan

"Trivial" means for us: of the form

(in support of Plancharel measure of rep.)

$$PGL_d(\mathbb{Q}_p) \longrightarrow \mathbb{C}^\times$$

$$g \longmapsto e^{\frac{2\pi i j}{d} \cdot val(det(g))} \qquad j = 0, \dots, d-1$$

Take $G = PGL_d(\mathbb{Q}_p)$

$K = PGL_d(\mathbb{Z}_p)$

$L^2(\Gamma\backslash G) = \oplus V_i^K$

How does a G-rep. V s.t. $V^K \neq 0$ look like?

An example (principal series): Take $z_1, \dots, z_d \in \mathbb{C}^\times$

$\chi_{\vec{z}} : B \to \mathbb{C}^\times$

$$\chi_{\vec{z}}(B) = \prod_{i=1}^{d} z_i^{val(b_{ii})}$$

normalized

$$I(\vec{z}) := \widetilde{Ind}_B^G \chi_{\vec{z}} = \{ f : G \to \mathbb{C} \mid f(bg) = \delta^{-1/2}(b)\, \chi_{\vec{z}}(b)\, f(g) \}$$

(B = Borel gp : upper triangular)

since $BK = G$,

$\dim I(\vec{z})^K \leq 1$: $\quad f(g) = f(bk) = (\delta\chi)(b)\, f(1)$

Exercise $\dim I(\vec{z})^K = 1$ (we saw this)

Thm Every irr. V s.t. $V^K \neq 0$ is of that form.

Hecke op's

Look at $A_i = K \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & p & & & \\ & & & p & & \\ & & & & \ddots & \\ & & & & & p \end{pmatrix} K$, $\underbrace{p \; p \dots p}_{i \text{ times}}$

p, i times

$$= \bigsqcup \begin{pmatrix} p & * & * & * & * & * \\ & 1 & & & & \\ & & 1 & p & * & * & * \\ & & & 1 & & \\ & & & & p & * \\ & & & & & 1 \end{pmatrix} K \qquad \text{where } * \in \mathbb{Z}/p$$

p right of $*$

1 above

1

Let $f_{\vec{z}}$ be the unique vector in $I(\vec{z})^K$

s.t. $f_{\vec{z}}(1) = 1$

Since $\dim I(\vec{z})^K = 1$, we know that $A_i f_{\vec{z}} = \lambda_i f_{\vec{z}}$

③

$$(A_1 f_{\vec z})(1) = \sum_{i=1}^{d} f\left(1 \underbrace{\begin{pmatrix}1&&&\\&1&&\\&&p&*\,*\,*\\&&&1\ a_1\end{pmatrix}}_{\in B}\right) = \sum_{i=1}^{d} \underbrace{\delta^{-1/2}\begin{pmatrix}1&&&\\&1&&\\&&p&*\,*\,*\\&&&1\ 1\end{pmatrix}}_{= \ p^{i-\frac{d+1}{2}}} \overbrace{\chi_{\vec z}\begin{pmatrix}1&&&\\&1&&\\&&p&*\,*\\&&&1\ 1\end{pmatrix}}^{z_i} \overbrace{f(1)}^{1}$$

$\to$ modular funtion of Borel

$$= \sum p^{d-i} p^{i-\frac{d+1}{2}} z_i = p^{\frac{d-1}{2}}(z_1+\ldots+z_d) \ \boxed{\equiv \ \lambda_1}$$

$$(A_2 f_{\vec z})(1) = \sum_{i<j} f\left(1\begin{pmatrix}1&&&&\\&1\ p\ *\,*\,\square\,*\,*&&&\\&&1&&\\&&&1\ p\,*\,*&\\&&&&1\\&&&&&1\end{pmatrix}\right) =$$

$\uparrow$ i row | $\uparrow$ j row

$$= \sum_{i<j} \underbrace{p^{d-i-1+d+j}}_{***}\ \underbrace{p^{i-\frac{d+1}{2}+j-\frac{d+1}{2}}}_{\delta}\ \underbrace{z_i z_j}_{\chi}$$

$$= p^{d-2}\sum_{i<j} z_i z_j$$

$$\lambda_k(A_k f_{\vec z})(1) = p^{\frac{k(d-k)}{2}} \sigma_k(\vec z)$$

$\nearrow$ symmetric polynomial

<u>Definition</u> The Hecke algebra $\overset{\mathcal{H}}{\vee}$ of G (w.r.t. K) is
$C_c({}_K G/_K)$
with $(f * f')(g) = \int_G f(x) f'(x^{-1}g)\,dx$

If V is a rep. of G, then V^K is a rep. of $\mathcal{H}$
(Analog of gp-algebra)
If $e \in \mathcal{H}$, $v \in V^K$ the action is $ev = \int_G e(g)\, gv\, dg$

1_K is identity, assuming $\mu_G(K) = 1$

Haar measure on G

We actually studied $\mathcal{H}$!

$$\begin{tikzcd}
\text{We studied } G\text{-equivariant ops on } G/K \arrow[d, leftrightarrow] & & \text{composition} \arrow[dd, leftrightarrow] \\
\text{combo of } KgK \arrow[d, leftrightarrow] & \sum c_i Kg_iK \arrow[d, leftrightarrow] & \\
\mathcal{H} & \sum c_i 1_{Kg_iK} & \text{convolution}
\end{tikzcd}$$

<u>note</u> $KgK \circlearrowright v \in V^K$ was defined by decomposing:

$$KgK = \bigsqcup s_iK$$

so, $KgKv = \sum s_i v$

Now,
$$\begin{aligned} 1_{KgK}\, v &= \int_G 1_{KgK}(x)\, xv\, dx \\ &= \int_{KgK} xv\,dx = \sum \int_{s_iK} xv\,dx \\ &= \sum \int_K s_i \overbrace{kv}^{v} dx = \sum \int_K s_i v\, dk = \\ & \sum \overbrace{\mu(K)}^{=1} s_i v = \sum s_i v \end{aligned}$$

Cartan

We saw: $\mathcal{H} = \bigoplus_{0 \le \lambda_1 \le \lambda_2 \le \dots} K(p^{\vec{\lambda}})K = \mathbb{C}[A_1, \dots, A_{d-1}]$

where $A_j = K\underbrace{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & p & \\ & & & & p \end{pmatrix}}_{j}K$

Now, the action of $\ell \in \mathcal{H}$ on $I(\vec{z})^K$ is determined.

