

$X$  transitive  $G$ -set.  $x_0 \in X$ ,  $K = \text{Stab}_G x_0$

Every  $KSK$  induces a branching operator  $T: X \rightarrow 2^X$

Specifically if  $KSK = \coprod_{s \in S} sKs^{-1}$ , then  $T(gx_0) = \{gsx_0\}_{s \in S}$ . We also denote  $T \cap L^p(X)$  ( $1 \leq p \leq \infty$ )

By  $Tf(x) = \sum_{y \in T(x)} f(y)$ .

$$L^\infty(X) \cong L^\infty(G/K) = L^\infty(G)^K$$

$$\tilde{f}(yx_0) = f(gx_0) \longleftrightarrow f$$

$L^\infty(G)$  is a  $G$  representation by right translation:  $(gf)(g') = f(g'g)$

For any  $G$ -rep.  $V$  and  $H \leq G$  we write  $V^H$  the pointwise invariant vectors  $V^H = \{v \in V : hv = v \forall h \in H\}$

$$\text{So } L^\infty(G)^K = \{f \in L^\infty(G) : kf = f \forall k \in K\}$$

↓  
Probabilistic view —  $f$  can be thought of as a

Directly  $L^\infty(G)^K \rightarrow L^\infty(X)$

$$f \longmapsto \widehat{f}(g_{\alpha_0}) = f(g)$$

$$\begin{array}{ccc} L^\infty(G)^K & \xrightarrow{\quad ? \quad} & L^\infty(X) \\ \downarrow \text{double slash} & & \downarrow T \\ L^\infty(G) & \xrightarrow{\quad \tilde{\quad} \quad} & L^\infty(X) \end{array}$$

Let  $\alpha_S = \sum_{s \in S} s \in \mathbb{C}G$

Claim: If  $V$  is any  $G$ -rep, then  $\alpha_S$  takes  $V^K$  to itself,

namely  $\alpha_S(V^K) \subseteq V^K$ .

(2) For  $V = L^\infty(G)$ ,  $\alpha_S$  corresponds to  $T$  in the sense of the commutative diagram above ~~above~~ (under

$$V^K \cong L^\infty(G)^K \cong L^\infty(X)$$

↑                      ↑  
   $\alpha_S$                     T

Proof:

(1) We need to show for  $v \in V^K$  and  $k \in K$  that  $k\alpha_S v = \alpha_S v$ .

$$k\alpha_S v = k \sum_{s \in S} sv = \sum_{s \in S} ks v$$

~~the sum is over S~~

Since

$$k \coprod_{s \in S} sK = \coprod_{s \in S} \cancel{s}K = KSK = \coprod_{s \in S} sK$$

$\Rightarrow$  ~~there exists~~  $\forall s \in S \exists s' \in S$  and  $k \in K$  s.t.

$ks = s'k'$ . Furthermore  $s \mapsto s'$  is a permutation on  $S$

$$\Rightarrow k \alpha_s v = \sum_{s \in S} ks v = \sum_{s \in S} s' k' v = \sum_{s \in S} s' v = \alpha_{s'} v.$$

2) For  $V = L^\infty(G)$  right and  $\tilde{f} \in L^\infty(X) \mapsto f(g) := \tilde{f}(g_x)$

$$\alpha_s f(g) = \sum_{s \in S} sf(g) = \sum_{s \in S} f(g_s)$$

$$\widehat{\alpha}_s f(g_x) = \sum_{s \in S} f(g_s) = \sum_{s \in S} \tilde{f}(g_{sx}) = T\tilde{f}(g_x).$$

□

Now, for  $\Gamma \leq G$  we look at  $\Gamma^X \cong \Gamma^G/K$  we get

$$L^\infty(\Gamma^X) \cong L^\infty(\Gamma^G/K) \cong L^\infty(\Gamma^G)^K$$

$$T \downarrow$$

$$\downarrow \alpha_s$$

$$L^\infty(\Gamma^X) \cong L^\infty(\Gamma^G_K) \cong L^\infty(\Gamma^G)^K$$

Claim: For  $V = L^2(\Gamma \backslash G)$   $M \leq G$  right translation  $\alpha_s$  corresponds to  $T$  under  $\mathcal{V}^K \cong L^2(\Gamma \backslash X)^K$

The same proof as the one for (2) works with  $g$  replaced by  $\Gamma g$ . □

On the building  $\alpha_s \in C^* L^2(G) \rightsquigarrow \alpha_s \in L^2(\Gamma \backslash G)$   
 ↪ a finite graph/cycle

Why do we care?

Because we can decompose reps. to irreducible rep.

$$L^2(\Gamma \backslash G) = \bigoplus_{i \in I} V_i. \text{ From this we get } L^2(\Gamma \backslash X) = L^2(\Gamma \backslash G)^K$$

$$= \bigoplus_{i \in I} V_i^K$$

$$\begin{aligned} L^2(\Gamma \backslash X) &\cong L^2(\Gamma \backslash G)^K = \bigoplus_{i \in I} V_i^K \\ &\downarrow \quad // \quad \downarrow \quad // \quad \downarrow \quad \alpha_s|_{V_i^K} \\ L^2(\Gamma \backslash X) &\cong L^2(\Gamma \backslash G)^K = \bigoplus_{i \in I} V_i^K \end{aligned}$$

Since  $\alpha_s \in C^* G$ .

$\bigoplus V_i$  respects  $\alpha_s$ .

On the building

$$\alpha_s \mathbb{C} L^2(G) = \bigoplus_{V \in \widehat{G}} V^K$$

$$\alpha_s \mathbb{C} L^2(\mathbb{M}^G) = \bigoplus V_i^K$$

$\mathbb{M}^{G/K}$  is Ramanujan  $\Leftrightarrow$  every irreducible rep.  $V \subseteq L^2(\mathbb{M}^G)$  is also in the regular rep.