

$$G \overset{G}{\curvearrowright} X, x_0 \in X \quad K = \text{Stab}_G(x_0)$$

$T: X \rightarrow 2^X$ (semi) branching operator given by KSK , namely

$$KSK = \coprod_{s \in S} sK \quad \text{and} \quad T(gx_0) = \{gsx_0\}_{s \in S}$$

For $\Gamma \leq G$ T descends to $\Gamma^X \cong \Gamma^G/K$. To understand the spectrum of $\underbrace{T \text{ on } \Gamma^X}_{\text{meaning on } L^0(\Gamma^X)}$ we use rep. theory.

We saw: $L^\infty(\Gamma^X) \cong L^\infty(\Gamma^G)^K$ action by right trans.

$$\begin{array}{ccc} T & & \downarrow \alpha_s \\ \downarrow & & \downarrow \end{array}$$

$$L^\infty(\Gamma^X) \cong L^\infty(\Gamma^G)^K$$

$L^\infty(\Gamma^G)$ is a G -rep. by $gf(\Gamma_x) = P(g \Gamma_x g)$

$$\alpha_S = \sum_{s \in S} s \in \mathbb{Q}G.$$

Decompose $L^\infty(\Gamma^G) = \bigoplus_{i \in I} V_i$
 \downarrow
A G -rep = ($\mathbb{Q}G$ modules)

α_S decomposes on it

The same holds for L^2

$$L^2(\mathbb{R}^X) \cong L^2(\mathbb{R}^G)^K \cong \bigoplus_{i \in I} V_i^K$$

$$\downarrow T \qquad \downarrow \alpha_S \qquad \downarrow \bigoplus_{i \in I} \alpha_S|_{V_i}$$

$$L^2(\mathbb{R}^X) \cong L^2(\mathbb{R}^G)^K = \bigoplus_{i \in I} V_i^K$$

assure
is finite

$$\Rightarrow \text{Spec}(TGL^2(\mathbb{R}^X)) = \bigcup_{i \in I} \text{Spec}(\alpha_S|_{V_i^K}).$$

Note: only the isomorphism type of the V_i matter.

If $\bigoplus_{i \in I} V_i^K \cong \bigoplus_{i \in I} W_i^K$ with $W_i \cong V_i$ as G -rep. $\forall i \in I$

$$\text{Then } \bigoplus_{i \in I} V_i^K \cong \bigoplus_{i \in I} W_i^K$$

$$\downarrow \bigoplus_{i \in I} \alpha_S|_{V_i} \qquad \downarrow \bigoplus_{i \in I} \alpha_S|_{W_i^K} \xrightarrow{\text{it is enough}} \text{to understand}$$

$$\bigoplus_{i \in I} V_i^K \cong \bigoplus_{i \in I} W_i^K \qquad \text{those}$$

$$\text{Spec}(TGL^2(\mathbb{R}^X)) = \bigcup_{i \in I} \text{Spec}(\alpha_S|_{W_i^K})$$

Example: $G = \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ $K = \mathrm{PGL}_2(\mathbb{Z})$ $X = T_{p+1}$, $T = \mathrm{Adj}$
 $X_0 = \mathbb{I}$, $K(\frac{1}{p})K \subseteq$
 $\Gamma \leq G$ a subgroup s.t. ΓX is a finite graph ($(p+1)$ -regular).

$$K(\frac{1}{p})K = (\frac{1}{p})K \cup \bigcup_{j=0}^{p-1} (\frac{p^j}{p})K.$$

Facts: Let $z_1, z_2 \in \mathbb{F}$. Define a rep. of G as follows:

$$V_{\mathbb{Z}} = \left\{ f: G \rightarrow \mathbb{C} : \begin{array}{l} \text{Upper triangular } b \in G \quad \forall g \in G \\ f(bg) = \chi_{\mathbb{Z}}(b) f(g) \end{array} \right\}, \text{ where}$$

$$\chi_{\mathbb{Z}} \left(\begin{pmatrix} p^{m_1} & * \\ 0 & p^{m_2} \end{pmatrix} \right) = \left(\frac{z_1}{p} \right)^{m_1} \left(\frac{z_2}{p} \right)^{m_2}$$

This is the induction from B to G of $\chi_{\mathbb{Z}}$, denoted

$$\mathrm{Ind}_B^G(\chi_{\mathbb{Z}}).$$

$\chi_{\mathbb{Z}}$ is a $\underbrace{1\text{-dim rep}}_{\text{Hom: } G \rightarrow \mathbb{C}^*}$ of B upper triangular.

~~G~~ G acts on $V_{\mathbb{Z}}$ by multiplication from the right

$$(gf)(x) = f(xg)$$

Fact: For G, K, Γ as above if $L^2(\Gamma \backslash G) = \hat{\bigoplus}_{i \in I} V_i$
 then for all but finitely many $i \in I$ $V_i^K = 0$.

Proof: Follows from the diagram since \mathbb{M}^X is finite.

②

Furthermore: If $V_i^K \neq 0$ and V_i is irreducible, then $V_i = V_{\bar{z}}$ for some \bar{z} or $V_i \cong \mathbb{C}$ and the rep is either $g \cdot \alpha = \alpha$ $\forall \alpha \in \mathbb{C}$ or $g \cdot \alpha = (-1)^{\text{level}(g)} \alpha \forall \alpha \in \mathbb{C}$.

