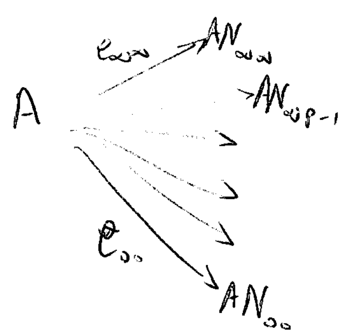
Lecture 10

The outgoing edges of I are $(N_i)_{i=1}^{\frac{p^d-1}{p-1}}$. For general A the outgoing edges are $(\text{fixed}(AN_i))_{i=\emptyset}^{\frac{p^d-1}{p-1}}$

(Bringing it back to X_p^d)

X_p^2 is a symmetric $(p+1)$-regular tree.

X_p^3 $\quad \underset{N_{\infty\infty}}{\begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix}} \quad \underset{N_{x\infty}}{\begin{pmatrix}1&&\\&p&x\\&&1\end{pmatrix}} \quad \underset{N_{xy}}{\begin{pmatrix}p&x&y\\&1&\\&&1\end{pmatrix}}$



p^2+p+1 outgoing neighbors.

Unlike the case $d=2$, the out-edges here are not the same as in-edges. E.g.,

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \xrightarrow{e_{xy}} \begin{pmatrix}&&\\&&\\&&\end{pmatrix}$$

↳ matrix with det. p^2 $\longrightarrow$ after fixing $\det = p^{2-3m} \neq 1$

There is no going back.

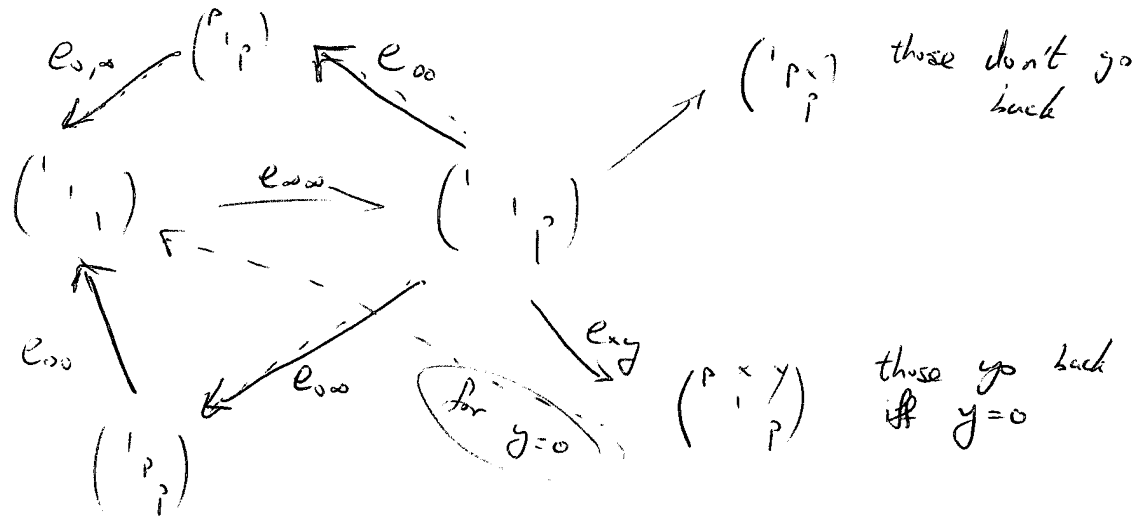
However

$$\begin{tikzcd} \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \arrow[r,"e_{\infty\infty}"] & \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \arrow[r,"e_{0,\infty}"] & \begin{pmatrix}1&&\\&p&\\&&p\end{pmatrix} \arrow[ll, bend left, "e_{0,0}"] \end{tikzcd}$$

we got paths of length 3.

$$\begin{tikzcd} \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \arrow[r,"e_{\infty,\infty}"] & \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix} \arrow[r,"e_{\infty,\infty}"] & \begin{pmatrix}1&&\\&1&\\&&p^2\end{pmatrix} \arrow[ll, bend left, dashed, "\nexists"] \end{tikzcd}$$

What are the triangles of $\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix}$



In total the edge is contained in p triangles.

