

Lecture

Building of PGL_d vertices

$$\textcircled{1} \quad \text{PGL}_d(\mathbb{Z}[\frac{1}{p}]) / \text{PGL}_d(\mathbb{Z})$$



Three ways to think about the vertices

$$\textcircled{2} \quad \text{Primitive } p\text{-lattices}$$

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$$\textcircled{3} \quad X_p^d = \left\{ \begin{pmatrix} p^{n_1} & & a_{1j} \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} : 0 \leq a_{ij} < p^{n_i} \right\}$$

a primitive matrix

edges

$$\textcircled{2} \quad L_1 \rightarrow L_2 \quad \text{if} \quad L_2 \leq L_1 \quad \text{or} \quad pL_2 \leq L_1$$

$$\textcircled{3} \quad \text{Define } \frac{p^d-1}{p-1} \text{ matrices } N_j = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & p^{d-1} & \cdots & \cdots & * \\ & & & \ddots & 0 \end{pmatrix} \quad 0 \leq j \leq \frac{p^d-1}{p-1}$$

then the outgoing neighbors of $A \in X_p^d$ are AN_j for $0 \leq j \leq \frac{p^d-1}{p-1}$,

where if $AN_j \notin X_p^d$ we fix it by recalling that $AN_j \in \text{PGL}_d(\mathbb{Z}[\frac{1}{p}])$ and hence has a unique rep. in X_p^d .

example

$$\begin{pmatrix} 8 & 5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ & 8 \end{pmatrix} \stackrel{\text{primitivity}}{\equiv} \begin{pmatrix} 4 & 5 \\ & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ & 4 \end{pmatrix}$$

one only need to divide by p and reduce $a_{ij} \bmod p^n$ to get the rep. in X_p^d

(2)

Claim: X_p^2 is a tree.

Instead of writing $N_1 \dots N_{p-1}$ we write

$$N_0 = \begin{pmatrix} p & \\ 1 & \end{pmatrix}, N_1 = \begin{pmatrix} p & \\ 1 & \end{pmatrix}, \dots, N_{p-1} = \begin{pmatrix} p & p-1 \\ 1 & \end{pmatrix} \text{ and } N_\infty = \begin{pmatrix} 1 & \\ p & \end{pmatrix}$$

(1) X_p^2 is symmetric, i.e. $\# A \mapsto B \iff B \mapsto A \quad \forall A, B \in X_p^2$.

Assume $A = \begin{pmatrix} p^m & a \\ 0 & p^n \end{pmatrix}$. Then $AN_0 = \begin{pmatrix} p^{m+1} & a \\ 0 & p^n \end{pmatrix}$ and $(AN_0)N_\infty = \begin{pmatrix} p^{m+1} & pa \\ 0 & p^{n+1} \end{pmatrix} \equiv \begin{pmatrix} p^m & a \\ 0 & p^n \end{pmatrix} = A$

$$\Rightarrow A \# \mapsto AN_0 \mapsto A.$$

Similarly $A \mapsto AN_j = \begin{pmatrix} p^{m+1} & ja + jp^m \\ 0 & p^n \end{pmatrix}$ and $AN_j N_\infty = \begin{pmatrix} p^{m+1} & pa + jp^{m+1} \\ 0 & p^{n+1} \end{pmatrix} \equiv \begin{pmatrix} p^m & a \\ 0 & p^n \end{pmatrix} = A$

So $A \mapsto AN_j \mapsto A$, for $j = 1, \dots, p-1$

Finally, $A \mapsto AN_\infty = \begin{pmatrix} p^m & pa \\ 0 & p^{n+1} \end{pmatrix} \stackrel{\text{fix}}{\equiv} \begin{pmatrix} p^m & pa \bmod p^{m+1} \\ 0 & p^{n+1} \end{pmatrix}$ (+ maybe divide by p)