③ If $V_i^K \neq 0$, then $\dim V_i^K = 1$, so α acts on V_i^K as a scalar. Each V_i contributes one eigenvalue to Adj.

If V_i is trivial $g \cdot \alpha = \alpha$, then $V_i^K = V_i = \mathbb{C}$ \mathbb{C}_{triv}

If V_i is the determinant $g \cdot \alpha = (-1)^{\text{level}(g)} \alpha = (-1)^{\text{ord}_p(\det(g))} \alpha$, then \mathbb{C}_{det}
 $V_i^K = V_i$ since $\det(g \in \text{PGL}_2(\mathbb{Z})) = \pm 1$

Let's compute α s on those \mathbb{C}_{triv} and \mathbb{C}_{det} .

$v \in V = \mathbb{C}_{\text{triv}}$ s.t. $v \neq 0$. What is $\alpha_S v = \binom{1}{p} v + \sum_{j=0}^{p-1} \binom{p}{j} v = (p+1) v$
 \downarrow
 triv action

α s acts on \mathbb{C}_{triv} by multiplication by $(p+1)$.

Whenever some $V_i \leq L^2(\mathbb{M}^X)$ is $\cong \mathbb{C}_{\text{triv}}$ $(p+1) \in \text{Spec}(\text{Adj}(\mathbb{M}^X))$

$V = C_{\det}, \sigma \in V.$

$$\alpha_S v = \binom{p}{p} v + \sum_{j=0}^{p-1} \binom{p-j}{j} v = - (p-1) v$$

If $C_{\det} \leq L^2(p)$ then \mathbb{F}^X is bipartite.

We got $C_{\det}, C_{\text{triv}}$ are the trivial eigenvalues.

In general an eigenvalue is trivial if it comes from a 1-dim representation

If $V \cong V_{\mathbb{Z}}$ then take $\sigma \in V_{\mathbb{Z}}^K$ s.t. $f(g) = 1$
 and $V_{\mathbb{Z}} \neq 0$

Then $\forall g \in G \quad g = b \cdot k$ with $b \in B, k \in K$. Now, we get

Let $f \in V_{\mathbb{Z}}^K$. Then $\forall g \in G$ we can write $g = b \cdot k$
 with $b \in B$ and $k \in K \Rightarrow f(g) = f(bk) = \chi_{\mathbb{Z}}(b) f(k) = \chi_{\mathbb{Z}}(b) f(k)$.

$\Rightarrow V_{\mathbb{Z}}^K$ is \leq one dim.

It is one dim by defining $f(bk) = \chi_{\mathbb{Z}}(b)$ if it is well defined it is in $V_{\mathbb{Z}}^K$.

It is well defined

$b'k = b'k' \Rightarrow \cancel{b'b} = \cancel{b'b'} \in \text{BnK}$ but if $g \in \text{BnK}$

then $\chi_{\bar{z}}(g) = 1$ since $g \in \text{BnK} \Rightarrow g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

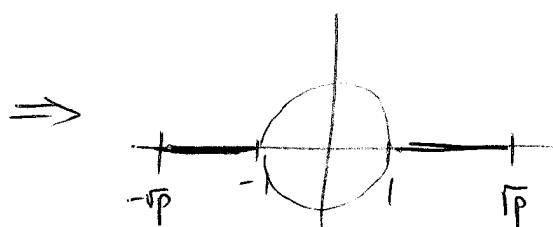
e.v. $\forall \bar{z} \quad V_{\bar{z}}^k = \bigoplus f_{\bar{z}}$, where $f_{\bar{z}}(\zeta_k) = \chi_{\bar{z}}(\zeta)$

$$\alpha_s f_{\bar{z}} = \binom{1}{p} f_{\bar{z}} + \sum_{j=0}^{p-1} \binom{p}{j} f_{\bar{z}} = \lambda f_{\bar{z}}$$

$$\alpha_s f_{\bar{z}}(I) = f_{\bar{z}} \left(\binom{1}{p} \right) + \sum_{j=0}^{p-1} f_{\bar{z}} \left(\binom{p}{j} \right) = \chi_{\bar{z}} \left(\binom{1}{p} \right) + \sum_{j=0}^{p-1} \chi_{\bar{z}} \left(\binom{p}{j} \right)$$

$$= \sqrt{p} \frac{z_2}{\sqrt{p}} + p \left(z_1 \cancel{\frac{z_1}{\sqrt{p}}} \right) = \sqrt{p} (z_1 + z_2).$$

Fact: if $V_{\bar{z}}$ has a unitary structure, then either
 $z_i \in S'$ or $z_i \in [-1, \sqrt{p}]$ (We have such a structure
from our Hilbert space)



Case a $z_i \in S'$
Case b $z_i \in [-1, \sqrt{p}]$

Case a $\alpha = 2 \operatorname{Re}(z_1) \sqrt{p} \in [-2\sqrt{p}, 2\sqrt{p}]$ (Ramanujan)

Case b $\lambda_1 = 1 \rightarrow 2\sqrt{p}$ $\lambda \in [2\sqrt{p}, p+1]$
 $\lambda_1 = \sqrt{p} \rightarrow p+1$

\Rightarrow - Satake parameter.

