NBRW on $PGL_2(\mathbb{Z}[\frac{1}{p}]) = G$

$\mathcal{B} = T_{p+1} \Rightarrow$ [redacted] $\mathcal{B}^1_{\text{directed}} = G/E$ $\quad E = \left\{\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in K\right\}$

For $\Gamma \leq G$ $\quad X = \Gamma\backslash\mathcal{B}$ finite $\quad X_I^1 \longleftrightarrow \Gamma\backslash G/E$

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{i\in I}} V_i \longrightarrow L^2(X_I^{(1)}) = L^2(\Gamma\backslash G)^E = \bigoplus_{i\in I} V^E$$

decomp of $L^2(\Gamma\backslash G)$ as a G-rep.

We know the unitary rep: $\mathbb{C}_{triv}$, $\mathbb{C}_{det}$ $g^\alpha = (-1)^{\text{ord}_p(\det g)\cdot\alpha}$

$$V_{\vec{z}} = \{f: G \to \mathbb{C} : \text{upper triangular } b \text{ } \forall g\}$$
$$f(bg) = \chi_{\vec{z}}(b) f(g)$$

$$\chi_{\vec{z}}\left(\begin{pmatrix} p^{n_1} & * \\ & p^{n_2} \end{pmatrix}\right) = \left(\frac{z_1}{\sqrt{p}}\right)^{n_1} (z_2\sqrt{p})^{n_2} \qquad \chi_{\vec{z}} : B \longrightarrow \mathbb{C}^* \text{ is a character.} \quad (z_1^{-1} = z_2)$$

either: (a) $|z_1| = 1$

(b) $z_1 \in [1, \sqrt{p}]$

X is Ramanujan if every $V_{\vec{z}}$ which occurs in $L^2(\Gamma\backslash G)$ is of type (a)

NBRW:

$$T\left(\overset{\cdot}{v}\longrightarrow\overset{\cdot}{w}\right)=\left\{\overset{\cdot}{w}\longrightarrow\overset{\cdot}{u}\ :\ u\neq v\right\}$$

T as double E-coset:

$e_0=\{(1,p)\to(1,1)\}$

$$T\left((1,p)\to(1,1)\right)=\left\{(1,1)\rightleftarrows\binom{p\ j}{1}\right\}_{j=0,\dots p-1}=E\binom{p}{1}E$$

$$Tge_0=gE\binom{p}{1}Ee_0=gE\binom{p}{1}e_0$$

$L^2(\Gamma\backslash G)\rightsquigarrow$ $L^2(X_I^{(1)})\overset{\simeq}{=}L^2(\Gamma\backslash G)^E=\bigoplus_{i\in I}V_i^E\simeq\bigoplus_{i\in I}V_{\vec{z}_i}^E$ (depends on e_0)

(T on $L^2(X_I^{(1)})$; α_S on $\bigoplus_{i\in I}V_{\vec{z}_i}^E$)

$$\alpha_S=\sum_{j=0}^{p-1}\binom{p\ j}{1}$$

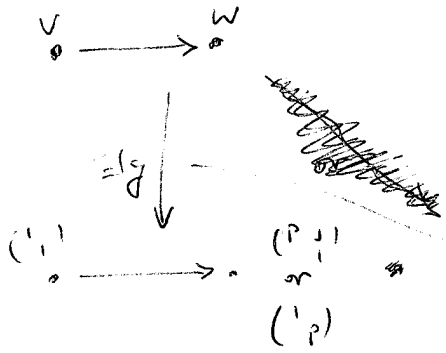
$f\in V_{\vec{z}}^K$ $\qquad f(g)=f(bk)=\chi_{\vec{z}}(b)f(1)$

($g=bk$ some $b\in B$, $k\in K$)

$f\in V_{\vec{z}}^E$ $\qquad f(g)=f(bt_ie)=\chi_{\vec{z}}(b)f(t_i^{-1})$ $\qquad \dim V_{\vec{z}}^E\le|B\backslash G/E|$ # groups in place of E.

if $G=\coprod_{i\in I}BgE$ ←

For our E: $G = BE \longleftrightarrow B$ acts transitively on directed edges.



in $B \cap K = \{(1,k)\} \Rightarrow$ doesn't take $(1,p)$ to (p,j)

We got edges $= B((1,) \to (1,p)) \sqcup B((1,) \to (p,))$

$\longleftrightarrow$ G is a disjoint union $G = BE \sqcup B(1,)E$

$\Longrightarrow$ $\dim V_{\bar{z}}^{E} \le 2$ and each $f \in V_{\bar{z}}$ is determined by $f(I)$ and $f((1,))$

Actually $\dim V_{\bar{z}}^{E} = 2$. Take $t_1 = (1,)$ $t_2 = (,1)$ $j = 1,2$

$f_i(t_j) = \delta_{ij}$ [HW: Show f_i are well defined]

NBRW sees

$$(\alpha_S f_i)(t_1) = \sum (p,j) f_i(I) = \sum f_i(p,j) = \sum_{\bar{z}} \chi((p,j))$$

$$= \sum_{j=0}^{p-1} \frac{z_1}{\sqrt{p}} = \sqrt{p}\, z_1$$

$$\alpha_s f_1(t_2) = \sum \binom{r}{i,j} f_1(i,j) = \sum P((r,i,j))$$