We saw (Gelfand trick): $\mathcal{H}$ is commutative,

so, every irr.⊛ $\mathcal{H}$-rep is 1-dim (use Schur's lemma)

so $I(\vec{z})^K$ are irr. $\mathcal{H}$-reps

⊛ <u>unitary</u> – this is easier, and suffices

From principal series, we get for every $\vec{z} \in (\mathbb{C}^\times)^d$ a 1-dim rep $I(\vec{z})^K$ with str. hom.

$$\rho_{\vec z} : \mathcal{H} \to \mathrm{End}_{\mathbb{C}}(I(\vec z)^K) = \mathbb{C}$$

$$A_j \mapsto p^{\frac{j(d-j)}{2}} \cdot \sigma_j(\vec z)$$

Keeping the $\vec z$ as parameters, we get

$$\rho : \mathcal{H} \to \mathbb{C}^{(\mathbb{C}^{\times d})}$$

actually, $\rho : \mathcal{H} \to \mathbb{C}[z_1, \ldots, z_d]$

Take $\tilde G = GL_d(\mathbb{Q}_p)$

$\tilde K = GL_d(\mathbb{Z}_p)$

$\tilde{\mathcal H} = C_c(\tilde K \backslash \tilde G / \tilde K)$

$I(\vec z) = \widetilde{\mathrm{Ind}}_{\tilde B}^{\tilde G} \chi_{\vec z}$, $\lambda_i = A_i f_{\vec z}(1) = p^{\frac{i(d-i)}{2}} \cdot \sigma_i(\vec z)$

$$\mathcal H = \mathbb{C}[A_1, \ldots, A_{d-1}, \underset{\widehat{pI}}{A_d}, \underset{\widehat{p^{-1}I}}{A_d^{-1}}]$$

notes: $\tilde K \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \tilde K = p \tilde K$

$A_d f_{\vec z} = z_1 z_2 \cdots z_d = \sigma_d(\vec z)$ ← up to some factor p^ℓ

$A_d^{-1} f_{\vec z} = (p^\ell z_1 \cdots z_d)^{-1}$

$\mathcal H$ is still commutative; the new elems. are in the center.

We got $\rho : \tilde{\mathcal H} \to \mathbb{C}(z_1, \ldots, z_d)$ (rational functions)

⑥

Claim ①: $\rho : \tilde{\mathcal{H}} \twoheadrightarrow \mathbb{C}[z_1, \dots, z_d, \overline{z_1 \dots z_d}^{\,1}]^{Sym(d)}$

(fundamental symm. polynomials generate all symm. polys)

$$= \mathbb{C}[z_1^{\pm 1}, \dots z_d^{\pm 1}]^{Sym(d)}$$

Claim ②: That is an isomorphism.

$$\begin{tikzcd}
\tilde{\mathcal{H}} = \mathbb{C}[A_1, \dots A_d, A_d^{-1}] \arrow[r, two heads] & \mathbb{C}[z_1, \dots, z_d, \sigma_d(\vec{z})^{-1}]^{Sym(d)} \\
\mathbb{C}[x_1, \dots x_d, x_d^{-1}] \arrow[u, "x_i \mapsto A_i"']
\end{tikzcd}$$

; onto, since they generate a free algebra

Compose the maps: $x_i \mapsto \rho^m \sigma_i(\vec{z})$

By the Fund. thm. of symm. polynomials.

If an iso decomposes through two epimorphisms – they are iso. themselves //

Corr: $\tilde{\mathcal{H}} \overset{\sim}{=} \mathbb{C}[z_1^{\pm 1}, \dots z_d^{\pm 1}]^{Sym}$

Satake isomorphism

symm. Laurent polynomials

Next week: we show that every k-spherical rep. $(V^K \neq 0)$ is prin. series

FROM EXPANDER GRAPHS TO RAMANUJAN COMPLEXES – JUNE 4^{th} , 2018

We had

$$G = GL_d(\mathbb{Q}_p) \ ; \ K = GL_d(\mathbb{Z}_p)$$

For every $z_1, ..., z_d \in (\mathbb{C}^*)^d$ we defined $I(\overrightarrow{z}) = \overset{\sim}{Ind}{}_B^G(\chi_{\overrightarrow{z}})$ where B is the upper triangular in G . We also saw that $\dim I(\overrightarrow{z})^K = 1$, denote $\langle v \rangle = I(\overrightarrow{z})^K$ and then

$$A_j v = p^{\frac{j(d-j)}{2}} \sigma_j(\overrightarrow{z}) v$$

We denoted by $\mathcal{H}$ the Hecke Algebra defined as

$$\mathcal{H} = H_G^K = C_c(K \backslash G / K) = G - \text{inv branching ops. on } G/K$$

Whenever $G \curvearrowright V$ then $\mathcal{H} \curvearrowright V^K$ and $\varphi v = \int_G \varphi(g) g v dg$ and 1_K is the identity in H . We saw that $\mathcal{H} = \mathbb{C}\left[A_1, ..., A_d, A_d^{-1}\right]$.

Theorem 1. *(Satake)* $\mathcal{H} \cong \mathbb{C}\left[x_1^{\pm 1}, ..., x_d^{\pm 1}\right]^{sym}$.

Claim 2. If V is a - G irreducible representation, then $V^K \neq 0$ (V is K -spherical).

Thus $V \cong I(\overrightarrow{z})$. If $X = \Gamma \backslash \mathcal{B}$, then $X^0 \cong \Gamma \backslash G/K$ and $L^2(X^0) \cong L^2(\Gamma \backslash G)^K = \oplus V_i^K$.

Small interlude:

Claim 3. For $K \leq_{\text{compact}} G$ and open , If V is K spherical, irreducible of G , then V^K is irreducible (As $\mathcal{H}$-representation). Also V^K determines V .

Proof. V irr. rep. Let $W \leq_{\mathcal{H}} V^K$. Take $0 \neq w \in W$,

$$\forall v \in V \ : \ v = \sum \alpha_i g_i w$$

if $v \in V^K$, then

$$\begin{aligned} v = 1_K v = 1_K \sum \alpha_i g_i w &= \sum \alpha_i 1_K g_i w \\ = \sum \alpha_i 1_K g_i 1_K w &= \sum \alpha_i 1_{K g_i K} w \in W \end{aligned}$$

Thus $W = V^K$.