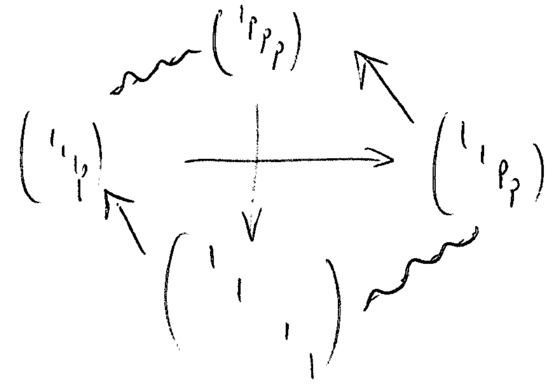
The building of $PGL_d(\mathbb{Z}[\frac{1}{p}])$ is defined as follows:

Vertices X_p^d

$(d-1)$-cells $\{v_1, \dots, v_d\}$ s.t. $\exists$ path $v_1 \to v_2 \to \dots \to v_d \to v_1$

pure $(d-1)$-dim. complex

$d=4$ tetraedra



one need to add two of the edges to get the tetraedra.

<u>2nd defn:</u> X_p^d is the flag complex of the graph with vertex set X_p^d and edges

$$A \longmapsto \text{fix}(AN) \quad \text{for} \quad N = \begin{pmatrix} 1_p p & & \\ & \ddots & \\ 0 & & 1_p \end{pmatrix}$$

p's or 1's on the diagonal and

if $a_{ii}=p$ $a_{jj}=1$ and $i<j$ then $a_{ij} \in \{0,\dots,p-1\}$ otherwise $a_{ij}=0$.

Apartments

The picture you get from restricting to diagonal vertices and edges.

$$X_1^d = \left\{ \begin{pmatrix} p^{n_1} & & 0 \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} : \begin{array}{l} \text{primitive,} \\ \min\{n_1,\dots,n_d\}=0 \end{array} \right\}$$

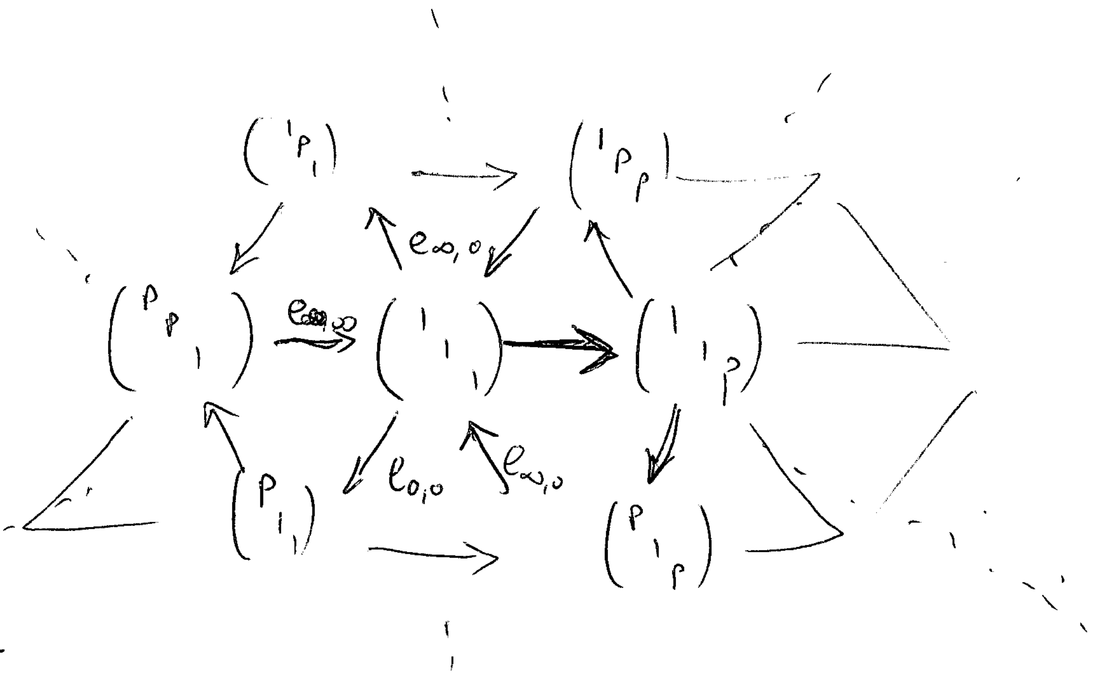
$$N = \left\{ \begin{pmatrix} p & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & p & \\ & & 1 \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \dots \right\}$$

