Lecture 12

Cells in the link of $I = \mathbb{Z}^d \longleftrightarrow \mathbb{Z}^d \supsetneq L_1 \supsetneq L_2 \supsetneq \dots \supsetneq L_j \supsetneq p\mathbb{Z}^d$

$$\Updownarrow$$

$\mathbb{F}_p^d \supsetneq V_1 \supsetneq V_2 \supsetneq \dots \supsetneq V_j \supsetneq \{0\}$

flags in $\mathbb{F}_p^d$!

We defined the spherical building of $\mathbb{F}_p^d$

cells $\equiv$ Flags in $\mathbb{F}_p^d$. We got $\underset{\in X_p^d}{\text{link}(v)} \cong$ spherical building of $\mathbb{F}_p^d$.

Claim: $G = PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts transitively on top $(d-1)$-cells.

Proof: We already know that G acts transitively on vertices so it suffices to show the stabilizer of a vertex acts transitively on $(d-1)$-cells containing it.

For example that $\text{stab}_G(I) \equiv K = PGL_d(\mathbb{Z})$ acts transitively on $(d-1)$-cells containing I.

$(d-1)$-cells containing $I \longleftrightarrow$ max flags in $\mathbb{F}_p^d$

Claim: K acts transitively on max flags by the mod p map.

and $PGL_d(\mathbb{F}_p)$ acts transitively on maximal flags in $\mathbb{F}_p^d$.

Thus if $GL_d(\mathbb{Z}) \twoheadrightarrow GL_d(\mathbb{F}_p)$ we are done.

This however is not the case. Instead look at

$$\underset{\substack{\swarrow \\ \text{comm.} \\ \text{ring}}}{SL_d^{\pm}(R)} = \{A \in M_d(R) : \det A = \pm 1\}$$

$SL_d^{\pm}(\mathbb{F}_p)$ acts transitively on maximal flags.

Now $SL_d^{\pm}(\mathbb{Z}) \twoheadrightarrow SL_d^{\pm}(\mathbb{F}_p)$ (*)

Claim:
In general, if R is an Euclidean domain, then

$$SL_d^{\pm}(R) = \Big\langle \underbrace{\underset{T_{a,i,j}}{\begin{pmatrix} 1 & & a \\ & \ddots & \\ & & 1 \end{pmatrix}}, \underset{S_{i,j}}{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}}_{(**)}, \underset{M_i}{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}} \Big\rangle$$

then (*) follows

Proof: $A \in SL_d^{\pm}(R)$ by Euclid algorithm on first row $A = \left(\begin{array}{c|c} a & 0\,0\,0\,0 \\ \hline * & * \end{array}\right)$

but $a \in R^* \Rightarrow$ Euclid on first column $A = \left(\begin{array}{c|c} a & 0 \cdots 0 \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right)$

This action can be done using the matrices in (**) Cont $\longrightarrow \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \\ & & & a_d \end{pmatrix}$

$\prod a_i = \pm 1$

$$\begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_d \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \xrightarrow{\text{col}} \begin{pmatrix} a & a \\ & b \end{pmatrix} \underset{T_{\frac{1-a}{a},1,2}}{\xrightarrow{\text{col}}} \begin{pmatrix} 1 & a \\ \frac{b-ab}{a} & b \end{pmatrix} \xrightarrow{\text{row}} \begin{pmatrix} 1 & a \\ 0 & ab \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}$$

move all to the last diagonal $\rightarrow$ it must be 1 $\square$

$S \subseteq G$

Cayley graph $\quad G$, $\quad Cay(G;S) = (V,E)$

$V = G \qquad E = \{(g, sg) : g \in G, s \in S\}$

This is an s-regular graph.

Too restrictive

> All graphs are directed

Schreier graph

G group $\quad H \leq G \quad S \subseteq G$

$Sch(G,H;S) = (V,E) \qquad V = G/H \qquad E = \{(gH, sgH) : g \in G, s \in S\}$

Too general

Hecke graph

$G \quad H \leq G \quad S \subseteq G$

$Hec(G,H;S) = (V,E) \qquad V = G/H \qquad E = \{(gH, gsH) : g \in G, s \in S\}$

This has transitive G action.

<u>Claim</u>: $X_p^d = \mathrm{Hecke}(G, K, \{(1_p)\})$ where $G = PGL_d(\mathbb{Z}[\tfrac{1}{p}])$, $K = PGL_d(\mathbb{Z})$

<u>Catch</u>: $S = \{s_1, \dots, s_m\}$

$$gK \to gs_1K, \quad gK \to gs_2K, \quad \dots, \quad gK \to gs_mK$$

maybe $gs_1K = gs_2K$

real problem if $gK = g'K$ it is not necessarily true that $g's_i = gs_i$ so we also need to go to

$gK \longrightarrow g's_iK$ for $1 \le i \le m$ and $g' \in G$ s.t. $gK = g'K$.