Let V_1, V_2 irreducible , $T \colon V_1^K \underset{\mathcal{H}}{\xrightarrow{\cong}} V_2^K$. Define $W = \{(v, Tv) \mid c \in V_1^K\} \subseteq V_1^K \times V_2^K = (V_1 \times V_2)^K$. Also define $U = \langle W \rangle_G$. Claim: $U^K = W$ (This is an exercise similar to the first part). But from here we get $U \neq V_1 \times 0, 0 \times V_2, 0, V_1 \times V_2$, and thus $V_1 \cong V_2$ (Schur up to semi-simplicity).

□

Back to $K = GL_d(\mathbb{Z}_p)$. We have a correspondencre

$$\{K - \text{spherical } G - \text{irr. rep.}\} \iff \{\mathcal{H} - \text{characters, } \chi \colon \mathcal{H} \to \mathbb{C}\} \iff \text{Hom}_{ring}(\mathcal{H}, \mathbb{C})$$

Let us understand the homomorphisms of the form $\mathbb{C}\left[x_1^{\pm}, ..., x_d^{\pm}\right]^{sym} \to \mathbb{C}$: they depend only on a set $\{z_1, ..., z_d\} \in (\mathbb{C}^*)^d$ by the choice $x_i \mapsto z_i$. Now for all such homomorphisms, $I(\overrightarrow{z})$ gives V^K, $\mathcal{H}$-rep with this hom. By Claim 3, the irr. rep. we started with is $\cong I(\overrightarrow{z})$.

Corollary 4. $L_2(X^0) = \oplus I(\overrightarrow{z})^K$. *We call* $z_1, ..., z_d$ *the Satake parameters if the irr. rep.*

$A_j = 1_{K\begin{pmatrix} p & & \\ & \ddots & \\ & & 1 \end{pmatrix}K}$ and $\mathrm{spec}A_j|_{X^0} = \left\{p^{\frac{j(d-j)}{2}}\sigma_j(\overrightarrow{z})\right\}$. If Adj is the adjacency matrix, then $Adj = \sum_{j=1}^{d-1} A_j$.

Theorem 5. *(Satake) When $G = PGL$, we have $I(\overrightarrow{z}) \leq_{weakly} L^2(G)$ iff $|z_i| \leq 1$.*

Remark 6. In this case $I(\overrightarrow{z}) \leq_{weakly} L^2(G) \iff I(\overrightarrow{z}) \in \bigcap_{\epsilon>0} L^{2+\epsilon}(G)$.

So X is Ramanujan on vertexes when $|z_i| = 1$ for all $\overrightarrow{z}$ in the sum $L_2(X^0) = \oplus I(\overrightarrow{z})^K$.

ADELE'S

$\hat{G} = unitary\ dual = \{cont.\ hom\colon\ G \to S^1\}$. E.g.

$$\hat{\mathbb{R}} = \{\xi_t\colon x \mapsto e^{2\pi i t x} \mid t \in \mathbb{R}\} \cong \mathbb{R}$$
$$\hat{\mathbb{Z}} = \{\xi_\alpha\colon n \to \alpha^n \mid \alpha \in S^1\} \cong S^1$$
$$\hat{S^1} = \{\alpha \mapsto \alpha^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$$
$$\widehat{\mathbb{Z}/n} \cong \mathbb{Z}/n$$

the main question is $\hat{\mathbb{Q}} =$? where $\mathbb{Q}$ is with the discrete topology. We can take all characters through $\mathbb{Q}_p$, meaning $\mathbb{Q} \to \mathbb{Q}_2 \to S^1$.

①

ארי ה עור 11/06

Adeles

$\hat{G} = \{ \chi : G \to S^1 \text{ cont.} \}$

$\hat{\mathbb{R}} \cong \mathbb{R}$ $\quad \chi(x) = e^{2\pi i x}$ $\quad \in \chi_\infty$

$\quad \chi_y(x) = \chi(yx) = e^{2\pi i y x}$ $\quad \in \chi_{\infty,y}$ $\quad \in$ names

For R top-ring, $\overset{\hat{R}^+}{\overset{\shortparallel}{\hat{R}}}$ is an R-mod by

$$(r\chi)(r') = \chi(rr')$$

A top-ring is self-dual

if $\hat{R}$ is a free R-mod of rank 1

i.e. $\exists \chi \in \hat{R}$ s.t. $\hat{R} = \{ \chi_y : x \mapsto \chi(yx) \mid y \in R \}$

We saw $\hat{\mathbb{Z}} = S^1$

$\hat{S^1} = \mathbb{Z}$

$\widehat{\mathbb{Z}/n} \cong \mathbb{Z}/n$ self-dual

Take the <u>discrete</u> topology on $\mathbb{Q}$: $\hat{\mathbb{Q}} = ?$

We can obtain characters:

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, hook] \arrow[dr, hook] \arrow[ddr, hook] & \mathbb{R} \arrow[r, "\chi_\infty"] & S^1 \\
& \mathbb{Q}_2 \arrow[ur, "\chi_2"] & \\
& \mathbb{Q}_3 \arrow[uur, "\chi_3"] & \\
& \vdots &
\end{tikzcd}$$

For example, a char.:

$\exists! \chi : \mathbb{Q} \to S^1$ s.t.

$$\chi\Big(\frac{a}{2^n} + \frac{c}{d}\Big) = e^{2\pi i \left(\frac{a}{2^n}\right)}$$
$$\overset{\uparrow}{(2 \nmid a, c, d)}$$

This χ doesn't come from $\hat{\mathbb{R}}$

②

Claim $\mathbb{Q}_p$ is self dual!

Take $\chi_p\Big(\overbrace{\sum_{n=n_0}^{\infty} a_n p^n}^{=:\alpha}\Big) =$ "$e^{2\pi i\alpha}$" $:=$

$e^{2\pi i\left(\sum_{n=n_0}^{-1} a_n p^n\right)}$ (so: $\chi_p(\mathbb{Z}_p) \equiv 1$)

note χ_p, as defined, factors through

$$\begin{tikzcd}\mathbb{Q}_p/\mathbb{Z}_p \equiv \mathbb{Z}(\tfrac{1}{p})/\mathbb{Z} \arrow[r, "e^{2\pi i\alpha}"] & S^1 \\ \arrow[u, phantom] \end{tikzcd}$$

$$\begin{tikzcd}\mathbb{Q}_p/\mathbb{Z}_p \arrow[rr, bend right, "\chi_p"'] & & S^1\end{tikzcd}$$

(continuous - clearly!)