<u>Tits</u>

"Field with one element $\mathbb{F}_1$"

X_1^2 $$\cdots \begin{pmatrix}1&\\&p^2\end{pmatrix} \rightleftarrows \begin{pmatrix}1&\\&p\end{pmatrix} \underset{e_0}{\overset{e_\infty}{\rightleftarrows}} \begin{pmatrix}1&\\&1\end{pmatrix} \underset{e_\infty}{\overset{e_0}{\rightleftarrows}} \begin{pmatrix}p&\\&1\end{pmatrix} \underset{e_\infty}{\overset{e_0}{\rightleftarrows}} \begin{pmatrix}p^2&\\&1\end{pmatrix} \cdots$$ line

X_1^3



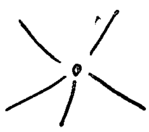
Triangular ~~Hexagonal~~ tesselation of the plane.

<u>Building</u>

local view - link

section - apartment

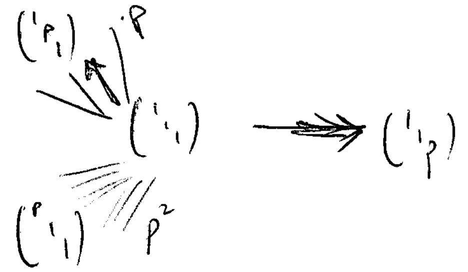
global view - entire building.

X_p^2 local  $p+1$ star Section:  line

T_{p+1}

Section: Triangular tesselation

See picture.



Back to the group $G = PGL_d(\mathbb{Z}[\frac{1}{p}])$

G acts on the building

vertices $\longleftrightarrow$ G/K $\qquad K = PGL_d(\mathbb{Z})$

$G \circlearrowright G/H$ by $g \circ g'H = gg'H$.

Facts:

G acts transitively on the vertices.

G acts transitively on 1-edges $\left(\begin{array}{c}\text{edges coming from} \\ N_i \quad i=1,\dots,\frac{p^d-1}{p-1}\end{array}\right)$

G acts transitively on $(d-1)$-cells.

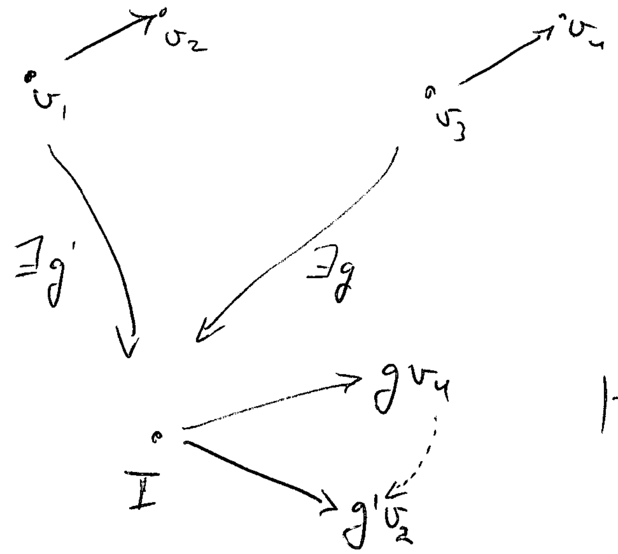
Thus $X^{(d-1)} = \left\{ g\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}, g\begin{pmatrix}1 & & \\ & \ddots & \\ & & p\end{pmatrix}, \dots, g\begin{pmatrix}1 & & \\ & p & \\ & & \ddots & \\ & & & p\end{pmatrix} \right\}_{g \in G}$

For any $g \in G$, the complex spanned by $g \cdot \{\text{diagonal matrices}\}$ is isomorphic to the diagonal matrices = fundamental apartment

these are all called apartments. ~~[illegible]~~

These are not all the " .

