Ramanujan graphs and complexes

20/11/17

Lecture 8

Integral lattice in $\mathbb{Z}^d$ - $\mathbb{Z}$ span of d linearly indep. vectors of $\mathbb{Z}^d$.

$=$ Subgroup of $\mathbb{Z}^d$ not contained in any (proper) subspace of $\mathbb{R}^d$.

$= A\mathbb{Z}^d$ for some $A \in M_d(\mathbb{Z})$ with $\det A \neq 0$

When is $A\mathbb{Z}^d = B\mathbb{Z}^d$? Iff $A = BC$ for some $C \in GL_d(\mathbb{Z})$.

Fix a prime p.

Bijections

lattices	matrices $M_d(\mathbb{Z})$
$A\mathbb{Z}^d \longleftarrow$	A
primitive lattice L: $\frac{1}{m}L \not\subseteq \mathbb{Z}^d$ $\forall m>1$ $\longleftrightarrow$	primitive matrix A: $\gcd(A_{ij}) = 1$
p-lattice: $\mathrm{covol}(L) = p^m$ for some $m \in \mathbb{N}$ $\longleftrightarrow$	$\lvert\det A\rvert = p^m$

$$A\mathbb{Z}^d \xleftrightarrow{\text{bij}} \{A \in M_d(\mathbb{Z}) : \det A \neq 0\}/GL_d(\mathbb{Z})$$

$\downarrow$ primitive

Primitive p-lattice

$$A\mathbb{Z}^d \xleftrightarrow{\text{bij}} \{A \in M_d(\mathbb{Z}) : \lvert\det A\rvert = p^m,\ \text{primitive}\}/GL_d(\mathbb{Z})$$

$$\xleftrightarrow{\text{bij}} GL_d(\mathbb{Z}[\tfrac{1}{p}])/GL_d(\mathbb{Z})$$

We are interested in $X_p^d = \{\text{all primitive } p\text{-lattices in } \mathbb{Z}^d\}$

Claim: $X_p^d \cong PGL_d(\mathbb{Z}[\tfrac{1}{p}])/PGL_d(\mathbb{Z})$

Each A in $PGL_d(\mathbb{Z}[\frac{1}{p}])$ has a unique scaling by some p^m which is integral and primitive.

Corollary: $PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts transitively on X_p^d.

Goal: Find rep. for the cosets.

Say A is a primitive, integral p-matrix $|\det(\cdot)| = p^m$ for $m \in \mathbb{N}$

Take $d=2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ elementry op. over $\mathbb{Z}$ do not change the column span $\Leftrightarrow$ the lattice $\longleftrightarrow$ left $GL_d(\mathbb{Z})$ coset.

$\begin{bmatrix} \text{switch columns} \\ \text{multi a column by } \pm 1 \\ \text{add column to another} \end{bmatrix}$ Using euclid's alg for gcd we obtain $\begin{pmatrix} x & y \\ 0 & \gcd(c,d) \end{pmatrix}$

$\det(\cdot) = x \cdot \gcd(c,d) = \pm p^m \Rightarrow$ x and $\gcd(c,d)$ are powers of $\pm p$

$\Rightarrow$ by multiplying column by ± 1 we can assume both are positive

$A\mathbb{Z}^d = \begin{pmatrix} p^m & z \\ 0 & p^n \end{pmatrix}\mathbb{Z}^d$ by adding / substracting column 1 from 2

we can assume that $0 \leq z \leq p^m - 1$. Finally,

Claim: $X_2^p \longleftrightarrow \left\{ \begin{pmatrix} p^n & a \\ 0 & p^m \end{pmatrix} : \begin{array}{l} 0 \le a < p^m \quad m,n \ge 0 \\ \text{either } n=0 \text{ or } m=0 \text{ or } n,m>0 \text{ and } p \nmid a \end{array} \right\}$

Ex: Prove $\longleftarrow$. Each such matrix gives a diff lattice.

For general d we can do the same

$$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \mathbb{Z}^d \longleftrightarrow \begin{pmatrix} p^m & a & b \\ 0 & p^n & c \\ 0 & 0 & p^l \end{pmatrix} \qquad \boxed{\begin{array}{l} 0 \le a,b \le p^m - 1 \\ 0 \le c \le p^n - 1 \\ \gcd(p^m, p^n, p^l, a, b, c) = 1 \end{array}}$$

$$X_d^p \longleftrightarrow \left\{ \begin{pmatrix} p^{n_1} & & a_{ij} \\ & \ddots & \\ 0 & & p^{n_d} \end{pmatrix} : \begin{array}{l} 0 \le a_{ij} \le p^{n_i} - 1 \\ \gcd(p^{n_i}, a_{ij}) = 1 \end{array} \quad 1 \le i \le d \right\}$$

X_d^p are the vertices of the Affine Bruhat-Tits building of type $\tilde{A}_{d-1}$ over $\mathbb{Z}[\frac{1}{p}]$.

