

$\mathbb{Q}_{10} + \times$

we showed it is a ring

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}_{10}$$

$\subseteq \mathbb{Q}$ \rightsquigarrow not comparable.
 $\subseteq \mathbb{R}$

Showed $\mathbb{Z} \not\subseteq \mathbb{Q}_{10}$

Claim: If $\alpha \in \mathbb{Q}_{10}$ and $\text{rnd}(\alpha) \in \{1, 3, 7, 9\} \subseteq (\mathbb{Z}/10)^*$ then α is invertible.

Proof: By multiplication by 10^k we can assume $\alpha = \dots \overset{\text{right most digit}}{\ldots} a_1 a_0$ $a_0 \neq 0$

$$\begin{array}{r} \dots \overset{\text{right most digit}}{\ldots} a_1 a_0 \\ \times 10 \\ \hline \dots \overset{\text{right most digit}}{\ldots} a_1 a_0 \end{array}$$

PA

~~also $2, 4, 5, 6, 8 \in \mathbb{Q}_{10}^*$~~

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Spoiler: \mathbb{Q}_{10} is not a field.

Claim: If p is prime, then \mathbb{Q}_p is a field.

Proof: $r_n(x) \in (\mathbb{Z}/p)^* = (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$

p -adic integers $P \in \mathbb{N}$

$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : \text{with } n \text{ digits } \cancel{\text{after}} \text{ to the right of the decimal point} \right\}$

\mathbb{Z}_p is a ring subring of \mathbb{Q}_{10} which is compact and uncountable

||
A seq of numbers in \mathbb{Q}_p converges if every digit eventually stabilizes (to an admissible limit) \times

~~\mathbb{Q}_{10} is not compact~~

↓
not exactly true

\mathbb{Z}_{10} has a lot of ~~most~~ of \mathbb{Q}_*

$$\left\{ \frac{a}{b} : (b_{10}) = 1 \right\} \subseteq \mathbb{Z}$$

also, every $x \in \mathbb{Z}_{10}$ with $x_0 \in (\mathbb{Z}_{10})^*$ is in \mathbb{Z}_{10}^*

→ If p is prime $\mathbb{Z}_p^* = \{x \in \mathbb{Z}_p : x \neq 0\}$

We get a decomposition of \mathbb{Q}_p^\times

every $\alpha \in \mathbb{Q}_p^\times$ can be written uniquely as $p^u \prod_{\substack{u \in \mathbb{Z} \\ u \in \mathbb{Z}_p^\times}} \zeta_p^u$.

$$\text{In particular } \frac{\text{GL}_1(\mathbb{Q}_p)}{\text{GL}_1(\mathbb{Z}_p)} = \mathbb{Q}_p^\times / \mathbb{Z}_p^\times \cong \mathbb{Z}$$

$$\mathbb{Q}_p^\times = \langle p \rangle \times \mathbb{Z}_p^\times$$

Hensel's Lemma

For $f(x) \in \mathbb{Z}_{\geq 1}$, when does f have a solution in $\mathbb{Q}_{10} \cap \mathbb{Z}_{10}$?

Observe: if f has a sol in \mathbb{Z} (in \mathbb{Q}) then it has a sol in \mathbb{Z}_m (in \mathbb{Q}_m) for every m (including $\begin{matrix} m=\infty \Rightarrow \mathbb{Q} \\ \mathbb{Q}_{\infty} = \mathbb{R} \\ \mathbb{Z}_{\infty} = \mathbb{Z} \end{matrix}$)

Since $\mathbb{Z} \hookrightarrow \mathbb{Z}_m$, $\mathbb{Q} \hookrightarrow \mathbb{Q}_m$ as rings.

Deep question: other direction.

Claim: TFAE

(1) $f(x)$ has a solution in \mathbb{Z}_{10}

(2) $\exists a_1 \in \mathbb{Z}$ s.t. $f(a_1) \equiv 0 \pmod{10^k}$ &

(3) " " " " " and $a_k \equiv a_{k-1} \pmod{10^{k-1}}$

e.g. $\{a_k\} = \{7, 67, 667, 6667, \dots\}$

for $f(x) = 3x^k + 1 \quad f(a_k) = 0 \pmod{10^k}$

indeed $f_{(n)}$ has a root in $\mathbb{Z}_{10} \dots 667$

Proof:

3 \Rightarrow 2 is obvious

1 \Rightarrow 3 there is a ring hom $\mathbb{Z}_{10} \xrightarrow{\text{mod } 10^k} \mathbb{Z}_{10^k}$

3 \Rightarrow 1 define α is the obvious way as the limit
of a_k . ($\alpha \pmod{10^k} = a_k$). Then $f(\alpha) \pmod{10^k} = f(a_k) = 0 \pmod{10^k}$

$\Rightarrow f(\alpha) = 0$.

2 \Rightarrow diagonal argument.



