

# Lecture 11

## Local structure

$$\text{Star}(v) = \{\sigma : v \in \sigma\} \cong \text{Cone}(\text{link}(v))$$

$$\text{Cone}(\triangle) = \text{tetrahedron}$$

$$\text{Cone}(X) = X \times I /_{(x,0) \sim (x',0)}$$

Recall:  $d$ -cells containing  $I = \mathbb{Z}^d$  correspond to chains

$$\mathbb{Z}^d \supseteq p L_1 \supseteq p^2 L_2 \supseteq \dots \supseteq p^{nd} L_d = p^{nd} \mathbb{Z}^d$$

$\iff d$ -cycles of 1-edges  $(\dots, p^x, \dots)$

Actually, we have  $\mathbb{Z}^d \supseteq L_1 \supseteq L_2 \supseteq \dots \supseteq L_d = p \mathbb{Z}^d$

because if  $p L_i \supseteq p L_{i+1}$  we get a contradiction (growth of  $p^d$  which is impossible as the total change in co-volume is  $p^d$ ).

$d$ -cells containing  $I \iff \mathbb{Z}^d \supseteq L_1 \supseteq L_2 \supseteq \dots \supseteq L_d \supseteq p \mathbb{Z}^d$   
 $\iff \text{IV iso-thm (correspondence thm)}$

$$\mathbb{F}_p^d = \mathbb{Z}^d / p \mathbb{Z}^d \supseteq L_1 / p \mathbb{Z}^d \supseteq \dots \supseteq L_d / p \mathbb{Z}^d \supseteq p \mathbb{Z}^d / p \mathbb{Z}^d = 0$$

(2)

⇕

$\mathbb{F}_p^d \supseteq V_1 \supseteq V_2 \dots \supseteq V_{d-1} \supseteq 0$  maximal flags in  $\mathbb{F}_p^d$  ~~maximal flags~~

cells containing  $\text{Id}$   $\longleftrightarrow$  Flags in  $\mathbb{F}_p^d$

dim  $\longleftrightarrow$  length-2

In particular  $\mathbb{F}_p^d \supseteq V \supseteq 0$  <sup>(non-trivial subspaces of  $\mathbb{F}_p^d$ )</sup> are in correspondence with neighboring vertices to  $\text{Id}$ ,

Def: The spherical building of  $GL_d(\mathbb{F}_p)$  is:

vertices - non trivial subspaces of  $\mathbb{F}_p^d$

edge - inclusion —  $\{V_1, V_2\}$  is an edge if  $V_1 \supseteq V_2$  or  $V_2 \supseteq V_1$

general cells - flag complex. In particular  $(d-2)$ -cells are maximal flags.

$\cong$  link of  $I$  is the affine building of  $PGL_d(\mathbb{Z}[\frac{1}{p}])$

(3)

$X_p^2 = (p+1)$  kg tree  $\rightarrow$   $lk(Id) \approx \begin{matrix} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{matrix}$   $(p+1)$ -points

$\longleftrightarrow$  non trivial subspace of  $\mathbb{F}_p^2$  =  $(p+1)$ -points no inclusion. (lines)

~~$\mathbb{F}_p^3$~~  : lines  $p^2+p+1$   
planes  $p^2+p+1$

Every line is contained in  $(p+1)$ -planes

$\Rightarrow$  Spherical ~~graph~~ building of  $GL_3(\mathbb{F}_p)$  is a  $(p+1)$ -reg bip. graph with  $2(p^2+p+1)$  vertices.

this is an excellent expander  $\xrightarrow{\text{Guraland theory}}$  Good <sup>Spec</sup> expansion for  $X_p^2$ .

$G = PGL_d(\mathbb{Z}[\frac{1}{p}])$  acts transitively on vertices  $G/K = GL_d(\mathbb{Z})$  <sup>by group action</sup>

1-edges,  $(d-1)$ -cell (top), apartments (by defn)  
 $\swarrow$   
saw

$(d-1)$ -cells! Since  $G$  acts trans. on vertices, it is left to show that  $Stab(Id) = K = GL_d(\mathbb{Z})$  acts transitively on  $(d-1)$  cells containing  $\mathbb{I}$

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Such  $(d-1)$ -cells correspond to maximal flags in  $\mathbb{F}_p^d$ .  
The action of  $GL_d(\mathbb{Z})$  is by its map to  $GL_d(\mathbb{F}_p)$

$$A \longmapsto A \bmod p$$

$GL_d(\mathbb{F}_p)$  acts trans. on max flags in  $\mathbb{F}_p^d$  (convince yourself)

However  $GL_d(\mathbb{Z}) \rightarrow GL_d(\mathbb{F}_p)$  is not onto ( $\det A = \pm 1$ )

Nevertheless  $SL_d(\mathbb{Z}) \rightarrow SL_d(\mathbb{F}_p)$  and the latter acts transitively on max flags.