Claim $\hat{\mathbb{Q}}_p = \{\chi_{p,\alpha} \mid \alpha \in \mathbb{Q}_p\}$

$\chi_{p,\alpha}(\beta) = \chi_p(\alpha\beta)$

Pf Say $\chi \in \hat{\mathbb{Q}}_p$. By continuity, for large k,

$\chi(p^k\mathbb{Z}_p) \subseteq B_{1/10}(1)$

but $p^k\mathbb{Z}_p$ is a gp $\Rightarrow$ $\chi(p^k\mathbb{Z}_p)$ is a gp.

$\Rightarrow \chi(p^k\mathbb{Z}_p) \equiv 1$

Take minimal such k.

Now, $\chi' = \chi \circ p^k$, $\chi'|_{\mathbb{Z}_p} \equiv 1$, $\chi'(\frac{1}{p}) \neq 1$

$((\omega_p)^p = 1)$ $\Rightarrow \chi'(\frac{1}{p}) = \omega_p^{a_1}$ for some $a_1 = 1 \to p-1$

$\chi'(\frac{1}{p^2}) = \omega_{p^2}^{a_2}$ $\quad a_2 \bmod p = a_1$

$\vdots$

$\chi'(\frac{1}{p^n}) = \omega_{p^n}^{a_n}$ $\quad a_n \bmod p^{n-1} = a_{n-1}$

So that $\alpha = \lim a_n \in \mathbb{Z}_p^\times$

check: $\chi_{p,p^k u} = \chi$

(**note** $\mathbb{Q}_p^\times = p^{\mathbb{Z}} \cdot \mathbb{Z}_p^\times$)

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_{p \le \infty} \xrightarrow{\chi_p} S^1$$

We can also multiply chars

$$\mathbb{Q} \xrightarrow[\chi_2 \cdot \chi_\infty]{} S^1$$

$$(\chi_2 \chi_\infty)\left(\frac{a}{2^n} + \frac{b}{c}\right) = e^{2\pi i\left(\frac{a}{2^n} + \frac{a}{2^n} + \frac{b}{c}\right)}$$

Denote: $P = \{\infty, 2, 3, 5, 7, \_\_\}$

So we have:

$$\hat{\mathbb{Q}} \supseteq \bigcup_{\substack{S \subseteq P \\ |S| < \infty}} \prod_{p \in S} \hat{\mathbb{Q}}_p$$

However, $\overset{\#}{\chi} := \prod_{p \le \infty} \chi_p \; ; \; \chi \in \hat{\mathbb{Q}}$

Well defined:

Decompose r

$$\chi(r) = \chi\left(\frac{a_2}{2^{m_2}} + \frac{a_3}{3^{m_3}} + \_\_ + \frac{a_{p_0}}{p_0^{m_{p_0}}}\right) =$$

$$\left(e^{2\pi i r}\right) \prod_{p \le p_0} e^{2\pi i\left(\frac{a_p}{p^{m_p}}\right)} =$$

$$e^{2\pi i\left(r + \frac{a_2}{p^{m_2}} + \_\_ + \frac{a_{p_0}}{p_0^{m_{p_0}}}\right)}$$

So we were wrong: Turns out: $\chi = \chi_{\infty, 2}$

However $\chi_{\infty, 7} \cdot \prod_{p < \infty} \chi_{p, p}$ doesn't come from finite products.

For any sequence $(a_\infty, a_2, a_3, \_\_) \in \prod_{p \le \infty} \mathbb{Q}_p$ we try to define

$$\chi_{\mathbb{Q}, \alpha} = \prod_{p \le \infty} \chi_{p, a_p}$$

Take $\chi_{\mathbb{Q},\alpha}(1) = \prod_{p\le\infty} \chi_{p,a_p}(1)$

We want to construct an example where, evaluating for every rational, we get a finite product, but of unbounded length.

Point: If $a_p \in \mathbb{Z}_p$ for almost all p, then $\chi_{\mathbb{Q},\alpha}$ is well-defined:

$\exists p_0 \ \forall p > p_0$ $a_p \in \mathbb{Z}_p \wedge r \in \mathbb{Z}_p$

$$\chi_{\mathbb{Q},\alpha}(r) = \prod \chi_{p,a_p}(r) = \prod \chi_p(a_p r) = \prod_{p\le p_0} \chi_p(a_p r)$$

$\chi_p|_{\mathbb{Z}_p} \equiv 1$

$\Rightarrow$ for any $\alpha \in \prod_{p\le\infty} \mathbb{Q}_p$ s.t. $\alpha_p \in \mathbb{Z}_p$ almost always, we got $\chi_{\mathbb{Q},\alpha} : \mathbb{Q} \to S^1$

These are the $\boxed{\text{adeles}}$:

$$\mathbb{A} \subset \prod_{p\le\infty} \hat{\mathbb{Q}}_p$$

[Exercise for any other α, $\exists q \in \mathbb{Q}$ s.t. the infinite prod. does not converge $\Rightarrow$ not well defined]

(Definition) Now $\forall \alpha \in \mathbb{A}^{\in \prod_{p\le\infty}\mathbb{Q}_p}$, $\chi_\alpha : \mathbb{Q} \to S^1$

$$\chi_\alpha(r) = \chi_{\infty,a_\infty}(r) \prod_{p<\infty} \chi_{p,\alpha_p}(r)$$
$$= \chi_\infty(-\alpha_\infty r) \cdot \prod_{p<\infty} \chi_p(\alpha_p r)$$

$\left(\vec{e_j} \text{ j-th coord} \right)$

Claim The following seq. is exact:

$$1 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{A} \longrightarrow \hat{\mathbb{Q}} \longrightarrow 1$$

(embedded diagonally in $\prod \mathbb{Q}_p$) $(\alpha_\infty = \alpha_2 = \ldots \in \mathbb{Q})$

If $\alpha \in \mathbb{Q}$ then

$$\chi_\alpha(r) = \chi_1(\alpha r) \text{, and } \chi_1|_{\mathbb{Q}} \equiv 1$$

note: If $r \in \mathbb{Q}$ then

$$\chi_1(r) = e^{2\pi i\left(-r + \overbrace{\frac{a_2}{2^{m_2}} + \_\_\_ + \frac{a_{p_0}}{p_0^{m_{p_0}}}}^{= r}\right)} = 1$$

[<u>Homework</u> If $\alpha \notin \mathbb{Q} \overset{\text{diag}}{\subseteq} A$ then $\chi_\alpha|_{\mathbb{Q}} \not\equiv 1$]

What about A? A is self-dual!