Claim: X_2^p is a $(p+1)$-reg tree.

Edges (First attempt): We say that $\{L_1, L_2\}$ is an oriented edge if $L_2 \overset{p}{<} L_1$ or $pL_2 \overset{p}{\le} L_1$, where $X \overset{p}{\le} Y$ means $X < Y$ and $[Y:X] = p$.

e.g. $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \overset{3}{\le} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{tikzcd} & \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \arrow[ur] \arrow[dr] & & \\ & \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} & \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \end{tikzcd}$$

$A\mathbb{Z}^2 \overset{3}{\le} \mathbb{Z}^2 \Rightarrow \det A = 3$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \xrightarrow{e_\infty} \begin{pmatrix}1&0\\0&3\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \xrightarrow{e_0} \begin{pmatrix}3&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \xrightarrow{e_1} \begin{pmatrix}3&1\\0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \xrightarrow{e_2} \begin{pmatrix}3&2\\0&1\end{pmatrix}$$

$d=3$

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \xrightarrow{e_{\infty\infty}} \begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix}$$

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \xrightarrow{e_{\infty a}} \begin{pmatrix}1&&\\&p&a\\&&1\end{pmatrix} \qquad 0 \leq a \leq p-1$$

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \xrightarrow{e_{bc}} \begin{pmatrix}p&b&c\\&1&0\\&&1\end{pmatrix} \qquad b, c = 0, \dots p-1$$

in X_d^p there are $p^{d-1} + p^{d-2} + \dots + p + 1$ edges leaving $\mathbb{Z}^d$

There is no $\mathbb{Z}^2 \overset{p}{<} L$, so when is $p\mathbb{Z}^2 \overset{p}{<} L$ $\quad \begin{pmatrix}p&0\\0&p\end{pmatrix} \overset{p}{<} L$

$\Rightarrow$ covol $L = p$

$$\begin{pmatrix}p&0\\0&1\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}p&1\\0&1\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix}p&p-1\\0&1\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&p\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

The bijection between ~~directed~~ edges in $d=2$ does not exists in $d \geq 3$.

Claim: $\forall A \in X_p^d$ A has out deg $\frac{p^d-1}{p-1}$ with the following endpoints $A \cdot A_1, \ldots, A \cdot A_{\frac{p^d-1}{p-1}}$, where $(A_i)_{i=1}^{\frac{p^d-1}{p-1}}$ are the out neighbors of $\mathbb{Z}^d$.

When thinking about X_p^d as $PGL_d(\mathbb{Z}[\frac{1}{p}])$

e.g. $\begin{pmatrix}1&\\&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&\\&3\end{pmatrix}$

primitive lattice generated

$$\begin{pmatrix}3&\\&1\end{pmatrix} \longrightarrow \begin{pmatrix}3&\\&1\end{pmatrix}\begin{pmatrix}1&\\&3\end{pmatrix} = \begin{pmatrix}3&\\&3\end{pmatrix} \overset{\downarrow}{\equiv} \begin{pmatrix}1&\\&1\end{pmatrix}$$

Claim: $\Gamma = PGL_d(\mathbb{Z}[\frac{1}{p}])$ acts on the graph thus obtained. Furthermore, if we have started with the edge $\begin{pmatrix}1&\\&1\end{pmatrix} \to \begin{pmatrix}p&\\&1\end{pmatrix}$ and required $PGL_d(\mathbb{Z}[\frac{1}{p}])$ action the graph we construct, we will be forced to have all these edges. (Since Γ has the elements $\begin{pmatrix}1&1\\&1\end{pmatrix}\begin{pmatrix}1&2\\&1\end{pmatrix}\cdots\begin{pmatrix}1&p-1\\&1\end{pmatrix}\begin{pmatrix}&1\\1&\end{pmatrix}$, those are row operation taking $\begin{pmatrix}1&\\&1\end{pmatrix}$ to itself and $\begin{pmatrix}p&\\&1\end{pmatrix}$ to $\begin{pmatrix}p&1\\&1\end{pmatrix} \cdots \begin{pmatrix}p&p-1\\&1\end{pmatrix}\begin{pmatrix}1&\\&p\end{pmatrix}$)

$$\begin{pmatrix}1&j\\0&1\end{pmatrix}\begin{pmatrix}1&\\&1\end{pmatrix} = \begin{pmatrix}1&j\\0&1\end{pmatrix} \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad A \equiv B \text{ if } A \in B \cdot GL_d(\mathbb{Z}).$$