

Lecture 2

26/3/18

from last week $\left\{ \begin{array}{l} G = \mathrm{PGL}_d(\mathbb{Q}_p), \quad K = \mathrm{PGL}_d(\mathbb{Z}_p). \\ (\square) k = (\square) k \text{ if } A \in \mathrm{GL}_d(\mathbb{Z}_p) \text{ is Triang.} \end{array} \right.$

$(\square) k = (\square) k$ if $A \in \mathrm{GL}_d(\mathbb{Z}_p)$ is Triang. Then the diag. elements are units (\mathbb{Z}_p^\times) .

Lattices

$\mathbb{Z}^d \leq \mathbb{R}^d \rightarrow \mathbb{Z}_p^d \leq \mathbb{Q}_p^d$: submodule over inf. ring inside vector space.

$\rightarrow \mathrm{SL}_d(\mathbb{Z}) \leq \mathrm{SL}_d(\mathbb{R})$: discrete cusp/cofinite sgp. of top. gp.
we will use this notion.

$\mathbb{Z}_p^d \leq \mathbb{Q}_p^d$ not discrete, not cusp. $[\mathbb{Q}_p^d / \mathbb{Z}_p^d \cong \mathbb{Z}[\frac{1}{p}] / \mathbb{Z}$, discrete]

$\mathbb{Z}_p^d \leq \mathbb{Q}_p^d$ similar.

$\mathbb{Z} \leq \mathbb{Q}_p$ neither. cusp lattice!

$\mathbb{Z}[\frac{1}{p}]^d \leq \mathbb{Q}_p^d$. $\mathbb{Q}_p^d / \mathbb{Z}[\frac{1}{p}]^d \cong \mathbb{Z}_p^d / \mathbb{Z}[\frac{1}{p}]^d$ cusp.
 $\{ \pm p^k \mid k \in \mathbb{Z} \}$

$\mathbb{Z}[\frac{1}{p}] \leq \mathbb{Q}_p$ not discrete ($p^k \rightarrow 0$).

$\mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}]) \leq \mathrm{PGL}_d(\mathbb{Q}_p)$. cocompact: we saw $\mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}])$ acts trans. on $(\mathcal{B}_p^d)^0 = \frac{\mathrm{PGL}_d(\mathbb{Q}_p)}{\mathrm{PGL}_d(\mathbb{Z}_p)}$,

so $\mathrm{PGL}_d(\mathbb{Z}_p) \mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PGL}_d(\mathbb{Q}_p)$.

$\rightarrow \mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}]) \cong \mathrm{PGL}_d(\mathbb{Z}_p) \cap \mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}])$ cusp.

not discrete: $(1 \ p^k) \rightarrow (1, 1)$.

$\Gamma(n) := \mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}](n)) = \underbrace{\{A \in \mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}]) \mid A \equiv I \pmod{n}\}}_{(p \nmid n)} = \ker(\mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}]) \rightarrow \mathrm{PGL}_d(\mathbb{Z}/n\mathbb{Z}))$
↓
Primitive p-matrices in $M_d(\mathbb{Z})$

$\Gamma(n) \leq \mathrm{PGL}_d(\mathbb{Q}_p)$. not discrete: $(1 \ p^k) \rightarrow (1, 1)$.

cusp: finite index in $\Gamma(n) = \mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}])$. This bounds # orbits in \mathcal{B}^0 .

another argument: $\Gamma(n) \cap K = \mathrm{PGL}_d(\mathbb{Z})(n) = \ker(\mathrm{PGL}_d(\mathbb{Z}) \rightarrow \mathrm{PGL}_d(\mathbb{Z}/n\mathbb{Z}))$ is finite,
 $\mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}]) = \text{const.} \quad \text{stab}(v_0) = \mathrm{PGL}_d(\mathbb{Z}_p)$

but discrete \cap cusp = finite.

$$PU_d(\mathbb{Z}[\frac{1}{p}]) \cap PGL_d(\mathbb{Q}_p) \cong PGL_d(\mathbb{Q}_p) \text{ a discrete cocompact lattice.}$$

} Previous

$$U_d(R) = \{A \in M_d(R[i]) \mid A^T A = I\}$$

\uparrow
non. ring
 $(A^*)_{i,j} = \overline{A_{ji}}$
 $a_{bi} = a_{-bi}$

The Thing: $\mathbb{Z}[\frac{1}{p}] \leq R \times \mathbb{Q}_p$.