For $\alpha \in A$, define $\chi_\alpha \in \hat{A}$ by

$$\chi_\alpha(\beta) = \prod_{p \leq \infty} \chi_p(\overbrace{\alpha_p \beta_p}^{= (\alpha\beta)_p})$$

this is a finite prod., since $\alpha, \beta \in A$.

(We claim that all chars are of this type.)

It's clear that $A \underset{\text{injective}}{\hookrightarrow} \hat{A}$

This mapping is also <u>onto</u>: If $\chi \in \hat{A}$,

consider $\chi|_{\mathbb{Q}_p \hookrightarrow A} \in \hat{\mathbb{Q}_p}$

$\Rightarrow \chi|_{\mathbb{Q}_p} = \chi_{p, \alpha_p}$ (we already know they are of this form)

① Almost all α_p are integral (by continuity)

② $\chi = \chi_\alpha$

We got: $\hat{A} \cong A$.

What about $\widehat{\mathbb{Q}\backslash A}$?

$$\widehat{\mathbb{Q}\backslash A} = \{\chi \in \hat{A} \mid \chi|_{\mathbb{Q}} \equiv 1\}$$

$$= \{\chi_\alpha \in A \mid \chi_\alpha|_{\mathbb{Q}} \equiv 1\}$$

"Generalized Functions" Vol. 6 – Gelfand et. al check! ↗ $= \{ \chi_\alpha \in \hat{\mathbb{A}} \mid \alpha \in \mathbb{Q} \} = \hat{\mathbb{Q}}$

⇒ By Pontryagin duality:

$$\hat{\hat{\mathbb{Q}}} = \mathbb{Q} \backslash \mathbb{A}$$

1) $\mathbb{Q} \underset{\substack{\text{discrete}\\ \text{co-compact}}}{\leq} \mathbb{A}$ (by the diag. embedding)

Pf $[0,1) \times \prod_{p<\infty} \mathbb{Z}_p$ is a fund. domain for $\mathbb{Q}$ in $\mathbb{A}$

since $\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \ldots, \alpha_{p_m}, \underbrace{\alpha_{p_{m+1}}, \ldots}_{\overset{\uparrow}{\mathbb{Z}_p}})$

$+ (-\alpha_{p_m} \pmod{p_m})$

$\vdots$

$+ \ldots$

$+ (-\lfloor \alpha_\infty \rfloor)$

Normalize the Haar measure

$\mu([0,1) \times \prod \mathbb{Z}_p) = 1$

Topology on $\mathbb{A}$: Neighborhoods of 0:

$(-\varepsilon, \varepsilon) \times \prod_{p \neq \infty} p^{m_p} \cdot \mathbb{Z}_p$ s.t. almost all $m_p = 0$

NTS: If $q_n \to 0$ (in this top), then $q_n \equiv 0$ for large n.

Since:

$(\mathbb{Q} \cap (-\frac{1}{2}, \frac{1}{2})) \times \mathbb{Z}_p = 0$

①

שיעור של אורי - 18/06/2018

$\mathbb{A} = \{ (\alpha_p)_{p \leq \infty} \mid \alpha_p \in \mathbb{Q}_p,$ for almost all p $\alpha_p \in \mathbb{Z}_p \}$

- the Adeles.

Notation: $= \prod'_{p \leq \infty} \mathbb{Q}_p$ - The prime ′ denotes a "restricted product"

Topology on $\mathbb{A}$: $\overbrace{\mathbb{R} \times \prod_{p \leq \infty} \mathbb{Z}_p}^{=\mathcal{U}}$

has product top. and is declared open.

For $\alpha \in \mathbb{A}$ $\alpha\mathcal{U}$ is also declared open.

nbd of x (a point) α_2: $* * * ! ! ! ! , ! !$ "!" are determined

α_3: $* * ! ! !$

$\vdots$

from some point $\rightarrow$ $\mathbb{Z}_p$

$\mathbb{Z}_p$

$\vdots$

$\mathbb{Q} \xhookrightarrow{\text{diag.}} \mathbb{A}$ co-cpt lattice

since $\left((-\tfrac{1}{2}, \tfrac{1}{2}) \times \prod \mathbb{Z}_p \right) \cap \mathbb{Q} = \{0\}$ it is disc.

also, $\left([0,1) \times \prod \mathbb{Z}_p \right) + \mathbb{Q} = \mathbb{A}$

m: normalized Haar measure on $\mathbb{A}$ s.t.

$$m\left([0,1) \times \prod \mathbb{Z}_p \right) = m(\mathbb{Q} \backslash \mathbb{A}) = 1$$

<u>Weak Approximation</u>: $\forall$ finite $S \subseteq P$ ← primes $+\infty$

$\mathbb{Q}$ is dense in $\prod_{p \in S} \mathbb{Q}_p$

$\left((\Leftarrow) \quad \mathbb{Q} + \prod'_{p \notin S} \mathbb{Q}_p \text{ is dense in } \mathbb{A} \right)$

<u>Pf</u> If $\infty \notin S$: $\mathbb{Q}$ is dense in $\prod_{p \leq p_0} \mathbb{Q}_p$

we are trying to simultaneously solve

$x \equiv \alpha_2 (2^{n_2})$

$\vdots$

$x \equiv \alpha_p (p^{n_p})$

Multiply by appropriate integers to obtain integral equations; Use CRT; Divide by denominators to solve original equations.

<u>If $\infty \in S$</u>, or generally, there are norms $|\cdot|_p$, $p \in S$

we want $x \in \mathbb{Q}$ s.t. $\forall p \in S$ $|x - \alpha_p|_p < \varepsilon$

find $t_p \in \mathbb{Q}$, $p \in S$, s.t. $1 < |t_p|_p$,

$|t_p|_q < 1$ $S \ni q \neq p$

take $x = \prod_{p \in S} \frac{t_p^r \alpha_p}{1 + t_p^r}$ for r large enough

[<u>exercise</u>: fill in the details]

<u>Strong Approximation</u>: $\mathbb{Q}$ is dense in

$$\mathbb{A}^p \overset{\text{def}}{=} \prod'_{p \neq l \leq \infty} \mathbb{Q}_l \quad \left(\Leftrightarrow \ \mathbb{Q} + \mathbb{Q}_p \text{ dense in } \mathbb{A} \right)$$

<u>pf</u> for $p = \infty$ (otherwise - somewhat more technical proof)

we show that $\mathbb{Q}$ is ~~dense~~ dense in $\prod'_{p < \infty} \mathbb{Q}_p$

x must satisfy $x \equiv \alpha_2 \ (2^{n_2})$

$\vdots$

$x \equiv \alpha_p \ (p^{n_p})$

$\forall l > p$ $x \in \mathbb{Z}_l$

solve using CRT.

for $p \neq \infty$, read about it!

