

Lecture

HW: $G = \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$, $K = \mathrm{PGL}_2(\mathbb{Z})$. Show

$$\{A \in G : \mathrm{level}(A) = m\} = K\left(\begin{pmatrix} 1 & \\ & p^m \end{pmatrix}\right)K.$$

Hecke graphs

Combinatorial operators $V = G/K \in \{-\{gK, gSK\}\}_{S \in S}$

Show: From S we can create $S \subseteq S'$ s.t.

$$KS'K \subseteq SK$$

and then the out neighbors of gK are $\{gS'K\}_{S' \in S'}$

Hecke($G, K, \{\begin{pmatrix} 1 & \\ & p \end{pmatrix}^S\} = X_p^2$) is $(p+1)$ -reg.

$$S = \left\{ \begin{pmatrix} 1 & \\ & p \end{pmatrix} \right\} \quad S' = \left\{ \begin{pmatrix} 1 & \\ & p^j \end{pmatrix}, j=0, \dots, p-1 \right\}$$

Set $X = G/K$ a comb branching op. on X
a G -equi map $T: X \rightarrow \mathbb{Z}$

$$\forall x \in X \quad \forall g \in G \quad T(gx) = gT(x)$$

Since $G \curvearrowright X$ trans. $T(x_0)$ determines T (any fixed choice of element in X , e.g. K)

For $x_0 = K$

-2 -

~~such x_0 's~~, Tx_0 must satisfy $\cancel{K} T(x_0) = T(x_0) \quad \forall k \in K$

Since $kT(x_0) = T(kx_0) = T(x_0)$.

~~as x_0 is a K -fixed point~~

Actually, any choice of a K -fixed set $S \subseteq X$ determines a unique combinatorial branching operator with $Tx_0 = S$.
(Check)

Rem:

More generally given $G \backslash X$ transitive action and $x_0 \in X$ we define $K = \text{Stab}_G(x_0)$

directed Hecke graph \longrightarrow branching rule (out neighbors)

K balanced set \longleftrightarrow K fixed \tilde{S} set.

S s.t. $KS = SK$ $K\tilde{S} = \tilde{S}$

here $S \subseteq G$ here $\tilde{S} \subseteq X = G/K$

$$\begin{array}{ccc} S = \tilde{S}K & \longleftrightarrow & \tilde{S} \\ S & \longrightarrow & SK/K \end{array}$$

G equivariant branching operator on trans. G -set X



sets K which are invariant w.r.t. the stabilizer of
a fixed vertex $x_0 \in X$



(union of)
double cosets of K .

Once again

$\overset{\text{trans}}{G} G^x X$, pick $x_0 \in X$. $K = \text{stab}(x_0)$

Take some bi- K -inv set $M \subseteq G$ ($\hookrightarrow M$ is a
(union of double
 K cosets))

decompose $M = \coprod_{s \in S} sK$ (thus defining S)

and then S is a K -balanced set \rightarrow Hecke graph

$$T_{x_0} = \{sx_0g^{-1} \}_{s \in S} \rightarrow T(gx_0) = \{gx_0g^{-1}\}_{g \in S}$$

gives a branching operator.

Look at $G = \mathrm{PGL}_2(\mathbb{Z}_{\ell^2})$, $K = \mathrm{PGL}_2(\mathbb{Z})$

$X = G/K$ $(p+1)$ -reg tree

What are comb. operators on X .

$$T_x = B_r(x)$$

$$T_x = S_r(x)$$

$$T_x = \{y : \mathrm{dist}(x, y) \in \{3, 7, 100\}\}$$

$T_x = \text{finite union of spheres}$ those are all of them.

$$X = T_L$$

$$G = \mathrm{Sym}(X) = \mathrm{Aut}(X)$$

what comb. op. are there? Union of spheres

If $y \in T_{x_0}$ and $\mathrm{dist}(x_0, y) = r$ then $S_r(x_0) \subseteq T_{x_0}$

Since $\forall y' \in S_r(x_0) \exists k \in K = \mathrm{Stab}_G(x_0) \text{ s.t. } ky = y'$

Hence $y' \in T_{x_0} = T(kx_0) = T(x_0)$

We want to study the behavior of T by spectral means.

Namely, define $A_T \subseteq L^2(X)$ by $(A_T f)(x) = \sum_{y \in T_x} f(y)$

Then, e.g., if $\text{Spec}(A_T) = \{|T_x|\}, \text{ small ev.}\}$, then T is
 "if X is finite"
 rapidly mixing.

Now, we can also talk about polynomials $A_T^2 - A_T^3$
 or, more generally, the ring of G -equiv. functions on $L^2(X)$.
 with finite support

For $X = T_k$, either $G = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ or $G = \text{Sym}(T_k)$, the regular adj. operator generates all these operators.

$\iff \forall r \in \mathbb{N}$, the operator $A_T f(x) = \sum_{y \in S_r(x)} f(y)$ is a poly in A_1 .

E.g. $A_2 = A_1^2 - k A_1^0$.

$$A_3 = A_1^3 - \square A_1$$

$$A_4 = A_1^4 - \square A_1^2 + \square A_1^0$$

Legendre polynomials.