<u>Define</u>: S is K balanced if $KSK = SK$

First note that $\mathrm{Hecke}(G, K, S)$ we actually have

$$gkK = gK \longrightarrow gksK$$

so, if $KSK = SK = \coprod_{s \in S'} sK$

balancification of S

$\Rightarrow$ edges of the graph are (gK, gsK) for $s \in S'$

[Out going neighbors of gK are $\{gsK\}$ for $s \in S'$]

$\boxed{\text{If } S \text{ is } K\text{-balanced then } S' = S} \Rightarrow$ The graph is $|S|$-regular

~~out neighbors~~

Back to the claim:

For $G = PGL_d(\mathbb{Z}[\tfrac{1}{p}])$ $K = PGL_d(\mathbb{Z})$ $S = \{\begin{pmatrix} 1 & \\ & p \end{pmatrix}\}$ we can take $S' = \{ \begin{pmatrix} 1 & \\ & p \end{pmatrix}, \begin{pmatrix} p & j \\ & 1 \end{pmatrix} \;\; j = 0, \dots, p-1 \}$

and S' is K-balanced and gives the $(p+1)$-regular tree.

First check that $X_p^2 = Hec(G, K; S')$

need to show $K \begin{pmatrix} 1 & \\ & p \end{pmatrix} K = \coprod_{j=0}^{p-1} \begin{pmatrix} p & j \\ & 1 \end{pmatrix} K \;\; \amalg \;\; \begin{pmatrix} 1 & \\ & p \end{pmatrix} K$

Recall: For $A \in PGL_d(\mathbb{Z}[\tfrac{1}{p}])$, the level of A is $\log_p \det A$ when A is scalled to be primitive.

In other words, if $A = \begin{pmatrix} p^m & a \\ & p^n \end{pmatrix} \cdot k$, then level $= m+n$.
$\gcd(p^m, p^n, a) = 1$

Claim $K \begin{pmatrix} 1 & \\ & p \end{pmatrix} K = \{ g \in G : \text{level}(g) = 1 \}$.

Proof:

(1) $K\begin{pmatrix}1 & \\ & p\end{pmatrix}K \subseteq$ level 1 because all the matrices on the left ~~have~~ det p. and are primitive.

(2) level 1 $= \coprod \begin{pmatrix}p & j\\ & 1\end{pmatrix}K \amalg \begin{pmatrix}1 & \\ & p\end{pmatrix}K$ because the general form is $\begin{pmatrix}p^m & a\\ & p^n\end{pmatrix}$ for $\gcd(p^m, p^n, a) = 1$. and the disjointness was left as an exercise

(3) $\coprod \begin{pmatrix}p & j\\ & 1\end{pmatrix}K \amalg \begin{pmatrix}1 & \\ & p\end{pmatrix}K \subseteq K\begin{pmatrix}1 & \\ & p\end{pmatrix}K$

we showed this. ~~when~~ $\underbrace{\begin{pmatrix} & 1\\ 1 & \end{pmatrix}}_{\in K}\begin{pmatrix}1 & \\ & p\end{pmatrix}K = \begin{pmatrix}p & \\ & 1\end{pmatrix}K$

$$\underbrace{\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}}_{\in K}\begin{pmatrix}1 & \\ & p\end{pmatrix}K = \begin{pmatrix}p & j\\ & 1\end{pmatrix}K.$$

Actually, $\mathrm{Hec}(G; K, g)$ if level $g = 1$ we get $\mathrm{Hec}\left(G, K, \begin{pmatrix}1 & \\ & p\end{pmatrix}\right)$

if level$(g) > 1$ we get a disconnected graph union of trees,

Note: In $\mathrm{Hec}(G, K, S)$ we have G-action on edges and vertices.

We will construct Ramanujan graphs as Hecke-Schreier graphs

$$HS(G, H, K, S) = Sch\left(Hec(G,K,S),\ H, S\right)$$

$$V = {}_H\backslash G/_K \qquad E = \{(HgK\ ,\ HgsK)\ \}$$

Cayley and Schreier graphs are edge labeled by S

Hecke graphs are not.

$\begin{pmatrix}1 & \\ & 1\end{pmatrix}K = IK$ —— $\begin{pmatrix}p & \\ & 1\end{pmatrix}$

—— $\begin{pmatrix}p & \\ & 1\end{pmatrix}$

$\begin{pmatrix} & 1\\ 1 & \end{pmatrix}K \quad \begin{pmatrix}1 & \\ & p\end{pmatrix}$

$\|$

IK

The ~~[illegible]~~labels we got are related to the specific representatives we choose the edges are not labeled.

<u>HW:</u>

Show that in PGL_2 $\{g : level(g) = n\} \equiv K\begin{pmatrix}1 & \\ & p^n\end{pmatrix}K.$

This is not true in higher dim.

~~a combinatorial operator on G/K~~

<u>Def:</u> A combinatorial branching map on G-set X is a map $T : X \to 2^X$ such that $T(gx) = g\,T(x)$ $\forall x \in X$, $g \in G$.

If X is transitive, then pick $x_0 \in X \Rightarrow T(x_0)$ determine T

$$T(x) = T(gx_0) = g\,T(x_0)$$

$\downarrow$

$\exists g$

Furthermore $T(x_0)$ is K-stable where $K = \mathrm{Stab}(x_0)$ because

$\forall k \in K$ $\quad k\,T(x_0) = T(kx_0) = T(x_0)$