<u>Proof of transitive action on edges</u>



It is left to see that

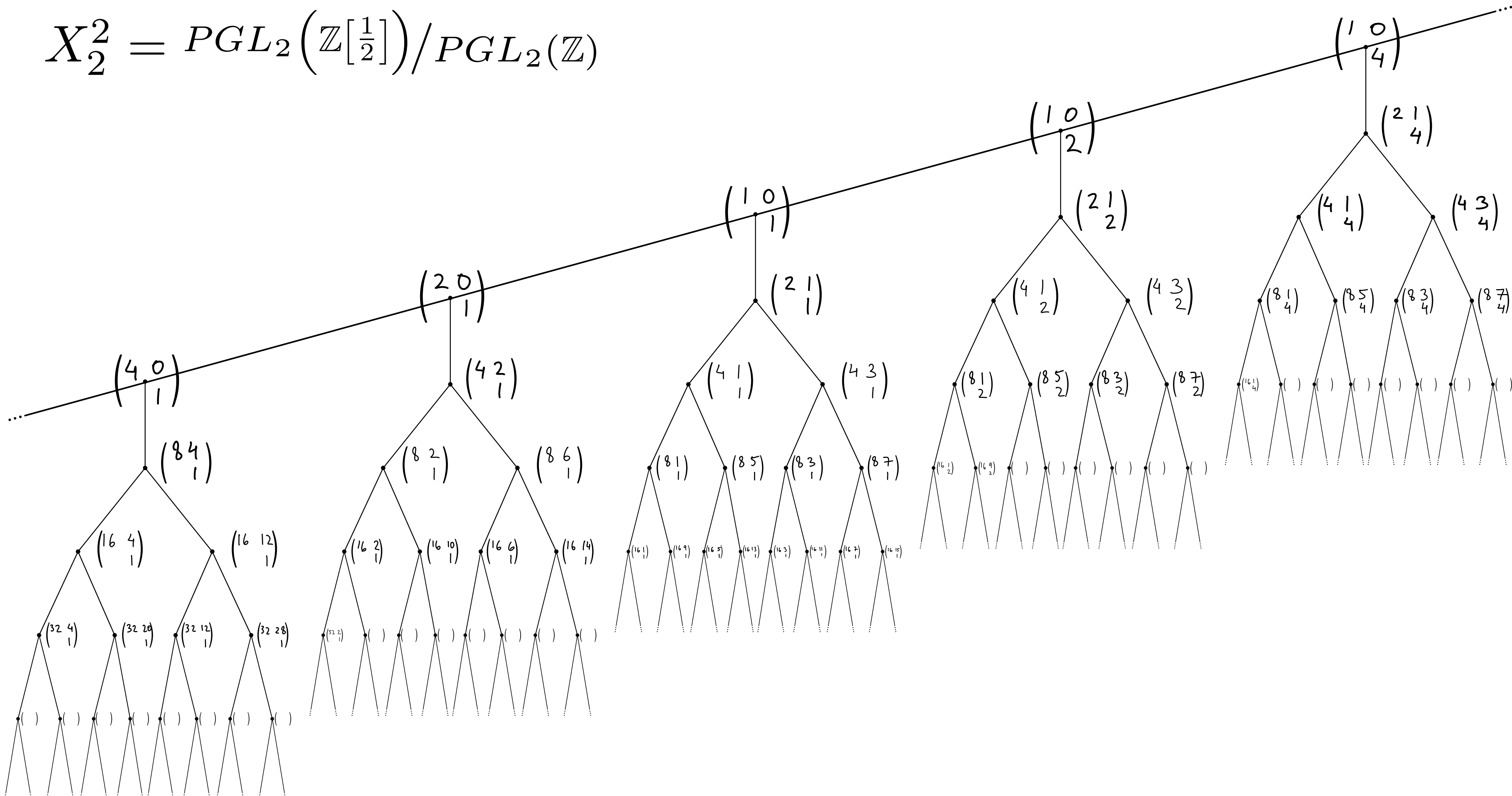
$\mathrm{Stab}(I)$ acts transitively on $N_1, \dots, N_{\frac{p^d-1}{p-1}}$ from the ~~left~~ left