Proof: discrete; we'll show 0 is not an acc. point.

$$\alpha_i \rightarrow 0 \iff \alpha_i \rightarrow 0 \text{ in } R \text{ & } \alpha_i \rightarrow 0 \text{ in } \mathbb{Q}_p.$$

$\alpha \in \mathbb{Z}[\frac{1}{p}] \rightarrow$ if $\alpha_i \rightarrow 0$ then $\alpha_i = \frac{m_i}{p^{l_i}}$, $l_i \rightarrow \infty$ and then $\alpha_i \rightarrow 0$ in \mathbb{Q}_p

Likewise: $\mathbb{Z}[\frac{1}{pq}] \overset{\text{disc.}}{\hookrightarrow} R \times \mathbb{Q}_p \times \mathbb{Q}_{q_p}$.

cpt.: $\mathbb{Z}[\frac{1}{p}] = \underbrace{[0,1] \times \mathbb{Z}_p}_{\text{cpt.}} = R \times \mathbb{Q}_p$. why? If $\alpha, \beta \in R$, sub. $\beta \bmod 1$ and get (r, \mathbb{Z}_p) .

Now sub. $[0,1]$ and get $(r - [r], \mathbb{Z}_p)$.

Likewise: $PGL_d(\mathbb{Z}[\frac{1}{p}]) \leq \underbrace{PGL_d(R \times \mathbb{Q}_p)}_{\text{non-cpt lattice}} \times PGL_d(\mathbb{Q}_p)$

Fact: $\exists Y \subseteq PGL_d(R)$ s.t. $PGL_d(\mathbb{Z})Y = PGL_d(R)$ [$PGL_d(\mathbb{Z})$ is a lattice in $PGL_d(R)$].
of finite volume

Example: $d=2$, $PGL_2(R) \cong \mathbb{H} \times PO(2)$

$$Y = \begin{array}{c} \text{Diagram of } \mathbb{H} \\ \hline \end{array} \times PO(2).$$

Now, take $PGL_d(\mathbb{Z}[\frac{1}{p}]) \subseteq \underbrace{PGL_d(R) \times PGL_d(\mathbb{Q}_p)}_{\text{lattice}}$. It is discrete (by same argument).

Cofinite: $PGL_d(\mathbb{Z}[\frac{1}{p}]) \cdot (Y \times PGL_d(\mathbb{Z}_p)) = \mathbb{Z}[\frac{1}{p}]$

If $(A, B) \in PGL_d(R) \times PGL_d(\mathbb{Q}_p)$, since $PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts trans. on $PGL_d(\mathbb{Q}_p)/PGL_d(\mathbb{Z}_p)$ we can move (A, B) to $(C, PGL_d(\mathbb{Z}_p))$. Now, $\exists D \in PGL_d(\mathbb{Z})$ s.t. $DC \in Y$ and $DPGL_d(\mathbb{Z}) \subseteq PGL_d(\mathbb{Z}_p)$

\rightarrow Now D is in $Y \times PGL_d(\mathbb{Z}_p)$

- Loc. cpl.
- ① Any topological gp. G has a unique ^{regular} measure μ satisfying $\mu(Ag) = \mu(A)$ $\forall g \in G$ (Haar measure).
- ② $\mu(Ag) = \Delta_G(g) \mu(A)$, Δ_G - the modular function of G .
- ③ $G \curvearrowright X$. Does X have a G -inv measure?
trans.
closed

$X \cong \Gamma^G$ for some sgp $\Gamma \leq G$. Is there G -inv measure on Γ^G ?
 $\mu(Ag) = \mu(A)$, $A \in \Gamma^G$
"stab _{Γ} ($x_0 \in X$)

Answer: if and only if $\Delta_G|_{\Gamma} \equiv \Delta_{\Gamma}$.

$\Delta_G \equiv 1$ for abelian/cpt/discrete/ $GL_n(\mathbb{R}, \mathbb{Q}_p, \dots)$.

$$\mu(\gamma) = \mu(g\gamma) = \Delta(g)\mu(\gamma)$$

Finally, if $\Gamma \leq GL_d(\mathbb{R}, \mathbb{Q}_p, \dots)$, then by ^{discrete} There is a unique G -inv measure μ on Γ^G . Say Γ is a lattice if $\mu(\Gamma^G) < \infty$.