$\begin{bmatrix} a & \\ & 1/a \end{bmatrix} \in SL_2$

This is w/s approx. for $\mathbb{A}^+$.

What about $\mathbb{A}^\times$? $SL_d(\mathbb{A})$? $GL_d(\mathbb{A})$?

namely: Given G (a gp. over $\mathbb{Q}$), ask:

is $G(\mathbb{Q})$ dense in $G\left(\prod_{p \in S} \mathbb{Q}_p\right) = \prod_{p \in S} G(\mathbb{Q}_p)$

e.g.: weak Is $SL_d(\mathbb{Q})$ dense in $SL_d(\mathbb{A})$?

or in $SL_d(\mathbb{R}\times\mathbb{Q}_2\times\mathbb{Q}_3)$?

Strong: Is $G(\mathbb{Q})$ dense in $G(\mathbb{A}^p)$?

[illegible]

$$\left(\Leftrightarrow \underbrace{G(\mathbb{Q})\cdot G(\mathbb{Q}_p)}_{\neq} \text{ dense in } G(\mathbb{A})\right)$$

$$G(\mathbb{Q}\cdot\mathbb{Q}_p)$$

In terms of equations: we are looking for a matrix

$$A \equiv A_2 \pmod{2^{n_2}}$$

$$\vdots$$

$$A \equiv A_p \pmod{p^{n_p}}$$

$$\left[A \in G\left(\prod_{\ell>p}\mathbb{Z}_\ell\right)\right] \leftarrow \text{Strong}$$

(note: $G\left(\prod_{\ell>p}\mathbb{Z}_\ell\right) \neq G\left(\prod_{\ell>p}{}'\mathbb{Q}_p\right)\cap\prod\mathbb{Z}_\ell$)

pf of s. approx. for ~~SL_d~~: SL_2:

~~[illegible]~~ $G := \overline{SL_2(\mathbb{Q})\cdot SL_2(\mathbb{Q}_p)} \overset{?}{=} SL_2(\mathbb{A})$

$$G \supseteq \overline{\left\{\begin{bmatrix}1 & \mathbb{Q}\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & \mathbb{Q}_p\\ 0 & 1\end{bmatrix}\right\}} = \overline{\begin{bmatrix}1 & \mathbb{Q}+\mathbb{Q}_p\\ 0 & 1\end{bmatrix}} = \left\{\begin{bmatrix}1 & \mathbb{A}\\ 0 & 1\end{bmatrix}\right\}$$

$$\supseteq \left\{\begin{bmatrix}1 & 0\\ \mathbb{A} & 1\end{bmatrix}\right\}$$

$$\forall \ell \quad G \supseteq \left\{\begin{bmatrix}1 & 0\\ \mathbb{Q}_\ell & 1\end{bmatrix}, \begin{bmatrix}1 & \mathbb{Q}_\ell\\ 0 & 1\end{bmatrix}\right\}$$

$$\Rightarrow G \supseteq SL_2(\mathbb{Q}_\ell)$$

$$\Rightarrow G \supseteq \cancel{\prod_{\ell>p_0} SL_2\left(\prod \mathbb{Q}_\ell\right)}\ \overline{\bigcup_{p_0\in\mathbb{N}}\prod_{\ell\leq p_0} SL_2(\mathbb{Q}_\ell)} = SL_2(\mathbb{A})$$

As for $GL_d(\mathbb{A})$ – No strong approx!

even $GL_1 = \mathbb{A}^\times$ has no s.a.

(*)

However, $G := \overline{GL_d(\mathbb{Q})\, GL_d(\mathbb{Q}_p)\, \underbrace{\overset{Ⓤ}{GL_d\Big(\mathbb{R} \times \prod_{\ell \neq \infty} \times \mathbb{Z}_\ell\Big)}}_{}} = GL_d(\mathbb{A})$

$\nearrow$ This open nbd. is big enough

pf: ~~$GL_d(\mathbb{Q})\, GL_d(\mathbb{Q}_p) \supseteq SL$~~ $G \supseteq SL_d(\mathbb{A})$

by S.A. for SL_d

Also, $\forall \alpha \in \mathbb{A}^* \quad \exists A \in G \quad \det(A) = \alpha$

since: $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \underbrace{\mathbb{Q}^\times \mathbb{Q}_p^\times \cdot \mathbb{R}^\times \prod_\ell \mathbb{Z}_\ell^\times}_{= \mathbb{A}^\times} \end{bmatrix}$

$\nearrow$ we saw a similar argument

note: $\overline{SU_d(\mathbb{Q})\, SU_d(\mathbb{R})} \neq SL_d(\mathbb{A})$

since $SL(\mathbb{Q}) \underset{\substack{\cancel{\text{discrete}} \\ \text{discrete}}}{\subseteq} SL_d(\mathbb{A})$, $SU_d(\mathbb{R})$ cpt.

Now, $G = SU_d$, SL_d have S.A. - i.e. If $\underset{p \le \infty}{G(\mathbb{Q}_p)}$ is non-cpt.

then $\overline{G(\mathbb{Q})\, G(\mathbb{Q}_p)} = G(\mathbb{A})$

S. A.

easy: linear: $\mathbb{A}$, SL_d

hard: quad: SU, SO, Spin $\leftarrow$ Kneser, Platonov

false: cubic: ell. curves

$$\mathbb{Q}^\times_{>0} = \mathbb{Z}^\times \backslash \mathbb{Q}^\times = \prod_{l<\infty} \mathbb{Z}_l^\times \backslash \mathbb{A}^{\infty} =$$

2nd iso. thm

since $\mathbb{Q}^\times \prod_{l<\infty} \mathbb{Z}_l^\times = (\mathbb{A}^\infty)^\times$, $\mathbb{Q}^\times \cap \prod \mathbb{Z}_l^\times = \mathbb{Z}^\times$

(take $d=1, p=\infty$, in $(*)$)

(we saw: $\mathbb{Q}^\times \mathbb{R}^\times \prod \mathbb{Z}_l^\times = \mathbb{A}^\times$)

