

# Ramanujan graphs and complexes

5/11/17

Today:  $T_k$  the  $k$ -regular tree and in particular  $\text{Spec}(A_{T_k})$

Tomorrow: hyperbolic plane and the spec of its adjacency matrix.

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For which  $\lambda$  is there  $f: V(T_k) \rightarrow \mathbb{C}$ , s.t.  $Af = \lambda f$ ?

Consider spherical functions: fix "center"  $v_0$  and look at functions which are constant on  $\sum_m \{v: \text{dist}(v, v_0) = m\}$ ,  $\forall m \in \mathbb{N}_0$ .

This is enough: If  $Af = \lambda f$ , pick  $v_0$  s.t.  $f(v_0) \neq 0$  and look on

$$f_{\text{sph}}(w) = \frac{1}{\#\sum_{v \in S_{\text{dist}(v_0, v)}(v_0)} f(v)} \equiv \frac{1}{|\{w: \text{dist}(v_0, w) = \text{dist}(v_0, v)\}|} \sum_{\substack{w_i \\ \text{dist}(v_0, w) \\ = \text{dist}(v_0, v)}} f(w)$$

- \*  $f_{\text{sph}}$  is not zero
  - \*  $\|f_{\text{sph}}\|_2^2 \leq \|f\|_2^2$
  - \*  $Af_{\text{sph}} = \lambda f_{\text{sph}}$ .
- exercise.
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Assume  $f$  is spherical and  $Af = \lambda f$ . w.l.o.g.  $f(v_0) = 1$

Denote  $f(n)$  the value of  $f$  at a vertex at dist  $n$  from  $v_0$

$$\Rightarrow \lambda = \lambda f(v_0) = Af(v_0) = kf(1)$$

$$\Rightarrow f(1) = \frac{\lambda}{k}$$

$$\lambda f_1 = (A_1) u = (k-1)f_2 + f_0 \rightarrow \text{get } f_2$$

RECURSION

$$f_0 = 1, f_1 = \frac{\lambda}{k}$$

$$f_n = \frac{\lambda f_{n-1} - f_{n-2}}{k-1}$$

Get:  $f_n = c_1 \left( \frac{2}{\lambda + \sqrt{\lambda^2 - \rho^2}} \right)^n + c_2 \left( \frac{2}{\lambda - \sqrt{\lambda^2 - \rho^2}} \right)^n$ , where  $\rho = 2\sqrt{k-1}$

~~There is a spherical function for every  $\lambda$ . Furthermore it is unique ~~assumed~~ (we used here the fact that  $f_0 = 1, f_1 = \frac{\lambda}{k}$ )~~

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When is  $f$  in  $L^2$ ?  $\Rightarrow \lambda \in \text{Spec}(A_{T_k})$

$$\|f\|_{S_n(\omega_0)}^2 = |f(n)|^2 \cdot |S_n(\omega_0)| = k(k-1)^{n-1} |f(n)|^2 \approx (k-1)^n |f(n)|^2$$

$$= \left( c_1 \left( \frac{2\sqrt{k-1}}{\lambda + \sqrt{\lambda^2 - \rho^2}} \right)^n + c_2 \left( \frac{2\sqrt{k-1}}{\lambda - \sqrt{\lambda^2 - \rho^2}} \right)^n \right)^2$$

$$= \left( c_1 \left( \frac{\rho}{\lambda + \sqrt{\lambda^2 - \rho^2}} \right)^n + c_2 \left( \frac{\rho}{\lambda - \sqrt{\lambda^2 - \rho^2}} \right)^n \right)^2$$

Case 1:

$$|\lambda| > \rho \Rightarrow \alpha \equiv \frac{\rho}{\lambda + \sqrt{\lambda^2 - \rho^2}} \quad \beta \equiv \frac{\rho}{\lambda - \sqrt{\lambda^2 - \rho^2}}$$

$\alpha\beta = 1 \Rightarrow$  Since both  $\alpha, \beta$  are real for  $|\lambda| > \rho$   
 $\Rightarrow \alpha \geq 1$  or  $\beta \geq 1$ , Furthermore  $|\lambda| > |\rho|$ , so  $\alpha > 1$   
 or  $\beta > 1$ .

$\Rightarrow$  exp growth  $\|f|_{S_n(w_0)}\| \rightarrow \infty$

Case 2:

$$|\lambda| \leq \rho \quad \alpha = \frac{\rho}{\lambda + i\sqrt{\rho^2 - \lambda^2}} \quad \beta = \bar{\alpha} \quad \text{Also } c_2 = \bar{c}_1$$

$$\text{So } \|f|_{S_n(w_0)}\|_2^2 = \left[ 2 \operatorname{Re} \left( c_1 \left( \frac{\rho}{\lambda + i\sqrt{\rho^2 - \lambda^2}} \right)^n \right) \right]$$

Observe:  $\left| \frac{\rho}{\lambda + i\sqrt{\rho^2 - \lambda^2}} \right| = 1$

$$\Rightarrow \|f|_{S_n(w_0)}\|_2^2 = \left( 2 \operatorname{Re} (c_1 \alpha^n) \right)^2$$

For infinitely many  $n$ 's  $\|f|_{S_n(w_0)}\|_2^2 \geq \delta > 0$ ,

$$\Rightarrow \sum_{n=0}^{\infty} \|f|_{S_n(w_0)}\|_2^2 = \infty \quad \text{for } \delta \text{ many } n\text{'s this is at least } \delta.$$

$A_{-k}$  has no  $L^2$ -eigenfunctions

$$\text{Spec}(A_{-k}) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible} \}.$$

Fact. ~~As before~~ If  $A$  is self adjoint  $\lambda \in \text{Spec}(A)$

$$\Leftrightarrow \exists \text{ seq } f_n \in L^2 \text{ s.t. } \frac{\|(A - \lambda I)f_n\|}{\|f_n\|} \xrightarrow{n \rightarrow \infty} 0 \quad (*)$$

Fix  $k, \lambda$  as before. Define for  $m \in \mathbb{N}$

$$f_m(u) = \begin{cases} f(u) & \text{dist}(u_0, u) \leq m \iff u \in B_m(u_0) \\ 0 & \text{otherwise} \end{cases}$$

for the spherical e.function of  $\lambda$  we found before.

~~As before~~

$$(A - \lambda I)f_m = \begin{cases} 0 & n \leq m-1 \\ n = m \\ 0 & n = m+1 \\ 0 & n \geq m+2 \end{cases}$$

Tomorrow! We will show that for  $|k| \leq p$  those are approximated eigenfunctions.

↙  
a seq of  $f_n$  as above. (\*)