Write $a = jp^{m-1} + t \quad t \in \{0, \dots, p^m - 1\}$, then $pa \bmod p^m = pt$, then

$$(AN_\infty)N_j = \begin{pmatrix} p^m & pa \bmod p^m \\ 0 & p^{n+1} \end{pmatrix} \left(\begin{pmatrix} p & \\ 1 & \end{pmatrix}^j \right) = \begin{pmatrix} p^{m+1} & ja + jp^{m+1} \\ 0 & p^{n+1} \end{pmatrix} = \begin{pmatrix} p^{m+1} & pa \\ 0 & p^{n+1} \end{pmatrix} = \begin{pmatrix} p^m & a \\ 0 & p^n \end{pmatrix}$$

example $\begin{pmatrix} 4 & 3 \\ 2 & \end{pmatrix} \xrightarrow{\text{con}} \begin{pmatrix} 4 & 6 \\ 4 & \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 2 \\ 4 & \end{pmatrix}$

$$\begin{pmatrix} 4 & 1 \\ 2 & \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 2 \\ 4 & \end{pmatrix}$$

(3)

Define level structure on $\mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}])$ on X_p^d

$$\text{level}(A) = \log_p (\det A) \quad \text{for } A \in X_p^d$$

on $\mathrm{PGL}_d(\mathbb{Z}[\frac{1}{p}])$ the level of an element is the level of the $\mathrm{PGL}_d(\mathbb{Z})$ -rep in X_p^d .

$$\text{level} \begin{pmatrix} p^m & a \\ & p^n \end{pmatrix} = m+n$$

Claim: \exists a path of length = level(A) from I to A.

Proof: Instead we will show that there is such a path from A to I. By ① (symmetry) the claim will follow.

$$A = \begin{pmatrix} p^m & a \\ & p^n \end{pmatrix} \quad \begin{cases} \text{if } m > 0 \\ \text{if } m = 0 \quad \text{and } n > 0 \end{cases} \quad AN_0 = \begin{pmatrix} p^m & pa \pmod{p^m} \\ 0 & p^{n+1} \end{pmatrix} = \begin{pmatrix} p^{m-1} & a \pmod{p^{m-1}} \\ 0 & p^n \end{pmatrix} \rightarrow \text{level} = (m+n).$$

$$AN_0 = \begin{pmatrix} 1 & 0 \\ & p^n \end{pmatrix} \begin{pmatrix} p & \\ & 1 \end{pmatrix} = \begin{pmatrix} p & \\ & p^n \end{pmatrix} = \begin{pmatrix} 1 & \\ & p^{n-1} \end{pmatrix} \rightarrow \text{level} = n-1.$$

$$m=n=0 \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

③ Claim: For $A \neq \text{Id}$ in $(AN_i)_{i=0, \dots, p-1, \infty}$ there are p vertices at level = level(A) + 1 and 1 vertex at level = level(A) - 1.

(4)

Proof.

$$\text{Case 1} \quad \binom{P^n \ a}{1} \quad a > 0 \quad \text{level } n$$

$$\binom{P^n \ a}{1} N_j = \binom{P^{n+1} \ j P^n a}{1} \quad \text{level } n+1$$

$$\binom{P^n \ a}{1} N_\infty = \binom{P^n \ a}{P} = \binom{P^{n-1} \ a P^n}{1} \quad \text{level } n-1$$

$$\text{Case 2} \quad \binom{1 \ 0}{P^m} \quad m > 0 \quad \text{level } m$$

$$\binom{1 \ P^m}{P^m} N_j = \binom{P \ j}{P^m} \quad \begin{cases} \text{level } m+1 & j \neq 0 \\ \text{level } m-1 & j=0 \end{cases}$$

$$\binom{1 \ P^m}{P^m} N_\infty = \binom{1 \ 0}{P^{m+1}} \quad \text{level } m+1$$

$$\text{Case 3} \quad \binom{P^n \ a}{P^m} \quad p \nmid a \quad \text{level} = n+m$$

Exercise →

Cor: X_P^2 is a tree.