$\mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times \langle p \rangle$

$$\prod \mathbb{Z}_l^\times \backslash \prod{}' \mathbb{Q}_p^\times = \prod{}' \left(\mathbb{Z}_p^\times \backslash \mathbb{Q}_p^\times \right) = \prod_{p<\infty}{}' \mathbb{Z}$$

$\prod'$ means: trivial after some p_0

This is in a sense trivial! (UFD).

$$\underbrace{PGL_d(\mathbb{Z}) \backslash PGL_d(\mathbb{Q})} = PGL_d(\prod \mathbb{Z}_p) \backslash PGL_d(\mathbb{A}^\infty) = \prod PGL_d(\mathbb{Z}_p) \backslash \prod{}' PGL_d(\mathbb{Q}_p)$$

rational lattices
homostety classes of

$$= \prod{}' PGL_d(\mathbb{Z}_p) \backslash PGL_d(\mathbb{Q}_p) = \prod_{p<\infty}{}' (\mathcal{B}_p^d)^0$$

trivial cosets after some $p_0 < p$

restriction = from some point, take the root, $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, of the tree

Recall: שיעור לג אורי - 25/06/2018

$G = PGL_d$

$$G(\mathbb{Z})\backslash G(\mathbb{Q}) = \text{Rational lattices up to scaling}$$

$$\wr\wr\ \circledast$$

$$G(\overbrace{\prod_{p<\infty}\mathbb{Z}_p}^{\hat{\mathbb{Z}}})\backslash G(\mathbb{A}^\infty) \qquad \left[\mathbb{A}^\infty = \prod_{p<\infty}{}' \mathbb{Q}_p\right]$$

$\circledast$ Follows from the 2nd iso. thm.:

$G(\mathbb{Q})G(\hat{\mathbb{Z}}) = G(\mathbb{A}^\infty)$ - Due to strong approx. and det. computation.

$$\cong \prod{}' \underbrace{G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)}_{=(\mathcal{B}_d^p)^{(0)}}$$

Now, $G(\mathbb{Q})G(\mathbb{R})\overbrace{G(\hat{\mathbb{Z}})}^{\text{open}} \subseteq G(\mathbb{A})$ (By strong approx.)

Equivalently, $G(\mathbb{Q})G(\hat{\mathbb{Z}}) = G(\mathbb{A}^\infty)$

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}}) = G(\mathbb{Q})\backslash\left(G(\mathbb{A}^\infty)/G(\hat{\mathbb{Z}}) \times G(\mathbb{R})\right)$$

$$= G(\mathbb{Q})\backslash\left[G(\mathbb{Q})/G(\mathbb{Z}) \times G(\mathbb{R})\right] \overset{\text{Claim}}{=} G(\mathbb{Z})\backslash G(\mathbb{R})$$

Generally: $\overset{\text{diagonal action}}{G\backslash}\left[G/H \times \overset{G\text{-set}}{X}\right] \cong H\backslash X$

[If $I \trianglelefteq R \circlearrowleft \overset{(R\text{-mod})}{M}$ then $R/I \otimes_R M = IM\backslash M$

(Known thm., analogous to our claim.)

Since: $xr \otimes y \mapsto x \otimes ry$

$\Leftrightarrow (gx.y) \mapsto (x, g^{-1}y)$]

E.G. $G = GL_2(\mathbb{R})$

$$G(\mathbb{Z}) \backslash \overbrace{G(\mathbb{R}) / SO(2)}^{\mathfrak{h} \in \text{ hyperbolic Plane}} \quad - \quad \text{Modular curve}$$

A Hecke operators (τ is a lattice)

$$(T_p f)(\tau) = f(p\tau) + \sum_{j=0}^{p-1} f\left(\frac{\tau+j}{p}\right)$$

for $f : {}_{SL_2(\mathbb{Z})}\backslash \mathfrak{h} \to \mathbb{C}$

Or even for: $T_p : L^2\left({}_{GL_2(\mathbb{Z})}\backslash GL_2(\mathbb{R})\right) \circlearrowright$

$$\cong$$

$$\mathcal{H}^{GL_2(\hat{\mathbb{Z}})}_{GL_2(\mathbb{A})} \overset{\text{Acts}}{\circlearrowright} L^2\left({}_{SL_2(\mathbb{Q})}\backslash SL_2(\mathbb{A}) / SL_2(\hat{\mathbb{Z}})\right)$$

<u>Note</u> $\forall p$ prime $\mathcal{H}^{\overbrace{GL_2(\mathbb{Z}_p)}^{=K_p}}_{GL_2(\mathbb{Q}_p)} \subseteq \mathcal{H}^{\overbrace{GL_2(\hat{\mathbb{Z}})}^{=\hat{K}}}_{GL_2(\mathbb{A})}$

$\in$

$$T_p \rightarrow \mathbb{1}_{\hat{K}(1,1,\ldots,1,\left[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right],1,1,\ldots)\hat{K}}$$

$\nearrow GL_2(\mathbb{Q}_p)$

$$= \mathbb{1}_{(1,\ldots,1,\left[\begin{smallmatrix}1&\\&p\end{smallmatrix}\right],1\ldots1)\hat{K}} + \sum_{j=0}^{p-1} \mathbb{1}_{(1,\ldots,\left[\begin{smallmatrix}p&j\\0&1\end{smallmatrix}\right],1,\ldots)\hat{K}}$$

Note that $T_p \in \mathcal{H}^{GL_2(\hat{\mathbb{Z}})}_{GL_2(\mathbb{A})}$

In previous notation:

$$(T_p f)(\overset{GL_2(\mathbb{R})}{\overset{\in}{g}}) = (T_p f)(\overset{\subset GL_2(\mathbb{A})}{(g,1,1,\ldots)})$$

$$= f((g,\ldots,1\left[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right],1,\ldots)) + \sum_j f((g,\ldots\left[\begin{smallmatrix}p&j\\&1\end{smallmatrix}\right]1,\ldots))$$

(We want to use $GL_2(\mathbb{Q})\ GL_2(\hat{\mathbb{Z}})$ to get back to $(SL_2(\mathbb{R}) \times 1,1,\ldots)$)

$$= f(\left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right]g, \left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right], \ldots, \underset{}{\overset{p\text{-th place}}{I}}, \left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right], \ldots)$$

using $\overrightarrow{GL_2(\mathbb{Q})}$

$$= f(\left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right]g, I, I, \ldots) + f(\left[\begin{smallmatrix}1/p&-j/p\\0&1\end{smallmatrix}\right]g, 1,1,\ldots)$$

using $\overrightarrow{GL_2(\mathbb{Z})}$

$$= f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1/p \end{bmatrix} g\right) + \sum_j f\left(\begin{bmatrix} 1/p & -j/p \\ 0 & 1 \end{bmatrix} g\right)$$

And this is (check) $= T_p$ - the classical op. defined by Hecke.