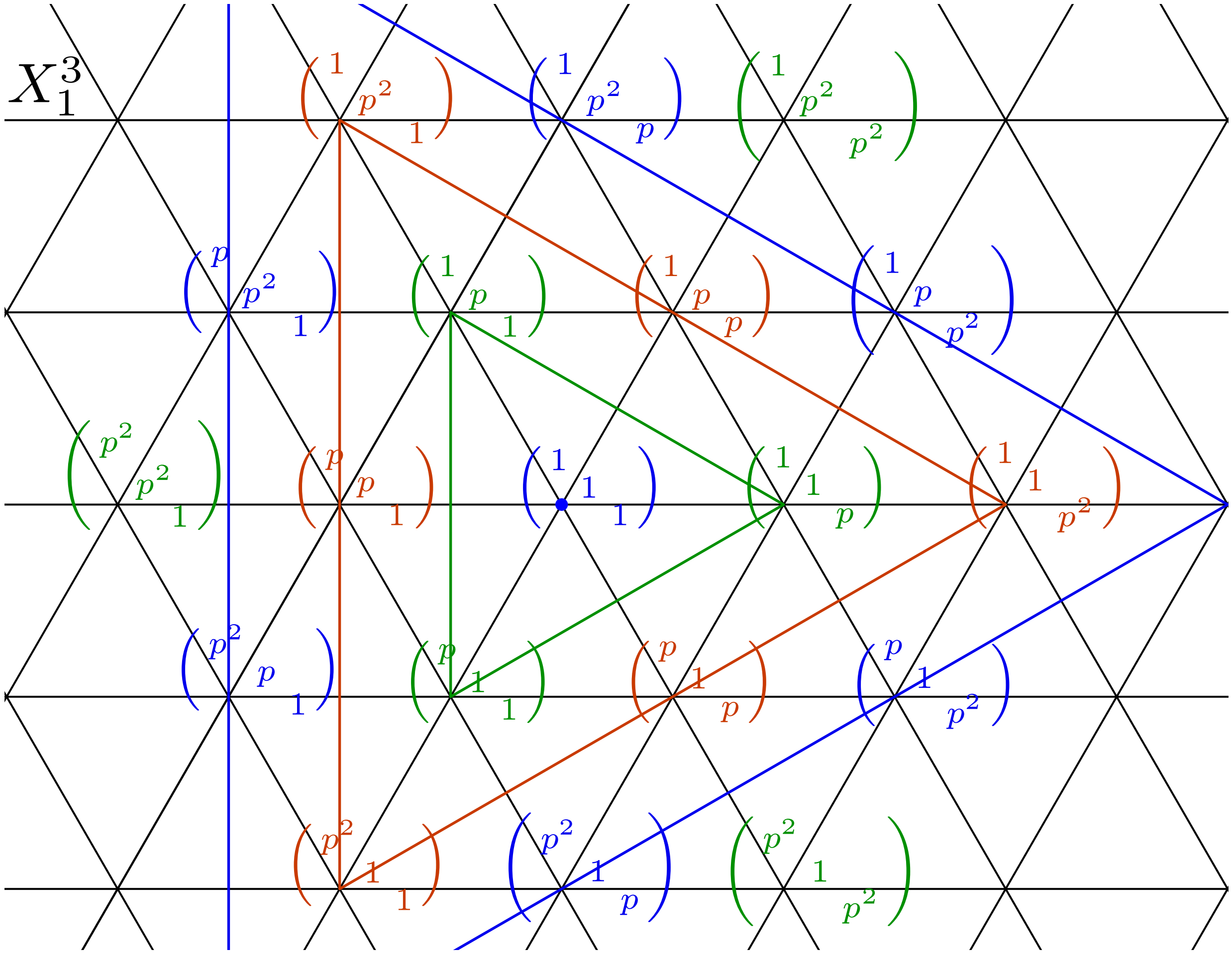
exercise.