Hensel's lemma

Let $f \in \mathbb{Z}[x]$. If $\exists a_0 \in \{0, \dots, 9\}$ s.t. $f(a_0) \equiv 0 \pmod{10}$
and $f'(a_0) \in (\mathbb{Z}/10)^*$, then f has a root in \mathbb{Z}_{10} .

$$\left(\sum_{i=0}^n a_i x^i \right)' = \sum_{i=0}^n i a_i x^{i-1}$$

examples:

$$f(x) = mx - 1 \quad m \in \mathbb{Z} \quad (m, 10) = 1$$

take $a_0 = \text{inv of } m \text{ in } \mathbb{Z}_{10}$

$$f(a_0) = m \cdot a_0 - 1 \equiv 0 \pmod{10}$$

$$f'(a_0) = m \in (\mathbb{Z}_{10})^*$$

$\Rightarrow \exists x \in \mathbb{Z}_{10} \text{ s.t. } f(x) = 0$. This is of course l.

$$f(x) = x^2 + x + 8$$

$$a_0 = 1 \quad f(a_0) = 0 \pmod{10}$$

$$f'(a_0) = 2 \cdot a_0 + 1 = 3 \in (\mathbb{Z}_{10})^*$$

$x^2 + x + 8$ has a solution in \mathbb{Z}_{10} , but not in \mathbb{R} . \Rightarrow

We cannot embed \mathbb{Q}_{10} in \mathbb{R} .

Since the sol of

$$x^2 + x + 8 \text{ are } \frac{-1 \pm \sqrt{-31}}{2} \Rightarrow \sqrt{-31} \in \mathbb{Q}_{10}$$

$$f(x) = x^2 + 21$$

$$f'(x) = 2x \notin (\mathbb{Z}_{10})^*$$

\Rightarrow Hensel's lemma is not iff.

Prat: We will construct $\{a_k\}_{k=1}^{\infty}$, s.t. $f(a_k) = 0 \pmod{10^{k+1}}$
and $a_k \equiv a_{k-1} \pmod{10^k}$ by induction. The lemma
we proved before, then proves the existence
of a solution.

Taylor $f(x+h) = f(x) + f'(x)h + \left(\frac{f''(x)}{2!}h^2 + \dots + \left(\frac{f^{(k)}(x)}{k!}h^k\right)\right)$, where
 $k = \deg f$.

We have a_{k-1} , s.t. $f(a_{k-1}) = 0 \pmod{10^k}$.

We try to construct $a_k = a_{k-1} + d \cdot 10^k$ s.t.

$$f(a_k) = 0 \pmod{10^{k+1}}$$

$$(10^{k+1}) \stackrel{?}{=} f(a_k) = f(a_{k-1} + d \cdot 10^k)$$

$$= \sum_{j=0}^{\infty} f^{(j)}(a_{k-1}) \cdot (d \cdot 10^k)^j \stackrel{?}{=} f(a_{k-1}) + f'(a_{k-1})(d \cdot 10^k)$$

~~$\approx f(a_{k-1}) + d \cdot 10^k$ since $f'(a_{k-1}) \equiv 0 \pmod{10^k}$~~
we got

We need $f'(a_{k-1})d^{10^k} \equiv -f(a_{k-1}) \pmod{10^{k+1}}$

By assumption $f(a_{k-1}) \equiv 0 \pmod{10^k} \Rightarrow$ we need

$$f'(a_{k-1})d \equiv -\frac{f(a_{k-1})}{10^k} \pmod{10}$$

Since $f'(a_{k-1}) \in (\mathbb{Z}/10)^*$ we can take

$$d = -\frac{1}{f'(a_0)} \cdot \frac{f(a_{k-1})}{10^k} \pmod{10}$$

$$(f'(a_{k-1}) = f'(a_0) \pmod{10})$$



$$\sqrt{m} \in \mathbb{R} \quad m \geq 10$$

$$\begin{aligned} m &\in \mathbb{Z} \\ m &\equiv 1 \pmod{80} \\ m &\equiv 1, 4 \pmod{5} \end{aligned}$$

For p prime $p \neq 2$. $\sqrt{m} \in \mathbb{Z}_p \iff \sqrt{m} \in \mathbb{F}_p$.

$f(x) = x^2 - m$ take $a_0 = \sqrt{m} \in \mathbb{F}_p$ then $f'(a_0) = 2a_0 \neq 0 \pmod{p}$

~~use~~ \Rightarrow Hensel's Lemma.

\mathbb{Z}_{10} is not a field

\mathbb{Z}_{10} has 0-divisors.

$$f(x) = x^2 - x$$

$$a_0 \in \{0, 1, 5, 6\} \Rightarrow f(a_0) \equiv 0 \pmod{10}$$

$$f'(a_0) = 2a_0 \quad a_0 \neq 0 \Rightarrow f'(a_0) \not\equiv 0 \pmod{10}$$

\Rightarrow Hence $\exists \alpha \in \mathbb{Z}_{10}$ with $\alpha_0 \neq 0$ s.t. $\alpha(\alpha - 1) \equiv 0$

$\Rightarrow \alpha$ ~~itself~~ is a non trivial 0-divisor.