Take $G = PSL_d$ or $G = PSU_d$

$$1 = G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}}\mathbb{R}) = \overbrace{\phantom{G(\mathbb{Q}_p)}}^{\mathcal{B}(G(\mathbb{Q}_p))^0}$$

$$= G(\mathbb{Q}) \backslash \left[G(\mathbb{A}^{\infty,p}) / G\left(\prod_{\ell \neq p} \mathbb{Z}_\ell\right) \times G(\mathbb{R})/G(\mathbb{R}) \times G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \right]$$

(we want! ① $G(\mathbb{Q})G\left(\prod_{\ell \neq p} \mathbb{Z}_\ell\right) = G(\mathbb{A}^{\infty,p})$

and this holds, by S.A.: $G(\mathbb{Q}_p)$ is non-cpt.

② $G(\mathbb{Q})\, G\left(\prod_{\ell \neq p} \mathbb{Z}_\ell\right) = G(\mathbb{Z}[\tfrac{1}{p}])$ - clearly)

$$= G(\mathbb{Q}) \backslash \left[G(\mathbb{Q}) / G(\mathbb{Z}[\tfrac{1}{p}]) \times \mathcal{B}_p \right] = G(\mathbb{Z}[\tfrac{1}{p}]) \backslash \mathcal{B}_p$$

Jacobi's Thm

$a^2+b^2+c^2+d^2 = p \iff PGU_2^+(\mathbb{Z}[\tfrac{1}{p}])(2)$ ← we saw

$a^2+b^2 = 1 \iff |U_1(\mathbb{Z})| = ? = 4$

easy to show, but how to generalize the method?

<u>Siegel</u>! Solve in $\mathbb{R}$ and for all $\mathbb{Z}_p$

"p-adic density": Density of $U_1(\mathbb{Z}_p)$ in $M_1(\mathbb{Z}_p[i])$

[for $p = \infty$, take $\mathbb{Z}_p = \mathbb{R}$]

How to ~~compute~~ define density! $\lim_{r\to\infty} \frac{|U_1(\mathbb{Z}/p^r)|}{p^r} := \mu_p$

for $p \equiv 1 \pmod 4$:

$$U_1(\mathbb{Z}_p) \cong \overset{\mathbb{Z}_p^*}{\overset{\shortparallel}{GL_1(\mathbb{Z}_p)}} \subseteq \overset{\mathbb{Z}_p}{\overset{\shortparallel}{M_1(\mathbb{Z}_p)}} \qquad m_p = \frac{p+1}{p}$$

for $p \equiv 3 \pmod 4$, in $\mathbb{F}_p$, take $r=1$!

the only quad. extension

$$\begin{tikzcd} U_1(\mathbb{F}_p) \arrow[r, hook] \arrow[dr, equal] & M_1(\mathbb{F}_p[i]) = M_1(\mathbb{F}_{p^2}) \\ & \{\alpha : \underset{\text{norm}}{N(\alpha)} \overset{\circledast}{=} 1\} \arrow[r, hook] & \mathbb{F}_{p^2} \end{tikzcd} \qquad m_p = \frac{p+1}{p}$$

$\circledast$ $\alpha\bar{\alpha} = \alpha^{p+1}$

for $p=\infty$: Definition: $\lim\limits_{\varepsilon\to 0} \dfrac{|\{A\in M_1(\mathbb{C}) : |A^*A-1|<\varepsilon\}|}{2\varepsilon} =$

$$\lim_{\varepsilon\to 0} \frac{\pi(1+\varepsilon)-\pi(1-\varepsilon)}{2\varepsilon} = \pi$$

$$m_\infty \prod_{p\neq 2} m_p = \pi \cdot \prod_{p\equiv 1(4)} \left(1-\frac{1}{p}\right) \prod_{p\equiv 3(4)} \left(1+\frac{1}{p}\right)$$

$$= \pi \cdot \prod_{p\neq 2} \left(1 - \frac{i^{p-1}}{p}\right) = \pi \cdot \prod_{p\neq 2} \left(\frac{1}{1-\frac{i^{p-1}}{p}}\right)^{-1}$$

Euler sum/product

$$\overset{\Downarrow}{=} \pi \left[\sum_{\text{odd } n} \frac{i^{n-1}}{n}\right]^{-1} = 4$$

We omitted $m_2 = 2$

e.g. $G = U_d$

Idea By Tamagawa: $G(\mathbb{A})$ has a unique rational Haar measure.

[Rational ≡ Expessible in $\mathbb{Q}$ (entries)]

Take some Haar measure on $G(\mathbb{Q})$, μ

by tensoring with $\mathbb{Q}_p$ or $\mathbb{R}$, get μ_p on $G(\mathbb{Q}_p)$ $\forall p$

and μ_∞ on $G(\mathbb{R})$

Take $\mu_{/A} = \prod_{p\leq\infty} \mu_p$

If we change μ to $q\mu$ for $q \in \mathbb{Q}$,

$\mu_{/A}$ does not change:

$$\prod_{p\leq\infty} |q|_p = 1$$

p-adic norms

(unimodular)

This is called the Tamagawa measure: τ

Consider $\tau\left(G(\mathbb{Q})\backslash G(A)\right)$

"The Tamagawa number of G"

(True for $d=1,2,3,4$)

$$\tau\left(U_1(\mathbb{Q})\backslash U_1(A)\right) = \tau\left(U_1(\mathbb{Z})\backslash U_1(\hat{\mathbb{Z}}\mathbb{R})\right)$$

$$= \frac{\tau(\overbrace{U_1(\hat{\mathbb{Z}})\mathbb{R}}^{``\prod_{p\leq\infty} U_1(\mathbb{Z}_p)})}{|U_1(\mathbb{Z})|} = \frac{\prod_{p\leq\infty}\mu_p}{|U_1(\mathbb{Z})|} = \frac{8}{|U_1(\mathbb{Z})|} = 2$$

So, if we knew the Tam. num for U_d,

$\tau(U_d) = 2$ ...

and: $|U_d(\mathbb{Z})| = d! \cdot 4^d$

$$|\{A^*A = I : A \in M_d\,\mathbb{Z}[i]\}|$$