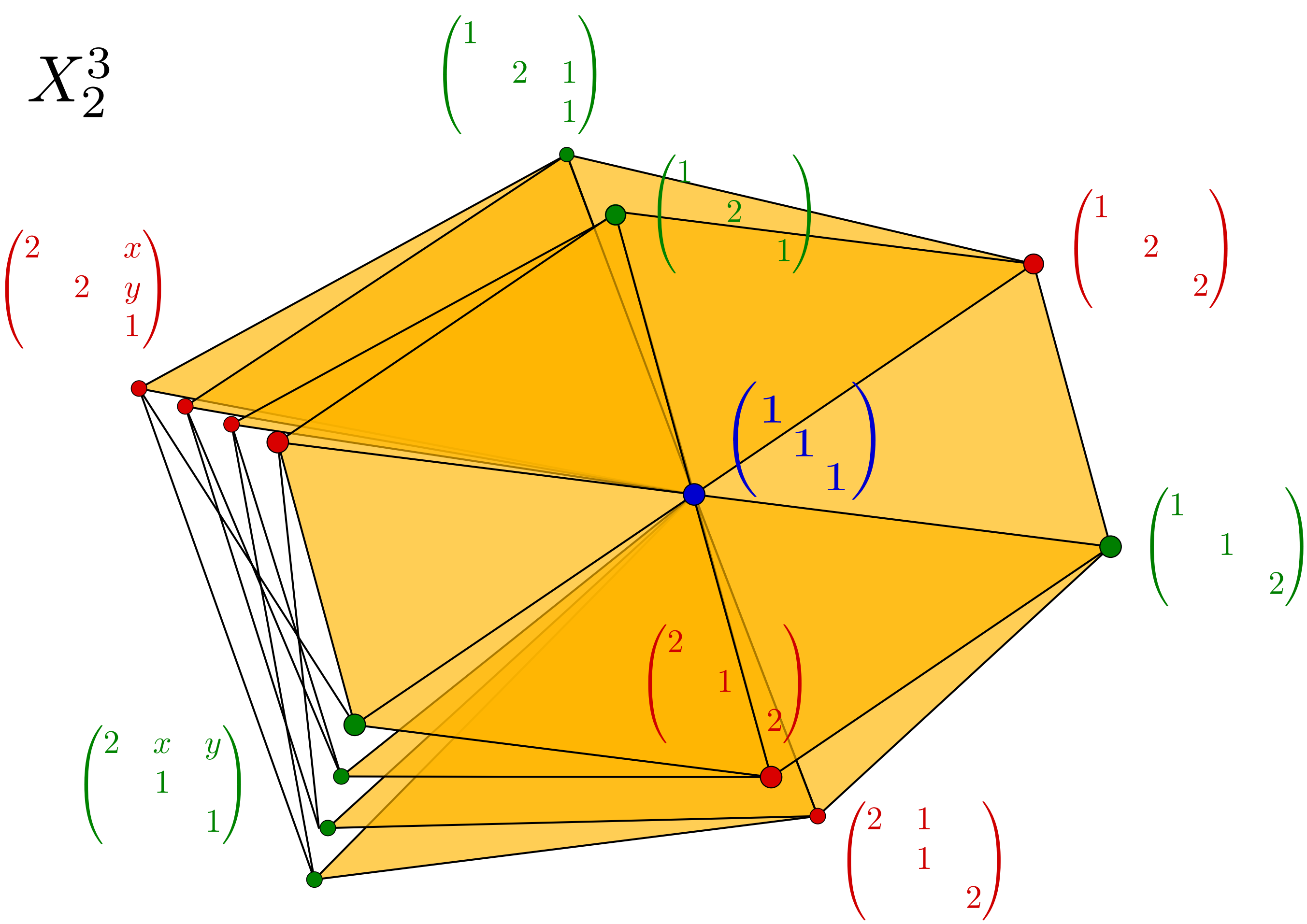
$\blacksquare$

$\mathrm{Stab}(I) = K = PGL_d(\mathbb{Z})$

$$X_2^2 = PGL_2\left(\mathbb{Z}[\tfrac{1}{2}]\right)/PGL_2(\mathbb{Z})$$




X_1^3


X_2^3